

## OSCILLATORY BEHAVIOR OF SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE NEUTRAL TERMS

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*Abstract.* The aim of the paper is to initiate a study of the oscillation of solutions of second order nonlinear differential equations with positive and negative nonlinear neutral terms. The results are illustrated by some examples.

### 1. Introduction

This paper is concerned with oscillatory behavior of all solutions of nonlinear second order delay differential equations with positive and negative neutral terms. The second term in the equation contains a delay as well. In particular, the equations under consideration here have the form

$$\left(r(t) (y'(t))^\beta\right)' + q(t)x^\gamma(\tau(t)) = 0, \quad t \geq t_0 \geq 1, \quad (1.1a)$$

where

$$y(t) := x(t) + p_1(t)x^{\alpha_1}(\sigma(t)) - p_2(t)x^{\alpha_2}(\sigma(t)). \quad (1.1b)$$

In the sequel, we will make use of the following conditions:

- (H<sub>0</sub>)  $\alpha_1, \alpha_2, \beta,$  and  $\gamma$  are the ratios of positive odd integers;
- (H<sub>1</sub>)  $p_1, p_2, q \in \mathcal{C}([t_0, \infty), [0, \infty))$  and  $q$  does not vanish identically on any half-line of the form  $[t_*, \infty)$  for any  $t_* > t_0$ ;
- (H<sub>2</sub>) the delay functions  $\tau, \sigma \in \mathcal{C}([t_0, \infty), \mathbb{R})$  are such that  $\tau(t) \leq t, \sigma(t) \leq t, \sigma'(t) > 0,$  and  $\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \sigma(t) = \infty$ ;
- (H<sub>3</sub>)  $r \in \mathcal{C}^1([t_0, \infty), (0, \infty))$  satisfies

$$R(t, t_0) := \int_{t_0}^t \frac{1}{r^{1/\beta}(s)} ds \rightarrow \infty \quad \text{as } t \rightarrow \infty;$$

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(H<sub>4</sub>)  $h(t) = \sigma^{-1}(\tau(t)) \leq t$ ,  $h'(t) \geq 0$ , and  $\lim_{t \rightarrow \infty} h(t) = \infty$ .

Let  $T \geq t_0$  be such that  $\tau(t) \geq t_0$  and  $\sigma(t) \geq t_0$  for  $t \geq T$ . By a *solution* of equation (1.1), we mean a function  $x \in \mathcal{C}([T, \infty), \mathbb{R})$  having the property that  $r(y')^\gamma \in \mathcal{C}^1([T, \infty), \mathbb{R})$  and which satisfies (1.1) on  $[T, \infty)$ . We only consider those solutions of (1.1) that exist on some half-line to the right and satisfy the condition

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \text{ for any } T_1 \geq T.$$

Moreover, we tacitly assume that equation (1.1) possesses such solutions. As is customary, a solution  $x$  of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and it is said to be *nonoscillatory* otherwise. The equation itself is termed oscillatory if all its solutions oscillate. We note that the equation is called half-linear if  $\gamma = \beta$ , sub-half-linear if  $\gamma \leq \beta$  and super-half-linear if  $\gamma \geq \beta$ .

In recent years, there has been considerable research activity concerning the oscillation and nonoscillation of solutions of various differential equations. The qualitative study of neutral differential equations, has besides its theoretical interest, significant practical importance. This is due to fact that they arise in various phenomena including problems concerning electric networks containing lossless transmission lines (as in high speed computers where such lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, and in the solution of variational problems with time delays. We refer the reader to Hale's monograph [7] for further applications in science and technology.

In reviewing the literature, it becomes apparent that results on the oscillatory behavior of second-order differential equations with a single sublinear neutral term are relatively scarce. For initial contributions on such equations, we refer the reader to [6] where some oscillation results for (1.1) were obtained in the linear case  $\beta = \gamma = 1$  and  $p_2(t) = 0$  using the Riccati transformation technique and some inequalities. On the other hand, Grace [5] established some new results for the case  $p_1(t) = 0$  and  $\alpha_2 = 1$ . Li and Rogovchenko [13, 14] have obtained oscillation results for neutral equations with  $p_2(t) = 0$  by comparing to first order equations and inequalities respectively. Additional results on equations with neutral terms can be found in [1, 2, 3, 4, 11, 13, 14, 16, 17, 18, 20]; while some of the equations studied in these papers have advanced and delayed arguments in their neutral terms and some have positive and some have negative terms, it is important to note that in all cases the neutral terms are linear functions.

To the best of our knowledge, there are no results for second-order differential equations with a neutral term of the form (1.1b) as studied in this paper. The aim of the present paper is to initiate the study of the oscillation problem for the nonlinear delay differential equation (1.1) under conditions (H<sub>0</sub>)–(H<sub>4</sub>) and different ranges on the values of  $\alpha_1$  and  $\alpha_2$ .

## 2. Main results

As usual, all functional inequalities considered in this paper are assumed to be satisfied for all sufficiently large  $t$ . Without loss of generality, in our proofs we only

need to be concerned with positive solutions of (1.1) since the proofs for eventually negative solutions are similar.

The following three lemmas are needed in the proofs of our main results.

LEMMA 1. Let  $q : [t_0, \infty) \rightarrow \mathbb{R}^+$ ,  $g : [t_0, \infty) \rightarrow \mathbb{R}$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions,  $f$  be nondecreasing with  $xf(x) > 0$  for  $x \neq 0$ , and  $g(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . If the first-order delay differential inequality (i.e.,  $g(t) \leq t$ )

$$y'(t) + q(t)f(y(g(t))) \leq 0$$

has an eventually positive solution, then so does the delay equation

$$y'(t) + q(t)f(y(g(t))) = 0.$$

*Proof.* This lemma is an extension of known results in [9, 10, 15] and the proof is immediate.  $\square$

LEMMA 2. (Young’s Inequality) Let  $X, Y$  be nonnegative,  $n > 1$ , and  $1/n + 1/m = 1$ . Then

$$XY \leq \frac{1}{n}X^n + \frac{1}{m}Y^m, \tag{2.1}$$

and equality holds if and only if  $Y = X^{n-1}$ .

LEMMA 3. ([8]) Let  $X, Y$  be nonnegative. Then

$$\begin{aligned} X^\lambda + (\lambda - 1)Y^\lambda - \lambda XY^{\lambda-1} &\geq 0 \quad \text{for } \lambda > 1 \\ X^\lambda - (1 - \lambda)Y^\lambda - \lambda XY^{\lambda-1} &\leq 0 \quad \text{for } 0 < \lambda < 1, \end{aligned} \tag{2.2}$$

where equality holds if and only if  $X = Y$ .

Our first oscillation result is contained in the following theorem.

THEOREM 1. Let  $0 < \alpha_1 < \alpha_2 \leq 1$ , conditions  $(H_0)$ – $(H_4)$  hold,

$$\lim_{t \rightarrow \infty} p_1(t) = 0, \tag{2.3}$$

$$\begin{aligned} 0 < p_2(t) < p < 1 &\quad \text{for } \alpha_2 = 1, \\ 0 < p_2(t) < p < \infty &\quad \text{for } 0 < \alpha_2 < 1. \end{aligned} \tag{2.4}$$

If there exist a constant  $\theta \in (0, 1)$  and a nondecreasing function  $\xi(t) : [t_0, \infty) \rightarrow \mathbb{R}^+$  such that  $h(t) \leq \xi(t) \leq t$  for  $t \geq t_0$  and both equations

$$W'(t) + \theta q(t)R^\gamma(\tau(t), t_1)W^{\gamma/\beta}(\tau(t)) = 0, \tag{2.5}$$

where  $t$  is large enough that  $\tau(t) \geq t_1$ , and

$$Z'(t) + \frac{q(t)}{p_2^{\gamma/\alpha_2}(h(t))}R^{\gamma/\alpha_2}(\xi(t), h(t))Z^{\gamma/\alpha_2\beta}(\xi(t)) = 0 \tag{2.6}$$

are oscillatory, then equation (1.1) is oscillatory.

*Proof.* Suppose, to the contrary, that  $x$  is an eventually positive solution of (1.1). Then there exists  $t_1 \in [t_0, \infty)$  such that  $x(\tau(t)) > 0$  and  $x(\sigma(t)) > 0$  on  $[t_1, \infty)$ . From (1.1), we have

$$\left( r(y')^\beta \right)'(t) = -q(t)x^\gamma(\tau(t)) \leq 0$$

on  $[t_1, \infty)$ , which implies that  $r(t)(y'(t))^\beta$  is nonincreasing and eventually does not change its sign on  $[t_2, \infty)$  for some  $t_2 \geq t_1$ . We claim that  $y'(t) > 0$  on  $[t_2, \infty)$ . Indeed, for the sake of a contradiction, assume that  $y'(t) < 0$  on  $[t_2, \infty)$ . Then there exists  $t'_2 \geq t_2$  such that

$$r(t)(y'(t))^\beta \leq r(t'_2)(y'(t'_2))^\beta := c_0 < 0 \quad \text{on } [t'_2, \infty).$$

Integrating the above inequality from  $t'_2$  to  $t$  and taking (H<sub>3</sub>) into account, we have

$$y(t) \leq y(t'_2) + c_0^{1/\beta} \int_{t'_2}^t r^{-1/\beta}(s) ds \rightarrow -\infty \quad \text{as } t \rightarrow \infty. \tag{2.7}$$

Since  $y(t) > -p_2(t)x^{\alpha_2}(\sigma(t))$ ,  $x(t)$  must be unbounded, and so there exists an increasing sequence  $\{s_k\}$  such that  $s_1 > t'_2$ ,  $\lim_{k \rightarrow \infty} s_k = \infty$ , and  $\lim_{k \rightarrow \infty} x(s_k) = \infty$ , where  $x(s_k) = \max\{x(u) : t_0 \leq u \leq s_k\}$ . Since  $\lim_{t \rightarrow \infty} \sigma(t) = \infty$ ,  $\sigma(s_k) > t_0$  for all sufficiently large  $k$ . Also, since  $\sigma(t) \leq t$ , we see that

$$x(\sigma(s_k)) \leq \max\{x(u) : t_0 \leq u \leq s_k\} = x(s_k).$$

Therefore,

$$y(s_k) \geq x(s_k) - p_2(s_k)x^{\alpha_2}(\sigma(s_k)) \geq \left( 1 - \frac{p_2(s_k)}{x^{1-\alpha_2}(s_k)} \right) x(s_k) > 0$$

for all sufficiently large  $k$ , which contradicts the fact that  $\lim_{t \rightarrow \infty} y(t) = -\infty$ . Hence, we have proven the claim, i.e.,  $y'(t) > 0$  on  $[t_2, \infty)$ .

Next, we have two cases to consider: either  $y(t) > 0$  or  $y(t) < 0$  for  $t \geq t_3$ , for some  $t_3 \geq t_2$ . First, suppose that  $y(t) > 0$  for  $t \geq t_3$ . Applying (2.1) from Lemma 2 with

$$n = \frac{\alpha_2}{\alpha_1} > 1, \quad X = x^{\alpha_1}(\sigma(t)), \quad Y = \frac{\alpha_1}{\alpha_2} \left( \frac{p_1(t)}{p_2(t)} \right), \quad \text{and} \quad m = \frac{\alpha_2}{\alpha_2 - \alpha_1},$$

and simplifying, we obtain

$$p_1(t)x^{\alpha_1}(\sigma(t)) - p_2(t)x^{\alpha_2}(\sigma(t)) \leq \frac{\alpha_2 - \alpha_1}{\alpha_1} \left( \frac{\alpha_1}{\alpha_2} p_1(t) \right)^{\alpha_2/(\alpha_2 - \alpha_1)} p_2^{\alpha_1/(\alpha_1 - \alpha_2)}(t) = P(t).$$

Thus, we see that

$$x(t) \geq \left( 1 - \frac{P(t)}{y(t)} \right) y(t).$$

Since  $y(t)$  is increasing,

$$y(t) \geq y(t_3) =: c_1 \tag{2.8}$$

for  $t \geq t_3$ . Hence,

$$x(t) \geq y(t) \left( 1 - \frac{P(t)}{c} \right).$$

Since in view of (2.3),  $\lim_{t \rightarrow \infty} P(t) = 0$ , for any fixed  $\varepsilon \in (0, 1)$  there exists  $t_\varepsilon \geq t_3$  such that

$$x(t) \geq \varepsilon y(t) \tag{2.9}$$

for  $t \geq t_\varepsilon$ . Thus, we have

$$\left( r(t) (y'(t))^\beta \right)' + \varepsilon^\gamma q(t) y^\gamma(\tau(t)) \leq 0. \tag{2.10}$$

Due to the fact that  $r^{1/\beta} y'$  is nonincreasing, we see that

$$y(t) \geq \int_{t_\varepsilon}^t r^{-1/\beta}(s) r^{1/\beta}(s) y'(s) ds \geq R(t, t_\varepsilon) r^{1/\beta}(t) y'(t). \tag{2.11}$$

Letting  $W(t) = r(t) (y'(t))^\beta$ , we have  $y'(t) = (W(t)/r(t))^{1/\beta}$ , and so  $y(t) \geq R(t, t_\varepsilon) W^{1/\beta}(t)$ . Using the above inequality in (2.10) gives

$$W'(t) + \varepsilon^\gamma q(t) R^\gamma(\tau(t), t_\varepsilon) W^{\gamma/\beta}(\tau(t)) \leq 0.$$

It follows from Lemma 1 that the corresponding differential equation (2.5) also has a positive solution, which is a contradiction.

Next, we consider the case where  $y(t) < 0$  for  $t \geq t_3$ . It follows from the definition of  $y$  that

$$z(t) = -y(t) = -x(t) - p_1 x^{\alpha_1}(\sigma(t)) + p_2(t) x^{\alpha_2}(\sigma(t)) \leq p_2(t) x^{\alpha_2}(\sigma(t)),$$

or

$$x(\sigma(t)) \geq \left( \frac{z(t)}{p_2(t)} \right)^{1/\alpha_2},$$

or

$$x(t) \geq \left( \frac{z(\sigma^{-1}(t))}{p_2(\sigma^{-1}(t))} \right)^{1/\alpha_2},$$

and so

$$\left( r(z')^\beta \right)'(t) = q(t) x^\gamma(\tau(t)) \geq q(t) \left( \frac{z(\sigma^{-1}(\tau(t)))}{p_2(\sigma^{-1}(\tau(t)))} \right)^{\gamma/\alpha_2} = \frac{q(t)}{p_2^{\gamma/\alpha_2}(h(t))} z^{\gamma/\alpha_2}(h(t)). \tag{2.12}$$

Clearly, we see that  $z'(t) = -y'(t) < 0$  for  $t \geq t_3$ . Now, for  $t_3 \leq u \leq v$ , we may write

$$z(u) - z(v) = - \int_u^v r^{-1/\beta}(s) r^{1/\beta}(s) z'(s) ds \geq R(v, u) \left( -r^{1/\beta}(v) z'(v) \right). \tag{2.13}$$

We let  $u = h(t)$  and  $v = \xi(t)$  in the above inequality to obtain

$$z(h(t)) \geq R(\xi(t), h(t)) \left( -r^{1/\beta}(\xi(t)) z'(\xi(t)) \right). \tag{2.14}$$

Using (2.14) in (2.12) gives

$$\begin{aligned} \left( r(z')^\beta \right)'(t) &\geq \frac{q(t)}{p_2^{\gamma/\alpha_2}(h(t))} z^{\gamma/\alpha_2}(h(t)) \\ &\geq \frac{q(t)}{p_2^{\gamma/\alpha_2}(h(t))} R^{\gamma/\alpha_2}(\xi(t), h(t)) \left( -r^{1/\beta}(\xi(t)) z'(\xi(t)) \right)^{\gamma/\alpha_2}. \end{aligned}$$

Setting  $Z(t) = -r(t)(z'(t))^\beta$ , we obtain

$$Z'(t) + \frac{q(t)}{p_2^{\gamma/\alpha_2}(h(t))} R^{\gamma/\alpha_2}(\xi(t), h(t)) Z^{\gamma/\alpha_2 \beta}(\xi(t)) \leq 0.$$

The remainder of the proof is similar to the case where  $y > 0$  and hence is omitted.  $\square$

**THEOREM 2.** Let  $0 < \alpha_1 < \alpha_2 \leq 1$  and conditions  $(H_0)$ – $(H_4)$ , (2.3), and (2.4) hold. If there exists a constant  $\theta \in (0, 1)$  such that (2.5) is oscillatory for any  $t_1 \geq t_0$  large enough with  $t \geq \tau(t) \geq t_1$  and

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t \frac{q(s)}{p_2^{\gamma/\alpha_2}(h(s))} R^{\gamma/\alpha_2}(h(t), h(s)) ds \begin{cases} > 1 & \text{if } \alpha_2 \beta = \gamma, \\ = \infty & \text{if } \alpha_2 \beta > \gamma, \end{cases} \quad (2.15)$$

then (1.1) is oscillatory.

*Proof.* Suppose to the contrary that  $x$  is an eventually positive solution of (1.1). Then there exists  $t_1 \in [t_0, \infty)$  such that  $x(\tau(t)) > 0$  and  $x(\sigma(t)) > 0$  on  $[t_1, \infty)$ . Proceeding as in the proof of Theorem 1, we obtain (2.12) and (2.13). Letting  $u = h(s)$  and  $v = h(t)$  in (2.13), we arrive at

$$z(h(s)) \geq R(h(t), h(s)) \left( -r^{1/\beta}(h(t)) z'(h(t)) \right).$$

Integrating (2.12) from  $h(t)$  to  $t$ , we have

$$\begin{aligned} Z(t) := -r(h(t)) (z'(h(t)))^\beta &\geq \int_{h(t)}^t \frac{q(s)}{p_2^{\gamma/\alpha_2}(h(s))} z^{\gamma/\alpha_2}(h(s)) ds \\ &\geq \left( -r^{1/\beta}(h(t)) z'(h(t)) \right)^{\gamma/\alpha_2} \int_{h(t)}^t \frac{q(s)}{p_2^{\gamma/\alpha_2}(h(s))} R^{\gamma/\alpha_2}(h(t), h(s)) ds \\ &= Z^{\gamma/\alpha_2 \beta}(t) \int_{h(t)}^t \frac{q(s)}{p_2^{\gamma/\alpha_2}(h(s))} R^{\gamma/\alpha_2}(h(t), h(s)) ds, \end{aligned}$$

that is,

$$Z^{1-\gamma/\alpha_2 \beta}(t) \geq \int_{h(t)}^t \frac{q(s)}{p_2^{\gamma/\alpha_2}(h(s))} R^{\gamma/\alpha_2}(h(t), h(s)) ds.$$

Taking the lim sup on both sides of the above inequality, we arrive at a contradiction to (2.15). The proof is now complete.  $\square$

Next, we have the following corollaries. The first two are based on a well known oscillation result in [9, Theorem 1].

COROLLARY 1. Let  $\alpha_1 < \alpha_2 = \beta = \gamma = 1$ , and conditions  $(H_0)$ – $(H_4)$ , (2.3), and (2.4) hold. If

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t q(s)R(\tau(s), t_1) ds > \frac{1}{e}$$

for any  $t_1 \geq t_0$  large enough, and there exists a nondecreasing function  $\xi(t) : [t_0, \infty) \rightarrow \mathbb{R}^+$  such that  $h(t) \leq \xi(t) \leq t$  for  $t \geq t_0$  such that

$$\liminf_{t \rightarrow \infty} \int_{\xi(t)}^t q(s)R(\xi(s), h(s)) ds > \frac{1}{e},$$

then (1.1) is oscillatory.

COROLLARY 2. Let  $0 < \alpha_1 < \alpha_2 \leq 1$ ,  $\gamma \leq \alpha_2 \beta$ , and conditions  $(H_0)$ – $(H_4)$ , (2.3) and (2.4) hold. If

$$\int_{t_0}^{\infty} q(s)R^\gamma(\tau(s), t_0) ds > \frac{1}{e}$$

and there exists a nondecreasing function  $\xi(t) : [t_0, \infty) \rightarrow \mathbb{R}^+$  such that  $h(t) \leq \xi(t) \leq t$  for  $t \geq t_0$  such that

$$\int_{t_0}^{\infty} q(s)R^{\gamma/\alpha_2}(\xi(s), h(s)) ds = \infty,$$

then (1.1) is oscillatory.

COROLLARY 3. Let  $0 < \alpha_1 < \alpha_2 \leq 1$  and conditions  $(H_0)$ – $(H_4)$ , (2.3) and (2.4) hold. If

$$\limsup_{t \rightarrow \infty} R(\tau(t), t_1) \left( \int_t^{\infty} q(s) ds \right)^{1/\beta} \begin{cases} > 1 & \text{if } \beta = \gamma, \\ = \infty & \text{if } \beta > \gamma, \end{cases} \quad (2.16)$$

for any  $t_1 \geq t_0$  with  $t \geq \tau(t) \geq t_1$  and (2.15) holds, then (1.1) is oscillatory.

*Proof.* Suppose to the contrary that  $x$  is an eventually positive solution of (1.1). Then there exists  $t_1 \in [t_0, \infty)$  such that  $x(\tau(t)) > 0$  and  $x(\sigma(t)) > 0$  on  $[t_1, \infty)$ . Proceeding as in the proof of Theorem 1, we obtain (2.10) and (2.11). Integrating (2.10) from  $t$  to  $u$  and passing to the limit as  $u$  approaches  $\infty$  in the resulting inequality, we obtain

$$r(\tau(t)) (y'(\tau(t)))^\beta \geq r(t) (y'(t))^\beta \geq \varepsilon^\gamma \int_t^\infty q(s)y^\gamma(\tau(s)) ds \geq \varepsilon^\gamma y^\gamma(\tau(t)) \int_t^\infty q(s) ds$$

Using the above inequality in (2.11), we get

$$y(\tau(t)) \geq R(\tau(t), t_1) r^{1/\beta}(\tau(t)) y'(\tau(t)) \geq \varepsilon^{\gamma/\beta} y^{\gamma/\beta}(\tau(t)) R(\tau(t), t_1) \left( \int_t^\infty q(s) ds \right)^{1/\beta},$$

that is,

$$y^{1-\gamma/\beta}(\tau(t)) \geq \varepsilon^{\gamma/\beta} R(\tau(t), t_1) \left( \int_t^\infty q(s) ds \right)^{1/\beta},$$

which clearly contradicts (2.16). The rest of the proof is similar to that of Theorem 2 and is omitted.  $\square$

The following results serve as analogs to Theorem 1 in cases of different ranges on  $\alpha_1$  and  $\alpha_2$ .

**THEOREM 3.** *Let  $\alpha_1 < 1$ ,  $\alpha_2 > 1$ , and conditions  $(H_0)$ – $(H_4)$  hold. Assume there exist a continuous function  $p(t) : [t_0, \infty) \rightarrow \mathbb{R}^+$  such that*

$$\lim_{t \rightarrow \infty} \frac{p^{\alpha_2}(t)}{p_2(t)} = \lim_{t \rightarrow \infty} \frac{p_1(t)}{p^{\alpha_1}(t)} = 0, \quad (2.17)$$

a constant  $\theta \in (0, 1)$ , and a nondecreasing function  $\xi(t) : [t_0, \infty) \rightarrow \mathbb{R}^+$  with  $h(t) \leq \xi(t) \leq t$  for  $t \geq t_0$  such that both equations (2.5) and (2.6) are oscillatory and

$$\int_{t_0}^{\infty} \frac{R^{\gamma/\alpha_2}(h(t), t_0)q(t)}{p_2^{\gamma/\alpha_2}(h(t))} dt = \infty. \quad (2.18)$$

Then a solution  $x(t)$  of (1.1) is either oscillatory or satisfies  $\lim_{t \rightarrow \infty} |x(t)| = \infty$ .

*Proof.* Suppose to the contrary that  $x$  is an eventually positive solution of (1.1) such that say  $x(\tau(t)) > 0$  and  $x(\sigma(t)) > 0$  on  $[t_1, \infty)$  for some  $t_1 \geq t_0$ . From (1.1), we have

$$\left( r(y')^\beta \right)'(t) = -q(t)x^\gamma(\tau(t)) \leq 0$$

on  $[t_1, \infty)$ , which implies that  $r(t)(y'(t))^\beta$  is nonincreasing and eventually does not change its sign on  $[t_2, \infty)$  for some  $t_2 \geq t_1$ . We shall distinguish the following four cases for  $t \geq t_2$ :

- (I)  $y(t) > 0$  and  $y'(t) < 0$ ; (II)  $y(t) > 0$  and  $y'(t) > 0$ ;  
 (III)  $y(t) < 0$  and  $y'(t) > 0$ ; (IV)  $y(t) < 0$  and  $y'(t) < 0$ .

First, we consider case (I). As in the proof Theorem 1 we arrive at (2.7), which yields a contradiction.

For case (II), from the definition of  $y$ ,

$$y(t) = x(t) + (p(t)x(\sigma(t)) - p_2(t)x^{\alpha_2}(\sigma(t))) + (p_1(t)x^{\alpha_1}(\sigma(t)) - p(t)x(\sigma(t))),$$

or

$$x(t) = y(t) - (p(t)x(\sigma(t)) - p_2(t)x^{\alpha_2}(\sigma(t))) - (p_1(t)x^{\alpha_1}(\sigma(t)) - p(t)x(\sigma(t))).$$

Applying (2.2) from Lemma 3 with

$$\lambda = \alpha_2 > 1, \quad X = p_2^{1/\alpha_2}(t)x(t), \quad \text{and} \quad Y = \left( \frac{1}{\alpha_2} p(t) p_2^{-1/\alpha_2}(t) \right)^{1/(\alpha_2-1)},$$



we have

$$p(t)x(\sigma(t)) - p_2(t)x^{\alpha_2}(\sigma(t)) \leq (\alpha_2 - 1)\alpha_2^{\alpha_2/(1-\alpha_2)} p_2^{1/(1-\alpha_2)}(t)p^{\alpha_2/(\alpha_2-1)}(t) = g_2(t).$$

If we apply (2.2) with

$$\lambda = \alpha_1 < 1, \quad X = p_1^{1/\alpha_1}(t)x(t), \quad \text{and} \quad Y = \left( \frac{1}{\alpha_1} p(t)p_1^{-1/\alpha_1}(t) \right)^{1/(\alpha_1-1)},$$

we have

$$p_1(t)x^{\alpha_1}(\sigma(t)) - p(t)x(\sigma(t)) \leq (1 - \alpha_1)\alpha_1^{\alpha_1/(1-\alpha_1)} p_1^{1/(1-\alpha_1)}(t)p^{\alpha_1/(\alpha_1-1)}(t) = g_1(t).$$

Thus, we see that

$$x(t) \geq \left( 1 - \frac{g_1(t) + g_2(t)}{y(t)} \right) y(t).$$

In view of (2.17), for any  $\varepsilon \in (0, 1)$  there exists  $t_\varepsilon \geq t_2$  such that (2.9) holds for  $t \geq t_\varepsilon$ . The rest of the proof for this case is similar to that of Theorem 1 and hence is omitted.

The nonexistence of the case (III) is shown as in the proof of Theorem 1. For case (IV), we use (2.12) with  $z(t) = -y(t)$ . Clearly,  $r(t)(z'(t))^\beta \geq r(t_3)(z'(t_3))^\beta =: c > 0$ . Thus,  $z(t) \geq c^{1/\beta}R(t, t_3)$ , which in view of (2.12) gives

$$\left( r(z')^\beta \right)'(t) \geq \frac{q(t)}{p_2^{\gamma/\alpha_2}(h(t))} z^{\gamma/\alpha_2}(h(t)) \geq c^{\gamma/\beta\alpha_2} \frac{R^{\gamma/\alpha_2}(h(t), t_3)q(t)}{p_2^{\gamma/\alpha_2}(h(t))}.$$

Now, we claim that  $\lim_{t \rightarrow \infty} r(t)(z'(t))^\beta = \infty$ . If not, then by integrating the above inequality from  $t_3$  to  $t$  and passing to the limit as  $t$  approaches  $\infty$ , we obtain a contradiction to the positivity of  $z'$ . From (H<sub>3</sub>) and the fact that  $z(t) \leq p_2(t)x^{\alpha_2}(t)$ , we also have  $\lim_{t \rightarrow \infty} z(t) = \lim_{t \rightarrow \infty} x(t) = \infty$ . The proof of the theorem is now complete.  $\square$

### 3. Examples

We conclude this paper with some examples to illustrate our results.

EXAMPLE 1. Consider the delay differential equation with mixed neutral terms

$$\left( \left( x(t) + p_1(t)x^{\alpha_1}(\sigma t) - \frac{1}{2}x(\sigma t) \right)' \right)' + \frac{q_0}{t^2}x(\lambda t) = 0, \quad t \geq t_0 \geq 1, \quad (3.1)$$

where  $\alpha_1 < 1$  and  $\gamma$  are quotients of odd positive integers,  $q_0 > 0$ ,  $\lambda, \sigma \in (0, 1)$ ,  $p_1(t) > 0$ , and  $p_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Here we have  $r(t) = 1$ ,  $p_2(t) = 1/2$ ,  $\alpha_2 = \beta = \gamma = 1$ ,  $R(t, t_0) = t - t_0$ , and  $h(t) = \lambda t / \sigma$ . It is clear that (H<sub>0</sub>)–(H<sub>4</sub>), (2.3), and (2.4) hold. With  $\xi(t) = 2\lambda t / \sigma$  and  $2\lambda / \sigma < 1$ , it is easy to see that if

$$\min \left\{ q_0 \lambda \ln \frac{1}{\lambda}, \frac{q_0 \lambda}{\sigma} \ln \frac{\sigma}{2\lambda} \right\} > \frac{1}{e}, \quad (3.2)$$

then the integral conditions in Corollary 1 hold, and so equation (3.1) is oscillatory.

EXAMPLE 2. Consider the equation

$$\left( t \left( x(t) + \frac{1}{t} x^{3/7} \left( \frac{t}{2} \right) - \frac{1}{2} x^{5/7} \left( \frac{t}{2} \right) \right) \right)' + q(t) x^\gamma \left( \frac{t}{4} \right) = 0, \quad t \geq t_0 \geq 1. \quad (3.3)$$

Here we have  $r(t) = t$ ,  $p_1(t) = 1/t$ ,  $p_2(t) = 1/2$ ,  $\alpha_1 = 3/7$ ,  $\alpha_2 = 5/7$ ,  $\beta = 1$ ,  $R(t, t_0) = \ln(t/t_0)$ , and  $h(t) = t/2$ . Conditions (H<sub>0</sub>)–(H<sub>4</sub>), (2.3), and (2.4) hold. We take  $\gamma \leq \alpha_2\beta = 5/7$  and  $\xi(t) = 3t/4$ . Then if

$$\int_{t_0}^{\infty} q(s) \ln \frac{s}{4t_0} ds > \frac{1}{e} \quad \text{and} \quad \int_{t_0}^{\infty} q(s) \left( \ln \frac{3}{2} \right)^{7\gamma/5} ds = \infty,$$

equation (3.3) is oscillatory by Corollary 2.

EXAMPLE 3. Consider the equation

$$\left( e^{-t} \left( x(t) + \frac{1}{t} x^{1/3} \left( \frac{t}{2} \right) - x^3 \left( \frac{t}{2} \right) \right) \right)' + \left( \frac{3}{4} e^{t/6} - e^{-7t/6} \left( \frac{2}{t^3} + \frac{5}{6t^2} - \frac{5}{6t} \right) \right) x \left( \frac{t}{3} \right) = 0, \quad t \geq t_0 \geq 1. \quad (3.4)$$

Here  $r(t) = e^{-t}$ ,  $p_1(t) = 1/t$ ,  $p_2(t) = 1$ ,  $\alpha_1 = 1/3$ ,  $\alpha_2 = 3$ ,  $\beta = \gamma = 1$ ,  $R(t, t_0) = e^t - e^{t_0}$ , and  $h(t) = 2t/3$ . Conditions (H<sub>0</sub>)–(H<sub>4</sub>), (2.3), and (2.4) are seen to hold. With  $p(t) = 1/t$ , condition (2.17) holds. Take  $\xi(t) = 5t/6$ . Then by [9, Theorem 1], equations (2.5) and (2.6) are oscillatory. It is also clear that condition (2.18) holds, so by Theorem 3, a solution  $x(t)$  of (3.4) is either oscillatory or  $\lim_{t \rightarrow \infty} |x(t)| = \infty$ . One such solution is  $x(t) = e^t$ .

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