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EXISTENCE THEORY AND STABILITY RESULTS FOR ψ–TYPE COMPLEX–ORDER IMPLICIT DIFFERENTIAL EQUATIONS

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Abstract. This paper concerns the existence and stability results for ψ-type complex-order implicit differential equations with boundary conditions. The results are based on the Banach contraction mapping principle. An example is presented to illustrate the main results.

1. Introduction

The concept of fractional derivative appeared for the first time in a famous correspondence between G.A. de L'Hôpital and G.W. Leibniz, in 1695. Many mathematicians have further developed this area and we can mention the studies of L. Euler (1730), J.L. Lagrange (1772), P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823), J. Liouville (1832), B. Riemann (1847), G.H. Hardy and J.E. Littlewood (1917), H. Weyl (1919) and E.R. Love (1938). In the past sixty years, fractional calculus had played a very important role in various fields such as physics, chemistry, mechanics, electricity, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. Further facts from historical point of view, and recent developments are reported on fractional calculus, one can refer to monographs Kilbas et al. [6], Lakshmikantham et al. [7], Miller and Ross [10] and Podlubny [11].

In the last decade, fractional calculus has been recognized as one of the best tools to describe long-memory processes. Such models are interesting for engineers and physicists but also for pure mathematicians. The most important among such models are those described by differential equations containing fractional-order derivatives. Their evolutions behave in a much more complex way than in the classical integer-order case and the study of the corresponding theory is a hugely demanding task. Although some results of qualitative analysis for fractional differential equations (FDEs) can be similarly obtained, many classical methods are hardly applicable directly to FDEs. New theories and methods are thus required to be specifically developed, whose investigation becomes more challenging. Compared the classical theory of differential equations, the

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research on the theory of FDEs with complex-order are only in their initial stage of development.

This paper is devoted to a rapidly developing area of the research on the qualitative theory of ψ-type complex-order differential equations (CDEs). In particular, we are interested in the basic theory of CDEs. Such basic theory should be the starting point for further research concerning the dynamics, control, numerical analysis and applications of CDEs. It is remarkable to mention that there are many researchers who also used complex-order derivatives. Fractional operators of complex-order were introduced in [9, 12]. Recently, Vivek et al. [14] studied the existence and Ulam stability results for neutral pantograph equations with complex-order. The authors also applied complex-order derivative to different kinds of differential equations, for example [2, 15, 16]. In [4], Vivek et al. investigated the existence and stability of solutions of boundary value problems for fractional integro-differential equations with complex order. Very recently, Almeida [1] introduced a new fractional derivative called a ψ-fractional derivative with respect to another function, which extended the classical fractional derivative and also studied some properties like semigroup law, Taylor's Theorem and so on. Thereafter, Harikrishnan et al. [5] initially studied a ψ-Hilfer fractional differential equation with complex order.

Following [5, 14, 4], we study ψ-type complex-order boundary value problems for the following implicit differential equations

$$\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t) = \mathscr{F}\left(t, u(t), \left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t)\right), \quad \text{for every} \quad t \in J := [0,T],\ T > 0, \tag{1}$$

$$au(0) + bu(T) = c, \tag{2}$$

where ${}^{c}\mathscr{D}^{\theta;\psi}$ is the ψ-type Caputo fractional derivative of order $\theta \in \mathbb{C}$, $\theta = m + i\alpha$. Let $\alpha \in \mathbb{R}^{+}$, $0 < \alpha < 1$, $m \in (0,1]$ and $\mathscr{F} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given continuous function. Here a, b, c are real constants with $a + b \neq 0$. Observing that (1)-(2) is equivalent to the integral equation

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\theta)} \int_0^t \psi'(s)\left(\psi(t) - \psi(s)\right)^{\theta-1} \mathscr{F}\left(s, u(s), \left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(s)\right) ds \\ & - \frac{1}{a+b}\left[\frac{b}{\Gamma(\theta)} \int_0^T \psi'(s)\left(\psi(T) - \psi(s)\right)^{\theta-1} \mathscr{F}\left(s, u(s), \left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(s)\right) ds - c\right]. \end{aligned} \tag{3}$$

Next, we consider nonlocal boundary value problems for ψ-type complex-order implicit differential equations of the form

$$\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t) = \mathscr{F}\left(t, u(t), \left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t)\right), \quad \text{for every} \quad t \in J := [0,T],\ T > 0, \tag{4}$$

$$u(0) + g(u) = u_0, \tag{5}$$

where $g : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function and u_0 is a real constant. It is seen that problem (4)-(5) is equivalent to the integral equation

$$u(t)=u_0-g(u)+\frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\mathscr{F}\left(s,u(s),\left({}^c\mathscr{D}^{\theta;\psi}u\right)(s)\right)ds. \tag{6}$$

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|_\infty=\sup\left\{|u(t)|:t\in J\right\}.$$

By $L^1(J)$ we denote the space of Lebesgue-integrable function $u:J\to\mathbb{R}$ with the norm

$$\|u\|_{L^1}=\int_0^T|u(t)|\,dt.$$

The study of a nonlocal Cauchy problem was first initiated by Byszewski [3]. The importance of nonlocal conditions, which are better than classical initial conditions, is explained in [3]. We have taken an example of non-local conditions as follows:

$$g(u)=\sum_{i=1}^{p}c_iu(t_i)$$

where c_i, $i=1,\ldots,p$ are constants and $0<t_1<t_2<\ldots<t_p\leqslant T$. The authors contribution of this article is given as follows:

(i) The new type ψ-complex-order derivative is proposed based on fractional calculus.

(ii) The sufficient conditions are derived to prove the existence and uniqueness of solution.

(iii) The different types of Ulam-Hyers stability are discussed.

The rest of this paper is structured as follows: In Section 2, we introduce some definitions, notations, and lemmas that are used throughout the paper. In Section 3, we will prove existence and stability results concerning problem (1)-(2). Section 4 is devoted to study the existence and stability of solution of the problem (4)-(5). Finally, an example is presented in Section 5 to illustrate our main results.

2. The basic ideas

We now introduce some definitions, preliminary facts about the fractional calculus, notations, and some auxiliary results that will be used later.

DEFINITION 1. [5] The ψ-type Riemann-Liouville fractional integral of order $\theta\in\mathbb{C}$, $(Re(\theta)>0)$ of a function $\mathscr{F}:(0,\infty)\to\mathbb{R}$ is

$$\mathscr{I}^{\theta;\psi}\mathscr{F}(t)=\frac{1}{\Gamma(\theta)}\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\mathscr{F}(s)ds,$$

where Γ is the gamma function.

DEFINITION 2. [5] For a function $\mathscr{F}$ given by on the interval J, the ψ-type Caputo derivative of fractional-order $\theta \in \mathbb{C}$, $(Re(\theta) > 0)$, is defined by

$$\left(\mathscr{D}^{\theta;\psi}\mathscr{F}\right)(t) = \frac{1}{\Gamma(n-\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{n-\theta-1}\mathscr{F}(s)ds,$$

when $n = [Re(\theta)] + 1$ and $[Re(\theta)]$ denotes the integral part of the real number θ.

DEFINITION 3. [6] The Stirling asymptotic formula of the gamma function for $z \in \mathbb{C}$ is

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{\frac{z-1}{2}} e^{-z}\left[1 + O\left(\frac{1}{z}\right)\right], \quad (|arg(z)| < \pi; |z| \to \infty);$$

moreover, for $(u, v \in \mathbb{R})$

$$|\Gamma(u+iv)| = (2\pi)^{\frac{1}{2}} |v|^{u-\frac{1}{2}} e^{-u-\pi|v|/2}\left[1 + O\left(\frac{1}{v}\right)\right], \quad (v \to \infty).$$

LEMMA 1. *(Gronwall's lemma) ([8, 13]) Let $\alpha > 0$ and let $\psi \in C^1((0,d],\mathbb{R})$ be a function such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in (0,d]$. Suppose that $d \geqslant 0$, z is a nonnegative function locally integrable on $(0,d]$, and w is nonnegative and locally integrable on $(0,d]$ with*

$$w(t) \leqslant z(t) + k\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} w(s)ds, \quad t \in (0,d].$$

Then

$$w(t) \leqslant z(t) + \int_0^t \sum_{n=1}^{\infty} \frac{[k\Gamma(\alpha)]^n}{\Gamma(n\alpha)} \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} z(s)ds, \quad t \in (0,d].$$

REMARK 1. ([8, 13]) Under the hypotheses of Lemma 1, let z be a nondecreasing function on $(0,d]$. Then we have

$$w(t) = z(t)\mathbb{E}_\alpha\left(k\Gamma(\alpha)\left(\psi(t)-\psi(s)\right)^{\alpha}\right), \quad t \in (0,d],$$

where, $\mathbb{E}_\alpha$ is the Mittag-Leffler function.

For the ψ-type complex-order implicit differential equations (1), we adopt the definitions in [5] of the Ulam-Hyers (U-H) stability, generalized U-H stability, U-H-Rassias stability and generalized U-H-Rassias stability.

Now we consider the Ulam stability for the problem

$$\left({}^c\mathscr{D}^{\theta;\psi}u\right)(t) = \mathscr{F}\left(t, u(t), \left({}^c\mathscr{D}^{\theta;\psi}u\right)(t)\right), \quad t \in [0,T], \tag{7}$$

and the following inequalities:

$$\left|\left({}^c\mathscr{D}^{\theta;\psi}z\right)(t) - \mathscr{F}\left(t, z(t), \left({}^c\mathscr{D}^{\theta;\psi}z\right)(t)\right)\right| \leqslant \varepsilon, \quad t \in [0,T], \tag{8}$$

$$\left|\left({}^{c}\mathscr{D}^{\theta;\psi}z\right)(t)-\mathscr{F}\left(t,z(t),\left({}^{c}\mathscr{D}^{\theta;\psi}z\right)(t)\right)\right|\leqslant\varepsilon\varphi(t),\quad t\in[0,T],\tag{9}$$

$$\left|\left({}^{c}\mathscr{D}^{\theta;\psi}z\right)(t)-\mathscr{F}\left(t,z(t),\left({}^{c}\mathscr{D}^{\theta;\psi}z\right)(t)\right)\right|\leqslant\varphi(t),\quad t\in[0,T].\tag{10}$$

DEFINITION 4. Equation (7) is U-H stable if there exists a real number $C_f>0$ such that for each $\varepsilon>0$ and for each solution $z\in C(J,\mathbb{R})$ of the inequality (8) there exists a solution $u\in C(J,\mathbb{R})$ of equation (7) with

$$|z(t)-u(t)|\leqslant C_f\varepsilon,\quad t\in J.$$

DEFINITION 5. Equation (7) is generalized U-H stable if there exists $\psi_f\in C([0,\infty),[0,\infty))$, $\psi_f(0)=0$ such that for each solution $z\in C(J,\mathbb{R})$ of the inequality (8) there exists a solution $u\in C(J,\mathbb{R})$ of equation (7) with

$$|z(t)-u(t)|\leqslant\psi_f\varepsilon,\quad t\in J.$$

DEFINITION 6. Equation (7) is U-H-Rassias stable with respect to $\varphi\in C(J,\mathbb{R})$ if there exists a real number $C_f>0$ such that for each $\varepsilon>0$ and for each solution $z\in C(J,\mathbb{R})$ of the inequality (9) there exists a solution $u\in C(J,\mathbb{R})$ of equation (7) with

$$|z(t)-u(t)|\leqslant C_f\varepsilon\varphi(t),\quad t\in J.$$

DEFINITION 7. Equation (7) is generalized U-H-Rassias stable with respect to $\varphi\in C(J,\mathbb{R})$ if there exists a real number $C_{f,\varphi}>0$ such that for each solution $z\in C(J,\mathbb{R})$ of the inequality (10) there exists a solution $u\in C(J,\mathbb{R})$ of equation (7) with

$$|z(t)-u(t)|\leqslant C_{f,\varphi}\varphi(t),\quad t\in J.$$

REMARK 2. A function $z\in C(J,\mathbb{R})$ is a solution of (4) if and only if there exists a function $g\in C(J,\mathbb{R})$ (which depend on z) such that

1. $|g(t)|\leqslant\varepsilon$, $t\in J$;
2. $\left({}^{c}\mathscr{D}^{\theta;\psi}z\right)(t)=\mathscr{F}\left(t,z(t),\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)\right)+g(t)$, $t\in J$.

REMARK 3. Clearly,

1. Definition 4 $\Rightarrow$ Definition 5.
2. Definition 6 $\Rightarrow$ Definition 7.

REMARK 4. A solution of the ψ-type complex-order implicit differential inequality

$$\left|\left({}^{c}\mathscr{D}^{\theta;\psi}z\right)(t)-\mathscr{F}\left(t,z(t),\left({}^{c}\mathscr{D}^{\theta;\psi}z\right)(t)\right)\right|\leqslant\varepsilon,\quad t\in J,$$

is called an ψ-type fractional ε-solution of the problem (7).

THEOREM 1. *(Banach's fixed point theorem). Let C be a non-empty closed subset of a Banach space X; then any contraction mapping T of C into itself has a unique fixed point.*

3. Existence and U-H stability of the boundary value problem

Let us start by defining what we mean by a solution of the problem (1)-(2).

DEFINITION 8. A function $u \in C(J,\mathbb{R})$ is said to be a solution of (1)-(2) if u satisfies the equation $\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t) = \mathscr{F}\left(t,u(t),\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)\right)$ on J, and the condition $au(0)+bu(T)=c$.

THEOREM 2. *Assume that:*

(H1) there exists two constants $K>0$ and $0<L<1$ such that

$$|\mathscr{F}(t,x,y)-\mathscr{F}(t,\overline{x},\overline{y})| \leqslant K|x-\overline{x}|+L|y-\overline{y}|, \quad \textit{for each } t\in J, \quad x,\overline{x},y,\overline{y}\in\mathbb{R}.$$

If

$$\frac{K(\psi(T))^m}{(1-L)m|\Gamma(\theta)|}\left(1+\frac{|b|}{|a+b|}\right)<1, \tag{11}$$

then, the problem (1)-(2) has a unique solution.

Proof. Let the operator $\mathscr{N}: C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be defined by

$$(\mathscr{N}u)(t) = \widetilde{A}_u + \frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1} g_u(s)ds,$$

where

$$g_u(t) = \mathscr{F}\left(t,\widetilde{A}_u+\mathscr{I}^{\theta;\psi}g_u(t),g_u(t)\right),$$

and

$$\widetilde{A}_u = \frac{1}{a+b}\left[c-\frac{b}{\Gamma(\theta)}\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{\theta-1} g_u(s)ds\right].$$

It is clear that the fixed points of $\mathscr{N}$ are solutions of (1)-(2).
Let $u,v\in C(J,\mathbb{R})$, and $t\in J$, then we have

$$\begin{aligned} &|(\mathscr{N}u)(t)-(\mathscr{N}v)(t)| \\ \leqslant &\frac{1}{|\Gamma(\theta)|}\int_0^t \left|\psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right| |g_u(s)-g_v(s)|\,ds \\ &+\frac{|b|}{|a+b|\,|\Gamma(\theta)|}\int_0^T \left|\psi'(s)\left(\psi(T)-\psi(s)\right)^{\theta-1}\right| |g_u(s)-g_v(s)|\,ds, \end{aligned} \tag{12}$$

and

$$\begin{aligned} |g_u(t)-g_v(t)| &= \left|\mathscr{F}\left(t,u(t),\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t)\right)-\mathscr{F}\left(t,v(t),\left({}^{c}\mathscr{D}^{\theta;\psi}v\right)(t)\right)\right| \\ &\leqslant K|u(t)-v(t)|+L|g_u(t)-g_v(t)|. \end{aligned}$$

Thus,

$$|g_u(t) - g_v(t)| \leqslant \frac{K}{1-L} |u(t) - v(t)|. \tag{13}$$

By replacing (13) in the inequality (12), we obtain

$$\begin{aligned} &|(\mathscr{N}u)(t) - (\mathscr{N}v)(t)| \\ \leqslant &\frac{K}{(1-L)|\Gamma(\theta)|} \int_0^t \left| \psi'(s) (\psi(t) - \psi(s))^{\theta-1} \right| |u(s) - v(s)| ds \\ &+ \frac{|b| K}{(1-L)|a+b||\Gamma(\theta)|} \int_0^t \left| \psi'(s) (\psi(t) - \psi(s))^{\theta-1} \right| |u(s) - v(s)| ds \\ \leqslant &\frac{K(\psi(T))^m}{(1-L)m|\Gamma(\theta)|} \|u - v\|_\infty + \frac{|b| K(\psi(T))^m}{(1-L)|a+b| m |\Gamma(\theta)|} \|u - v\|_\infty. \end{aligned}$$

Then,

$$\|(\mathscr{N}u) - (\mathscr{N}v)\|_\infty = \left[\frac{K(\psi(T))^m}{(1-L)m|\Gamma(\theta)|} \left(1 + \frac{|b|}{|a+b|} \right) \right] \|u - v\|_\infty.$$

From (11) it follows that $\mathscr{N}$ is a contraction and thus by Banach contraction mapping principle the implicit boundary value problem (1)-(2) has a unique solution. The proof is completed. □

THEOREM 3. *Assume that (H1) and* (11) *are satisfied. Then the problem* (1)-(2) *is U-H stable.*

Proof. Let $\varepsilon > 0$ and let $z \in C^1(J, \mathbb{R})$ be a function which satisfies the inequality:

$$\left| \left({}^c\mathscr{D}^{\theta;\psi} z \right)(t) - \mathscr{F}\left(t, z(t), \left({}^c\mathscr{D}^{\theta;\psi} z \right)(t) \right) \right| \leqslant \varepsilon, \quad t \in J, \tag{14}$$

and let $u \in C(J, \mathbb{R})$ be the unique solution of the following Cauchy problem

$$\begin{cases} \left({}^c\mathscr{D}^{\theta;\psi} u \right)(t) = \mathscr{F}\left(t, u(t), \left({}^c\mathscr{D}^{\theta;\psi} u \right) \right), & t \in J, \\ u(0) = z(0), \ u(T) = z(T). \end{cases}$$

So

$$u(t) = \widetilde{A}_u + \frac{1}{\Gamma(\theta)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\theta-1} g_u(s) ds.$$

On the other hand, if $u(T) = z(T)$ and $u(0) = z(0)$, then $\widetilde{A}_u = \widetilde{A}_z$. Indeed

$$\left| \widetilde{A}_u - \widetilde{A}_z \right| \leqslant \frac{|b|}{|a+b||\Gamma(\theta)|} \int_0^T \left| \psi'(s) (\psi(T) - \psi(s))^{\theta-1} \right| |g_u(s) - g_z(s)| ds$$

and by the inequality (13), we find

$$\begin{aligned}\left|\widetilde{A}_u-\widetilde{A}_z\right| &\leqslant \frac{|b|K}{(1-L)\,|a+b|\,|\Gamma(\theta)|}\int_0^T\left|\psi'(s)\,(\psi(T)-\psi(s))^{\theta-1}\right||u(s)-v(s)|\,ds\\ &=\frac{|b|K}{(1-L)\,|a+b|}\mathscr{I}^{\theta;\psi}\,|u(T)-z(T)|=0.\end{aligned}$$

Thus $\widetilde{A}_u=\widetilde{A}_z$.
Then, we have

$$u(t)=\widetilde{A}_z+\frac{1}{\Gamma(\theta)}\int_0^t\psi'(s)\,(\psi(t)-\psi(s))^{\theta-1}\,g_u(s)ds.$$

By integration of the inequality (14), we obtain

$$\left|z(t)-\widetilde{A}_z-\frac{1}{\Gamma(\theta)}\int_0^t\psi'(s)\,(\psi(t)-\psi(s))^{\theta-1}\,g_z(s)ds\right|\leqslant\frac{\varepsilon(\psi(T))^m}{m|\Gamma(\theta)|},$$

with

$$g_z(t)=\mathscr{F}\left(t,\widetilde{A}_z+\mathscr{I}^{\theta;\psi}g_z(t),g_z(t)\right).$$

We have for any $t\in J$,

$$\begin{aligned}|z(t)-u(t)|=&\left|z(t)-\widetilde{A}_z-\frac{1}{\Gamma(\theta)}\int_0^t\psi'(s)\,(\psi(t)-\psi(s))^{\theta-1}\,g_z(s)ds\right.\\ &\left.+\frac{1}{\Gamma(\theta)}\int_0^t\psi'(s)\,(\psi(t)-\psi(s))^{\theta-1}\,(g_z(s)-g_y(s))\,ds\right|\\ \leqslant&\left|z(t)-\widetilde{A}_z-\frac{1}{\Gamma(\theta)}\int_0^t\psi'(s)\,(\psi(t)-\psi(s))^{\theta-1}\,g_z(s)ds\right|\\ &+\frac{1}{|\Gamma(\theta)|}\int_0^t\left|\psi'(s)\,(\psi(t)-\psi(s))^{\theta-1}\right|\left|(g_z(s)-g_y(s))\,ds\right|.\end{aligned}$$

Using (13), we obtain

$$|z(t)-u(t)|\leqslant\frac{\varepsilon(\psi(T))^m}{m|\Gamma(\theta)|}+\frac{1}{|\Gamma(\theta)|}\int_0^t\left|\psi'(s)\,(\psi(t)-\psi(s))^{\theta-1}\right||z(s)-u(s)|\,ds,$$

and by the Lemma 1 and Remark 1, we get

$$|z(t)-u(t)|\leqslant\left(\frac{\varepsilon(\psi(T))^m}{m|\Gamma(\theta)|}\right)\mathbb{E}_m\left(\frac{K}{(1-L)\,|\Gamma(\theta)|}\Gamma(m)(\psi(T))^m\right).$$

Thus, the problem (1)-(2) is U-H stable. □

THEOREM 4. *Assume that (H1),* (11) *and*

(H2) there exists an increasing function $\varphi\in C(J,\mathbb{R}_+)$ *and there exists* $\lambda_\varphi>0$ *such that for any* $t\in J$

$$\mathscr{I}^{\theta;\psi}\varphi(t)\leqslant\lambda_\varphi\varphi(t)$$

are satisfied. Then, the problem (1)-(2) *is U-H-Rassias stable.*

Proof. Let $z \in C^1(J,\mathbb{R})$ be solution of the inequality

$$\left|\left({}^c\mathscr{D}^{\theta;\psi}z\right)(t) - \mathscr{F}\left(t, z(t), \left({}^c\mathscr{D}^{\theta;\psi}z\right)(t)\right)\right| \leqslant \varepsilon\varphi(t),\ t \in J, \tag{15}$$

and let $u \in C(J,\mathbb{R})$ be the unique solution of the Cauchy problem:

$$\begin{cases} \left({}^c\mathscr{D}^{\theta;\psi}u\right)(t) = \mathscr{F}\left(t, u(t), \left({}^c\mathscr{D}^{\theta;\psi}u\right)\right), & t \in J, \\ u(0) = z(0),\ u(T) = z(T). \end{cases}$$

We have

$$u(t) = \widetilde{A}_z + \frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1} g_u(s)ds,$$

where $g_u \in C(J,\mathbb{R})$ satisfies the equation:

$$g_u = \mathscr{F}\left(t, \widetilde{A}_z + \mathscr{I}^{\theta;\psi}g_u(t), g_u(t)\right),$$

and

$$\widetilde{A}_z = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\theta)}\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{\theta-1} g_z(s)ds\right].$$

By integration of (15), we obtain

$$\left|z(t) - \widetilde{A}_z - \frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1} g_z(s)ds\right| \leqslant \varepsilon\lambda_\varphi\varphi(t).$$

On the other hand, we have

$$\begin{aligned} |z(t)-u(t)| = &\left|z(t) - \widetilde{A}_z - \frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1} g_z(s)ds\right. \\ &\left. + \frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\left(g_z(s)-g_y(s)\right)ds\right| \\ \leqslant &\left|z(t) - \widetilde{A}_z - \frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1} g_z(s)ds\right| \\ &+ \frac{1}{|\Gamma(\theta)|}\int_0^t \left|\psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right| \left|\left(g_z(s)-g_y(s)\right)ds\right|. \end{aligned}$$

Using (13), we have

$$|z(t)-u(t)| \leqslant \varepsilon\lambda_\varphi\varphi(t) + \frac{K}{(1-L)|\Gamma(\theta)|}\int_0^t \left|\psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right| |z(s)-u(s)|ds.$$

By appling Lemma 1 and Remark 1, we get that for any $t \in J$:

$$|z(t)-u(t)| \leqslant \varepsilon\lambda_\varphi\varphi(t)\mathbb{E}_m\left(\frac{K}{(1-L)|\Gamma(\theta)|}\Gamma(m)(\psi(T))^m\right).$$

Thus, the problem (1)-(2) is U-H-Rassias stable. □

REMARK 5. Our results for the boundary value problem (1)-(2) are suitable for the following problems:

- Initial value problem: $a=1$, $b=0$, $c=0$.
- Terminal value problem: $a=0$, $b=1$, c arbitrary.
- Anti-periodic problem: $a=1$, $b=1$, $c=0$.

However, they are not for the periodic problem, i.e. for $a=1$, $b=-1$, $c=0$.

4. Existence and U-H stability of the nonlocal boundary value problem

LEMMA 2. *Let $\mathscr{F}: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function; then the problem* (4)-(5) *is equivalent to the following problem*

$$u(t)=u_0-g(u)+\mathscr{I}^{\theta;\psi}K_u(t)$$

where

$$K_u(t)=\mathscr{F}(t,u(t),K_u(t)).$$

THEOREM 5. *Assume that:*

(H3) there exist $K>0$, $0<\overline{K}<1$ and $0<L<1$ such that:

$$|\mathscr{F}(t,x,y)-\mathscr{F}(t,\overline{x},\overline{y})|\leqslant K|x-y|+\overline{K}|y-\overline{y}|,\quad \text{for any } x,y,\overline{x},\overline{y}\in\mathbb{R}$$

and $|g(u)-g(\overline{u})|\leqslant L|u-\overline{u}|$ *for any* $u,\overline{u}\in C(J,\mathbb{R})$.

If

$$L+\frac{K(\psi(T))^m}{(1-\overline{K})m|\Gamma(\theta)|}<1 \tag{16}$$

then, the boundary value probel (4)-(5) *has a unique solution on* J.

Proof. The proof of this theorem is similar to Theorem 2 and is omitted. □

THEOREM 6. *Assume that (H3) and the inequality* (16) *are satisfied; then the problem* (4)-(5) *is U-H stable.*

Proof. Let $\varepsilon>0$ and let $z\in C^1(J,\mathbb{R})$ satisfying the inequality

$$\left|\left({}^c\mathscr{D}^{\theta;\psi}z\right)(t)-\mathscr{F}\left(t,z(t),\left({}^c\mathscr{D}^{\theta;\psi}z\right)(t)\right)\right|\leqslant\varepsilon,\quad t\in J, \tag{17}$$

and let $u\in C(J,\mathbb{R})$ be the unique solution of the Cauchy problem

$$\begin{cases}\left({}^c\mathscr{D}^{\theta;\psi}u\right)(t)=\mathscr{F}\left(t,u(t),\left({}^c\mathscr{D}^{\theta;\psi}u\right)\right), & t\in J,\\ z(0)+g(u)=u_0.\end{cases}$$

Then,

$$u(t)=u_0-g(u)+\frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}K_u(s)ds,$$

where

$$K_u(t)=\mathscr{F}\left(t,u(t),K_u(t)\right).$$

By integration of the inequality (17), we find

$$\left|z(t)-u_0+g(z)-\frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}K_z(s)ds\right|\leqslant\frac{\varepsilon(\psi(T))^m}{m|\Gamma(\theta)|},$$

where $K_z=\mathscr{F}\left(t,z(t),K_z(t)\right)$. For every $t\in J$, we have

$$\begin{aligned}|z(t)-u(t)|\leqslant&\left|z(t)-u_0+g(z)-\frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}K_z(s)ds\right|\\&+\left|g(u)-g(z)+\frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\left(K_z(s)-K_u(s)\right)ds\right|\\\leqslant&\frac{\varepsilon(\psi(T))^m}{m|\Gamma(\theta)|}+|g(z)-g(u)|\\&+\frac{1}{|\Gamma(\theta)|}\int_0^t\left|\psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right||K_z(s)-K_u(s)|\,ds.\end{aligned}$$

After simple computation, we get

$$\begin{aligned}|z(t)-u(t)|\leqslant&\frac{\varepsilon(\psi(T))^m}{m|\Gamma(\theta)|}+L|z(t)-u(t)|\\&+\frac{K}{(1-\overline{K})\,|\Gamma(\theta)|}\int_0^t\left|\psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right||z(s)-u(s)|\,ds.\end{aligned}$$

Thus

$$\begin{aligned}|z(t)-u(t)|\leqslant&\frac{\varepsilon(\psi(T))^m}{(1-L)m|\Gamma(\theta)|}\\&+\frac{K}{(1-\overline{K})(1-L)\,|\Gamma(\theta)|}\int_0^t\left|\psi'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right||z(s)-u(s)|\,ds.\end{aligned}$$

Using Lemma 1 and Remark 1, we obtain for every $t\in J$

$$|z(t)-u(t)|\leqslant\frac{\varepsilon(\psi(T))^m}{(1-L)m|\Gamma(\theta)|}\mathbb{E}_m\left(\frac{K}{(1-L)(1-\overline{K})\,|\Gamma(\theta)|}\Gamma(m)(\psi(T))^m\right).$$

Thus, the problem (4)-(5) is U-H stable. □

THEOREM 7. *Assume that (H3), inequality* (16) *and*

(H4) there exist an increasing function $\varphi\in C(J,\mathbb{R}_+)$ *and* $\lambda_\varphi>0$ *such that*

$$\mathscr{I}^{\theta;\psi}\varphi(t)=\lambda_\varphi\varphi(t)\quad\textit{for each}\quad t\in J$$

are satisfied. Then the problem (4)-(5) *is U-H-Rassias stable.*

5. An example

Consider the following boundary value problem

$$\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t) = \frac{1}{10e^{t+2}\left(1+|u(t)|+|({}^{c}\mathscr{D}^{\theta;\psi}u)(t)|\right)}, \quad \text{for each } t \in [0,1], \tag{18}$$

$$u(0)+u(1)=0, \tag{19}$$

where $\theta = m + i\alpha$. Let $\alpha = \frac{1}{2}$ and $m = 1$.
Set

$$\mathscr{F}(t,x,y) = \frac{1}{10e^{t+2}\left(1+|x|+|y|\right)},\ t \in [0,1],\ x,y \in \mathbb{R}.$$

Clearly, the function $\mathscr{F}$ is continuous.
For any $x,y,\overline{x},\overline{y} \in \mathbb{R}$ and $t \in [0,1]$

$$|\mathscr{F}(t,x,y) - \mathscr{F}(t,\overline{x},\overline{y})| \leqslant \frac{1}{10e^2}\left(|x-\overline{x}|+|y-\overline{y}|\right).$$

Hence condition (H1) is satisfied with $K = L = \frac{1}{10e^2}$. Thus condition (11) is satisfied with $a = b = 1$, $c = 0$ and $\psi(T) = 1$. It follows from Theorem 2 that the problem (18)-(19) has a unique solution on J. Moreover, Theorem 3 implies that the problem (18)-(19) is U-H stable.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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REFERENCES

[1] R. ALMEIDA, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul. **44** (2017), 460–481.

[2] O. BAGHANI, D. VIVEK, K. KANAGARAJAN, *Theory of hybrid fractional differential equations with complex order*, Sahand Communications in Mathematical Analysis **15** (1) (2019), 65–76.

[3] L. BYSZEWSKI, *Theorems about the existence and uniqueness of solutions of semilinear evoluation nonlocal Cauchy problem*, J. Math. Anal. Appl. **162** (1991), 494–505.

[4] E. M. ELSAYED, K. KANAGARAJAN, D. VIVEK, *On the existence and stability of solution of boundary value problem for fractional integro-differential equations with complex order*, Filomat **32** (8) (2018), 1–10.

[5] S. HARIKRISHNAN, R.W. IBRAHIM, K. KANAGARAJAN, *On ψ-Hilfer fractional differential equation with complex order*, Universal Journal of Mathematics and Applications **1** (1) (2018), 33–38.

[6] A.A. KILBAS, H.M. SRIVASTAVA, J.J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[7] V. LAKSHMIKANTHAM, S. LEELA, J. VASUNDHARA DEVI, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[8] L. Liu, J.R. Wang, D. O'Regan, *Ulam-Hyers-Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations*, Adv. Difference Equ. (2019), **2019:50**.

[9] E.R. Love, *Fractional derivatives of imaginary order*, J. Lond. Math. Soc. (2) **2** (2-3) (1971), 241–259.

[10] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[11] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[12] S. G. Samko, A.A. Kilbas O. I. Marichev, *Fractional Integrals and Derivatives-Theory and Applications*, Gordon and Breach Science Publishers, Amsterdam, 1993.

[13] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator*, Differ. Equ. Appl. **11** (1) (2019), 87–106.

[14] D. Vivek, E. M. Elsayed, K. Kanagarajan, *Existence and Ulam stability results for a class of boundary value problem of neutral pantograph equations with complex order*, SEMA Journal, (2020).

[15] D. Vivek, K. Kanagarajan, S. Harikrishnan, *Dynamics and stability results of fractional pantograph equations with complex order*, Journal of Applied Nonlinear Dynamics **7** (2) (2018), 179–187.

[16] D. Vivek, K. Kanagarajan, S. Harikrishnan, *Dynamics and stability results of fractional integro-differential equations with complex order*, Discontinuity, Nonlinearity, and Complexity **7** (2) (2018), 119–127.

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