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ANALYTICAL APPROXIMATION OF TIME–FRACTIONAL TELEGRAPH EQUATION WITH RIESZ SPACE–FRACTIONAL DERIVATIVE

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Abstract. In this study, fractional reduced differential transform method (FRDTM) is developed to derive a semi- analytical solution of fractional partial differential equations which involves Riesz space fractional derivatives. We focus primarily on implementing the novel algorithm of FRDTM to Riesz space -fractional telegraph equation while the telegraph equation has fractional order. Some theorems with their proofs which are used for calculating differential transform of Riesz derivative operator are presented, as well as the convergence condition and the error bound of the proposed method are established. To illustrate the reliability and capability of the method, some examples are provided. The results reveal that the algorithm is very effective and uncomplicated.

1. Introduction

Fractional differential equations are generalization of integer order differential equations to fractional order equations. Partial differential equations of fractional order derivatives are the convenient methods to describe a lot of governing phenomena in physics, chemistry and biochemistry, control theory, fluid mechanics, quantum mechanics, viscoelasticity and other science. A great attention has been conducted to find analytical and numerical solutions of the fractional differential equations. Time fractional telegraph equation with Riesz space fractional derivative is a typical fractional diffusion- wave equation which is applied in signal analysis and modeling of the reaction diffusion and the random walk of suspension flows and so on. Lately numerical methods such as finite element approximation [1, 2], L1/L2 – approximation method, the standard/shifted Grunwald method and the matrix transform method [3], Chebyshev Tau approximation [4] and radial basis function approximation [5], the discontinuous Galerkin finite element method [6] are used for solving telegraph equation. Riesz derivative operator appears in some partial differential equations such as telegraph equation, wave equation, diffusion equation, advection-dispersion equation and other partial differential equations. Several methods are used to find an approximate solution for fractional telegraph equation with Riesz space fractional derivative [7,8,9].

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Differential transform method (DTM) is a reliable and effective method which was constructed by Zhou [10] for solving linear and nonlinear differential equations arising in electrical circuit problems. This method constructs an iterative procedure based on Taylor expansion to obtain an analytical solution in the form of a polynomial. DTM has extremely been extended to obtain numerical solution for wide range of ordinary differential equations and partial differential equations [11]. Moreover, the generalized differential transform method (GDTM) was used for fractional ordinary differential equations [12] and fractional partial differential equation [13]. DTM and GDTM were applied by Soltanizadeh, Vineet and Garg [14, 15, 16] to find exact and numerical solutions for telegraph equations without incorporating Riesz space derivative. Compared to the classical Taylor series method, DTM and GDTM needs less computational time work, however to conquer the demerit of multi part computations in DTM and GDTM, the reduced differentia transform method (RDTM) was introduced and used to find the approximate solution of PDEs [17], then it was improved to fractional reduced differential transform (FRDTM) to obtain an approximate solution of fractional PDEs [18].

Lately FRDTM was used to obtain analytical approximation of time-fractional telegraph equation [19] and multi term time-fractional diffusion equation [20] and space-time fractional order heat-like and wave-like partial differential equations [21]. These types of equations which previously were solved by FRDTM were not involved Riesz derivative operator. The main goal of this study is applying an improved scheme for FRDTM and RDTM based on the order of equation, to determine an approximate solution for such equations which are involving Riesz space fractional derivative. For the simplicity we consider a kind of telegraph equation with Riesz operator on a finite one dimensional domain in the form:

$$ {}_0^C D_t^{2\beta} u(x,t) + 2\kappa\, {}_0^C D_t^{\beta} u(x,t) + \vartheta^2 u(x,t) - \eta \frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma} = f(x,t), \quad 0 < \beta \leqslant 1 \quad (1.1)$$

$$ a \leqslant x \leqslant b, \ 0 \leqslant t \leqslant T, $$

subject to the initial conditions:

$$ u(x,0) = \varphi(x), \ \frac{\partial u(x,0)}{\partial t} = \psi(x), \ a \leqslant x \leqslant b, $$

and boundary conditions:

$$ u(a,t) = u(b,t) = 0, \qquad 0 \leqslant t \leqslant T, $$

where $\kappa > \vartheta \geqslant 0$ and $\eta > 0$ are constants.
Generally the Riesz space-fractional operator $\frac{\partial^\gamma}{\partial |x|^\gamma}$ over $[a,b]$ is defined by:

$$ \frac{\partial^\gamma}{\partial |x|^\gamma} u(x,t) = -\frac{1}{2\cos\left(\frac{\pi\gamma}{2}\right)} \frac{1}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial x^n} \int_a^b \frac{u(s,t)\,ds}{|x-s|^{\gamma-1}}, \quad n-1 < \gamma < n, n \in N. $$

In this paper we suppose that $1 < \gamma < 2$, and $u(x,t), f(x,t), \varphi(x)$ and $\psi(x)$ are real--valued and sufficiently well-behaved functions.

The rest of this study is organized as follows. In Section 2 we propose some preliminary definitions of fractional calculus. In Section 3 we give some knowledge of fractional reduced differential transform method. Section 4 discusses about applying RFDTM for fractional telegraph equation with Riesz operator accompanied with some theorems. In Section 5 convergence analysis and error bound of the method are discussed. In Section 6 some numerical examples are presented to demonstrate the efficiency and convenience of the theoretical results. The concluding remarks are given in Section 7.

2. Some definitions of fractional calculus

In this section some necessary definitions of fractional calculus are introduced. Since the Riemann-Liouville and the Caputo derivatives are often used, as well as the Riesz fractional derivative is defined based on left and right Riemann-Liouville and Caputo derivatives, we focus in these definitions of fractional calculus. Furthermore in the modeling of most physical problems, the initial conditions are given in integer order derivatives and the integer order derivatives are coincided with Caputo initial conditions definition; therefore the Caputo derivative is used in numerical algorithm.

DEFINITION 1. The left and right Riemann-Liouville integrals of order $\alpha > 0$ for a function $f(x)$ on interval (a,b) are defined as follows,

$$\begin{cases} {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \dfrac{f(s)}{(x-s)^{1-\alpha}} ds, \\ {}_xJ_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b \frac{f(s)}{(s-x)^{1-\alpha}} ds, \end{cases}$$

where $\Gamma(\mathrm{z}) = \int\limits_0^{\infty} \mathrm{e}^{-\mathrm{t}} \mathrm{t}^{\mathrm{z}-1} \mathrm{dt}$, $\mathrm{z} \in \mathrm{C}$ is the Gamma function.

DEFINITION 2. The left and right Riemann-Liouville derivatives of order $\alpha > 0$ for a function $f(x)$ defined on interval (a,b) are given as follows,

$$\begin{cases} {}_a^{RL}D_x^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int\limits_a^x (x-s)^{m-\alpha-1} f(s) ds, \\ {}_x^{RL}D_b^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int\limits_x^b (s-x)^{m-\alpha-1} f(s) ds, \end{cases}$$

where $m-1 < \alpha \leqslant m$.

REMARK 1. From the Riemann-Liouville derivatives definition and the definition of the Riesz space fractional derivative one can conclude for $0 \leqslant x \leqslant L$

$\frac{\partial^\gamma}{\partial |x|^\gamma} u(x,t) = -\zeta_\gamma ({}_0^{RL}D_x^\gamma + {}_x^{RL}D_L^\gamma) u(x,t)$, while $\zeta_\gamma = \frac{1}{2\cos(\frac{\pi\gamma}{2})}$, $\gamma \neq 1$, and, ${}_0^{RL}D_x^\gamma$ and ${}_x^{RL}D_L^\gamma$ are left and right Riemann-Liouville derivatives.

DEFINITION 3. The left and right Caputo derivatives of order α for function $f(x)$ is defined as:

$$\begin{cases} {}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int\limits_a^x (x-s)^{m-\alpha-1} \frac{d^m f(s)}{ds^m} ds, & m-1 < \alpha \leqslant m, \\ {}_x^C D_b^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int\limits_x^b (s-x)^{m-\alpha-1} \frac{d^m f(s)}{ds^m} ds, & m-1 < \alpha \leqslant m. \end{cases}$$

From the Caputo's derivative definitions we have

$$D^\alpha C = 0, \quad (C \text{ is a constant}) \tag{2.1}$$

$$ {}_0^C D_t^\alpha t^\beta = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta \leqslant \alpha \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha} & \text{for } \beta \in N_0 \text{ and } \beta \geqslant \alpha. \end{cases} \tag{2.2}$$

There are relations between Riemann-Liouville derivatives and Caputo derivatives as follows:

$$ {}_a^C D_x^\alpha f(x) = {}_a^{RL} D_x^\alpha f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(x-a)^{k-\alpha}}{\Gamma(1+k-\alpha)}, \tag{2.3}$$

$$ {}_x^C D_b^\alpha f(x) = {}_x^{RL} D_b^\alpha f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)(b-x)^{k-\alpha}}{\Gamma(1+k-\alpha)}. \tag{2.4}$$

It is clear that if in (2.3), $f^{(k)}(a) = 0, \quad k = 0,1,..,m-1$ then the left Riemann-Liouville derivatives and the left Caputo derivatives are equivalent, the same is hold for the right Riemann-Liouville derivatives and the right Caputo derivatives in (2.4) when $f^{(k)}(b) = 0, \quad k = 0,1,..,m-1$. For comprehensive properties of fractional derivatives and integrals one can refer to the literature [22, 23, 24, 25].

In general the Caputo derivative operators ${}_a^C D_t^\alpha$ and ${}_a^C D_t^\beta$ do not commute. The following lemma provides appropriate conditions for interchanging Caputo derivatives in some special cases.

LEMMA 1. *Let* $m-1 < \alpha < m, \quad n-1 < \beta < n$, m,n *are positive integers and* $f \in C^n[a,b]$. *Then*

$$ {}_aJ_t^\alpha \, {}_a^C D_t^\beta f(t) = {}_a^C D_t^{\beta-\alpha} f(t), \qquad \alpha \neq \beta, \tag{2.5}$$

$$ {}_aJ_t^{\alpha}\, {}_a^CD_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{\Gamma(k+1)} f^{(k)}(a), \tag{2.6} $$

$$ {}_a^CD_t^{\beta}\, {}_aJ_t^{\alpha} f(t) = {}_aJ_t^{\alpha-\beta} f(t), \quad \beta \leqslant \alpha \text{ or } \alpha < \beta,\ \alpha \in N, \tag{2.7} $$

$$ {}_a^CD_t^{\beta}\, {}_aJ_t^{\alpha} f(t) = {}_a^CD_t^{\beta-\alpha} f(t) + \sum_{k=0}^{n-m} \frac{(t-a)^{k+\alpha-\beta}}{\Gamma(k+1+\alpha-\beta)} f^{(k)}(a), \quad \alpha < \beta, \tag{2.8} $$

$$ {}_a^CD_t^{\beta}\, {}_a^CD_t^{m} f(t) = {}_a^CD_t^{m+\beta} f(t), \quad \beta > 0, \tag{2.9} $$

$$ {}_a^CD_t^{m}\, {}_a^CD_t^{\beta} f(t) = {}_a^CD_t^{m+\beta} f(t)) + \sum_{j=n}^{m+n-1} \frac{(t-a)^{j-m-\beta}}{\Gamma(1+j-m-\beta)} f^{(j)}(a), \tag{2.10} $$

The interchange of the Caputo derivative operators in (2.10) is allowed under the following conditions:

$$ f^{(j)}(a) = 0, \qquad j = n, n+1, \ldots, m+n-1,\ m = 0,1,2,\ldots. $$

When $n-1 < \beta < n$ *then* $0 < n-\beta < 1$. *For* $0 < n-\beta < 1$, *one has*

$$ {}_a^CD_t^{n-\beta}\, {}_a^CD_t^{\beta} f(t) = f^{(n)}(t), \qquad 0 < n-\beta < 1, \tag{2.11} $$

Let $0 < \alpha < 1$, $n-1 < \alpha+\beta < n$, *n is positive integer. Then one has*

$$ {}_a^CD_t^{\alpha}\, {}_a^CD_t^{\beta} f(t) = {}_a^CD_t^{\alpha+\beta} f(t), \qquad 0 < \alpha < 1, \tag{2.12} $$

Note that relations (2.5) to (2.8)) are also hold for Caputo right derivative. The proofs and details of Lemma1 are discussed by C. Li and F. Zeng [25].

3. Fractional reduced differential transform method (FRDTM)

Consider a function of two variables $u(x,t)$ and suppose that it can be represented as a product of two single-variable functions, i.e. $u(x,t) = f(x)g(t)$. If $u(x,t)$ is analytic and can be differentiated continuously with respect to space x and time t in the domain of interest, then the function $u(x,t)$ is represented as:

$$ u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha}, \tag{3.1} $$

where $0 < \alpha \leqslant 1$, $U_k(x)$ is called the spectrum function of $u(x,t)$ which is defined as follows

$$ U_k(x) = \frac{1}{\Gamma(\alpha k+1)} \left[(D_t^{\alpha})^k u(x,t) \right]_{(t=t_0)}, \tag{3.2} $$

while $(D_t^{\alpha})^k = D_t^{\alpha}.D_t^{\alpha} \ldots .D_t^{\alpha}$, $k - times$, and D_t^{α} represents the derivative operator of Caputo definition with respect to time t. Substituting (3.2) in (3.1) yields

$$ u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \left[(D_t^{\alpha})^k u(x,t) \right]_{(t=t_0)} (t-t_0)^{k\alpha}. \tag{3.3} $$

In practical application $u(x,t)$ is approximated by

$$u_N(x,t) = \sum_{k=0}^{N} U_k(x)(t-t_0)^{k\alpha}, \tag{3.4}$$

where N is the order of approximate solution. The exact solution can be obtained as

$$u(x,t) = \lim_{N\to\infty} u_N(x,t). \tag{3.5}$$

Based on the definitions (3.1) and (3.2), we have the following results.

THEOREM 1. *Suppose that $U_k(x)$, $V_k(x)$ and $W_k(x)$ are the FRDT of the functions $u(x,t)$, $v(x,t)$ and $w(x,t)$, respectively; then*

(a) If $u(x,t) = v(x,t) \pm w(x,t)$, *then* $U_k(x) = V_k(x) \pm W_k(x)$,

(b) If $u(x,t) = cv(x,t)$, $c \in R$, *then* $U_k(x) = cV_k(x)$,

(c) If $u(x,t) = v(x,t)w(x,t)$, *then* $U_k(x) = \sum_{r=0}^{k} V_k(x)W_{k-r}(x)$,

(d) If $u(x,t) = (x-x_0)^{m\beta}(t-t_0)^{n\alpha}$, *then* $U_k(x) = (x-x_0)^{m\beta}\,\delta(k-n)$,
where $\delta(k) = \begin{cases} 1, & k=0, \\ 0, & k \neq 0. \end{cases}$

(e) If $u(x,t) = D_t^{\alpha}v(x,t)$, $0<\alpha\leqslant 1$, *then* $U_k(x) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_{k+1}(x)$,

(f) If $u(x,t) = D_t^{N\alpha}v(x,t)$, $0<\alpha\leqslant 1$, *then* $U_k(x) = \frac{\Gamma(\alpha k+N\alpha+1)}{\Gamma(\alpha k+1)}V_{k+N}(x)$.

In case $\alpha = 1$, FRDTM is reduced to classical reduced differential transform method (RDTM) which is well addressed in [17].

4. Description of the method

To obtain fractional differential transform of Eq. (1.1) we need some theorems which are presented as follows. For simplicity we suppose $(x_0,t_0) = (0,0)$ in both Theorem 2 and Theorem 3.

THEOREM 2. *Suppose* $u(x) = \sum_{k=0}^{\infty} U(k)x^k$ *and* $v(x) = {}_0^{RL}D_x^{\gamma}u(x)$ $\quad 1<\gamma<2$, *is the left Riemann-Liouville derivative then* $v(x) = \sum_{k=0}^{\infty} U(k)\frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)}x^{k-\gamma}$.

Proof. By replacing $u(x)=\sum_{k=0}^{\infty}U(k)x^k$ in the left Riemann-Liouville derivative definition it is easy to achieve

$$\begin{aligned}
v(x)={}_0^{RL}D_x^{\gamma}u(x)&=\frac{1}{\Gamma(2-\gamma)}\frac{\partial^2}{\partial x^2}\int_0^x\frac{\sum_{k=0}^{\infty}U(k)\xi^k}{(x-\xi)^{\gamma-1}}\,d\xi\\
&=\frac{1}{\Gamma(2-\gamma)}\sum_{k=0}^{\infty}U(k)\frac{\partial^2}{\partial x^2}\int_0^x\frac{\xi^k}{(x-\xi)^{\gamma-1}}d\xi\\
&=\frac{1}{\Gamma(2-\gamma)}\sum_{k=0}^{\infty}U(k)\frac{\partial^2}{\partial x^2}\left(L^{-1}\left(L\left(\int_0^x\frac{\xi^k}{(x-\xi)^{\gamma-1}}d\xi\right)\right)\right)\\
&=\frac{1}{\Gamma(2-\gamma)}\sum_{k=0}^{\infty}U(k)\frac{\partial^2}{\partial x^2}L^{-1}(L(x^k*x^{1-\gamma}))\\
&=\frac{1}{\Gamma(2-\gamma)}\sum_{k=0}^{\infty}U(k)\frac{\partial^2}{\partial x^2}\left(L^{-1}\left(\frac{\Gamma(\mathrm{k}+1)}{\mathrm{s}^{\mathrm{k}+1}}.\frac{\Gamma(2-\gamma)}{\mathrm{s}^{2-\gamma}}\right)\right)\\
&=\sum_{k=0}^{\infty}U(k)\frac{\partial^2}{\partial x^2}\left(\frac{\Gamma(k+1)}{\Gamma(k+3-\gamma)}x^{k+2-\gamma}\right)\\
&=\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)}U(k)x^{k-\gamma}.
\end{aligned}$$

THEOREM 3. *Suppose* $u(x)=\sum_{k=0}^{\infty}U(k)x^k$ *and* $v(x)={}_x^{RL}D_l^{\gamma}u(x)$ $1<\gamma<2$, *is the right Riemann-Liouville derivative, then*
$v(x)=\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\binom{i}{k}(-1)^k(l)^{i-k}U(i)\frac{\Gamma(\mathrm{k}+1)}{\Gamma(\mathrm{k}+1-\gamma)}(l-x)^{k-\gamma}$.

Proof. By replacing $u(x)=\sum_{k=0}^{\infty}U(k)x^k$ in the right Riemann-Liouville derivative definition it is easy to obtain

$$\begin{aligned}
v(x)={}_x^{RL}D_l^{\gamma}u(x)&=\frac{1}{\Gamma(2-\gamma)}\frac{\partial^2}{\partial x^2}\int_x^l\frac{\sum_{k=0}^{\infty}U(k)\xi^k}{(\xi-x)^{\gamma-1}}\,d\xi\\
&=\frac{1}{\Gamma(2-\gamma)}\sum_{k=0}^{\infty}U(k)\frac{\partial^2}{\partial x^2}\int_x^l\frac{\xi^k}{(\xi-x)^{\gamma-1}}d\xi,
\end{aligned}$$

With replacing $\xi-x=t$,

$$=\frac{1}{\Gamma(2-\gamma)}\sum_{k=0}^{\infty}U(k)\frac{\partial^2}{\partial x^2}\left(\int_0^{l-x}\frac{(t+x)^k}{t^{\gamma-1}}dt\right)$$

$$= \frac{1}{\Gamma(2-\gamma)} \sum_{k=0}^{\infty} U(k) \frac{\partial^2}{\partial x^2} \left(L^{-1} \left(L \left(\int_0^y \frac{(l-(y-t))^k}{t^{\gamma-1}} dt \right) \right) \right)$$

$$= \frac{1}{\Gamma(2-\gamma)} \sum_{k=0}^{\infty} U(k) \frac{\partial^2}{\partial x^2} L^{-1} (L((l-y)^k * y^{1-\gamma}))$$

$$= \frac{1}{\Gamma(2-\gamma)} \sum_{k=0}^{\infty} U(k) \frac{\partial^2}{\partial x^2} L^{-1} \left(L \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^i (l)^{k-i} y^i * y^{1-\gamma} \right) \right)$$

$$= \frac{1}{\Gamma(2-\gamma)} \sum_{k=0}^{\infty} U(k) \frac{\partial^2}{\partial x^2} L^{-1} \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^i (l)^{k-i} L(y^i * y^{1-\gamma}) \right)$$

$$= \frac{1}{\Gamma(2-\gamma)} \sum_{k=0}^{\infty} U(k) \frac{\partial^2}{\partial x^2} L^{-1} \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^i (l)^{k-i} \left(\frac{\Gamma(\mathrm{i}+1)}{\mathrm{s}^{\mathrm{i}+1}} . \frac{\Gamma(2-\gamma)}{\mathrm{s}^{2-\gamma}} \right) \right)$$

$$= \sum_{k=0}^{\infty} U(k) \frac{\partial^2}{\partial x^2} \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^i (l)^{k-i} \left(\frac{\Gamma(\mathrm{i}+1)}{\Gamma(\mathrm{i}+3-\gamma)} (l-x)^{\mathrm{i}+2-\gamma} \right) \right)$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} (-1)^i (l)^{k-i} U(k) \frac{\Gamma(\mathrm{i}+1)}{\Gamma(\mathrm{i}+1-\gamma)} (l-x)^{\mathrm{i}-\gamma}$$

$$= \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \binom{i}{k} (-1)^k (l)^{i-k} U(i) \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} (l-x)^{k-\gamma}.$$

In both Theorem 2 and Theorem 3 L is Laplace transform operator, which is defined as $L(f) = \int_0^{\infty} f(t) e^{-st} dt, \quad s \in C,$ and (f*g) represents the convolution of functions f and g, which is defined as $(f * g)(t) = \int_0^t f(t-\tau) g(\tau) d\tau.$

These theorems support us to apply FRDTM for Eq (1.1).

Recall Eq (1.1), then with applying FRDTM, the attained recurrence relation is

$$\frac{\Gamma(\beta(k+2)+1)}{\Gamma(\beta k+1)} U_{k+2}(x) + 2\kappa \frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)} U_{k+1}(x) + \vartheta^2 U_k(x) - \eta \frac{\partial^\gamma U_k(x)}{\partial |x|^\gamma} = F_k(x),$$
$$k = 0, 1, 2, \ldots. \tag{4.1}$$

Where $F_k(x)$ is the FRDT of $f(x,t)$. Fractional reduced differential transform of initial conditions are

$$U_0(x) = \Phi(x), \; U_1(x) = \Psi(x), \tag{4.2}$$

where $\Phi(x)$ and $\Psi(x)$ are the FRDT of $\varphi(x)$ and $\psi(x)$, respectively. Relation (4.1) is a recurrence relation and we can calculate $U_k(x), \quad k = 0, 1, 2, \ldots$ from (4.1) and (4.2), as well $\frac{\partial^\gamma U_k(x)}{\partial |x|^\gamma}$ can be computed according to Theorem 2 and Theorem 3.

After calculating $U_k(x)$, $k = 0,1,2,\ldots,N$ from (4.1) and (4.2) with the inverse transformation we will get

$$u_N(x,t) = \sum_{k=0}^{N} U_k(x)(t-t_0)^{k\alpha}, \tag{4.3}$$

which is the approximate of $u(x,t)$. In the next section we will show that

$$\lim_{N\to\infty} u_N(x,t) = u(x,t).$$

5. Convergence analysis and error estimate of the method

THEOREM 4. *Suppose that* ${}_0D_t^{k\alpha}u(x,t) \in C([0,L]\times[0,T])$ *for* $k = 0,1,\ldots,N+1$ *where* $0<\alpha<1$, *let*

$$u_N(x,t) = \sum_{k=0}^{N} U_k(x)t^{k\alpha}.$$

Then $\lim_{N\to\infty} u_N(x,t) = u(x,t)$, *in addition, there exist a value* ξ, *with* $0<\xi<1$ *while the error term* $E_N(x,t)$ *is obtained from*

$$E_N(x,t) = \sup_{t\in[0,t]} \left| \frac{{}_0D_t^{(N+1)\alpha}u(x,\xi)\,t^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)} \right|.$$

Proof.

$$J^{k\alpha}{}_0D_t^{k\alpha}u(x,t) - J^{(k+1)\alpha}{}_0D_t^{(k+1)\alpha}u(x,t) \tag{5.1}$$

Based on the aforementioned points in Lemma1 for $0<\alpha<1$

$$= J^{k\alpha}\left({}_0D_t^{k\alpha}u(x,t) - J^{\alpha}{}_0D_t^{\alpha}\left({}_0D_t^{k\alpha}u(x,t)\right)\right) \tag{5.2}$$

With (2.8)

$$= J^{k\alpha}\left({}_0D_t^{k\alpha}u(x,0)\right) \tag{5.3}$$

$$= \frac{{}_0D_t^{k\alpha}u(x,0)\,t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{5.4}$$

With (3.2)

$$= U_k(x)t^{k\alpha}. \tag{5.5}$$

The N-th order approximation of $u(x,t)$ is

$$\sum_{k=0}^{N} U_k(x)t^{k\alpha} = \sum_{k=0}^{N} J^{k\alpha}{}_0D_t^{k\alpha}u(x,t) - J^{(k+1)\alpha}{}_0D_t^{(k+1)\alpha}u(x,t) \tag{5.6}$$

$$= u(x,t) - J^{(N+1)\alpha} {}_0D_t^{(N+1)\alpha} u(x,t) \tag{5.7}$$

$$= u(x,t) - \frac{1}{\Gamma((N+1)\alpha)} \int_0^t \frac{{}_0D_t^{(N+1)\alpha} u(x,\tau)}{(t-\tau)^{1-(N+1)\alpha}} d\tau \tag{5.8}$$

Applying integral mean value theorem we will have

$$= u(x,t) - \frac{{}_0D_t^{(N+1)\alpha} u(x,\xi)}{\Gamma((N+1)\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-(N+1)\alpha}} \tag{5.9}$$

$$= u(x,t) - \frac{{}_0D_t^{(N+1)\alpha} u(x,\xi)}{\Gamma((N+1)\alpha+1)} t^{(N+1)\alpha}. \tag{5.10}$$

Hence we will get

$$u(x,t) = \sum_{k=0}^{N} U_k(x) t^{k\alpha} + \frac{{}_0D_t^{(N+1)\alpha} u(x,\xi)}{\Gamma((N+1)\alpha+1)} t^{(N+1)\alpha}. \tag{5.11}$$

Therefore the error term is in the form

$$E_N(x,t) = u(x,t) - \sum_{k=0}^{N} U_k(x) t^{k\alpha} = \sup_{t\in[0,1]} \left| \frac{{}_0D_t^{(N+1)\alpha} u(x,\xi)}{\Gamma((N+1)\alpha+1)} t^{(N+1)\alpha} \right|. \tag{5.12}$$

When $N \to \infty$, *then* $E_N \to 0$, and $u(x,t)$ can be approximated by

$$u(x,t) \cong \sum_{k=0}^{N} U_k(x) t^{k\alpha}. \tag{5.13}$$

6. Numerical results

In this section, some examples are illustrated to show the applicability of the mentioned scheme. We noted that if the partial derivative in equation is integer order, RDTM is used and in the cases that the equation has fractional order derivatives then FRDTM is used. The examples are presented in both cases.

EXAMPLE 1. Consider the following Riesz -space fractional telegraph equation with constant coefficients [7]

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 20\frac{\partial u(x,t)}{\partial t} + 25u(x,t) - \frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma} = f(x,t), \tag{6.1}$$

where the initial and boundary conditions are

$$u(0,t) = u(1,t) = 0, \qquad 0 \leqslant t \leqslant T,$$
$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = x^2(1-x)^2, \quad 0 \leqslant x \leqslant 1,$$

and the inhomogeneous term is

$$f(x,y) = x^2(1-x)^2[24\sin t + 20\cos t] + \frac{\sin t}{2\cos\frac{\gamma\pi}{2}}\left\{\frac{\Gamma(5)}{\Gamma(5-\gamma)}\left(x^{4-\gamma}+(1-x)^{4-\gamma}\right)\right.$$
$$\left. +2\frac{\Gamma(4)}{\Gamma(4-\gamma)}\left(x^{3-\gamma}+(1-x)^{3-\gamma}\right)+\frac{\Gamma(3)}{\Gamma(3-\gamma)}\left(x^{2-\gamma}+(1-x)^{2-\gamma}\right)\right\}.$$

Under these assumptions, the exact solution of Eq (6.1) is $u(x,t) = x^2(1-x)^2\sin t$. With applying RDTM for Eq (6.1) we obtain the recurrence relation as follows

$$(k+2)(k+1)U_{k+2}(x) + 20(k+1)U_{k+1}(x) + 25U_k(x) - \frac{\partial^\gamma U_k(x)}{\partial |x|^\gamma} = F_k(x), \quad (6.2)$$

while

$$F_k(x) = x^2(1-x)^2[24S(k)+20C(k)] + \xi_\gamma S(k)\left\{\frac{\Gamma(5)}{\Gamma(5-\gamma)}\left(x^{4-\gamma}+(1-x)^{4-\gamma}\right)\right.$$
$$\left. +2\frac{\Gamma(4)}{\Gamma(4-\gamma)}(x^{3-\gamma}+(1-x)^{3-\gamma})+\frac{\Gamma(3)}{\Gamma(3-\gamma)}\left(x^{2-\gamma}+(1-x)^{2-\gamma}\right)\right\}.$$

Where $S(k)$ and $C(k)$ indicate differential transforms of $\sin t$ and $\cos t$, respectively which are calculated by

$$S(k) = \begin{cases} 0 & k \text{ is even} \\ \frac{(-1)^{\frac{k-1}{2}}}{k!} & k \text{ is odd,} \end{cases} \qquad C(k) = \begin{cases} 0 & k \text{ is odd} \\ \frac{(-1)^{\frac{k}{2}}}{k!} & k \text{ is even.} \end{cases} \quad (6.3)$$

RDTM of initial conditions are

$$U_0(x) = 0,\ U_1(x) = x^2(1-x)^2. \quad (6.4)$$

After simple calculation from (6.2) and (6.3) and (6.4) we will get

$$U_2(x) = 0,\ U_3(x) = \frac{-1}{3!}x^2(1-x)^2,\ U_4(x) = 0,$$
$$U_5(x) = \frac{1}{5!}x^2(1-x)^2, \ldots, U_k(x) = \begin{cases} 0, & k \text{ is even} \\ \frac{(-1)^{\frac{k-1}{2}}}{k!}x^2(1-x)^2, & k \text{ is odd.} \end{cases} \quad (6.5)$$

With inverse Transformation

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = x^2(1-x)^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots\right) = x^2(1-x)^2\sin t. \quad (6.6)$$

Which is the exact solution.

Chen et al. [7] proposed a class of unconditionally stable difference scheme (FD) based on the Pade approximation to solve the problem (6.1) at $t = 2.0$ with several choices for h and τ, where h and τ are the space and time step sizes, respectively. In Table 1, we compare $\|u-u_N\|_2$ at $t=2.0$ with different choice for N. The results show that our method is more accurate than the three schemes which were proposed on their work. The main advantage of this method is lower computational work than Chen et al. [7] , where with $\tau = 10^{-4}$ as a time step length they need to evaluate 2000 iterations to reach $t=2$.

Table 1: Comparison of $\|u-u_N\|_2$ for different values of h and τ for Example 1 at $t=2.0$.

N	Improved RDTM method	FD schemes [7]($\tau=10^{-4}$)			
		h	SchemeI	SchemeII	SchemeIII
5	3.5194e-4	0.25	8.5871-4	1.0948e-3	1.7573e-3
10	5.8880e-7	0.125	1.9827e-4	2.5363e-4	4.2490e-4
15	3.4711e-12	0.0625	5.0593e-5	6.1691e-5	1.0145e-4
20	3.3912e-16	0.03125	1.4007e-5	1.6003e-5	2.4455e-5

EXAMPLE 2. Consider the following fractional telegraph equation with Riesz space fractional derivative [1]

$$\begin{aligned} &{}_0^CD_t^{2\beta}u(x,t)+{}_0^CD_t^{\beta}u(x,t)+\left({}_0^{RL}D_x^{\gamma}+{}_x^{RL}D_l^{\gamma}u(x,t)\right)=f(x,t),\\ &1/2<\beta<1,\quad 0\leqslant x\leqslant 1,\quad 0\leqslant t\leqslant 1, \end{aligned} \tag{6.7}$$

with initial conditions

$$u(x,0)=x^2(1-x)^2,\quad \frac{\partial u(x,0)}{\partial t}=-4x^2(1-x)^2\quad 0\leqslant x\leqslant 1,$$

and boundary conditions

$$u(0,t)=u(1,t)=0,\qquad 0\leqslant t\leqslant 1,$$

the inhomogeneous term is

$$\begin{aligned} f(x,t)=&\left(\frac{8t^{2-2\beta}}{\Gamma(3-2\beta)}+\frac{8t^{2-\beta}}{\Gamma(3-\beta)}-\frac{4t^{1-\beta}}{\Gamma(2-\beta)}\right)x^2(1-x)^2\\ &+(2t-1)^2\left(\frac{2\left(x^{2-\gamma}+(1-x)^{2-\gamma}\right)}{\Gamma(3-\gamma)}-\frac{12\left(x^{3-\gamma}+(1-x)^{3-\gamma}\right)}{\Gamma(4-\gamma)}+\frac{24\left(x^{4-\gamma}+(1-x)^{4-\gamma}\right)}{\Gamma(4-\gamma)}\right). \end{aligned}$$

The exact solution of Eq (6.7) is $u(x,t)=\left(4t^2-4t+1\right)x^2(1-x)^2$.
Suppose $2\beta=1.6$, we choose $\alpha=0.2$ and using FRDTM and Theorem 2 and Theorem 3 we get

$$\frac{\Gamma(o.2k+2.6)}{\Gamma(0.2k+1)}U_{k+8}(x)+\frac{\Gamma(0.2k+1.8)}{\Gamma(0.2k+1)}U_{k+4}(x)+\left({}_0^CD_x^{\gamma}+{}_x^CD_1^{\gamma}\right)U_k(x)=F_k(x), \tag{6.8}$$

while

$$F_k(x) = x^2(1-x)^2\left(\frac{8\delta(k-2)}{\Gamma(3-2\alpha)} + \frac{8\delta(k-6)}{\Gamma(3-\alpha)} - \frac{4\delta(k-1)}{\Gamma(2-\alpha)}\right)$$
$$+ (4\delta(k-10) - 4\delta(k-5) + \delta(k))$$
$$\left(\frac{2\left(x^{2-\gamma}+(1-x)^{2-\gamma}\right)}{\Gamma(3-\gamma)} - \frac{12\left(x^{3-\gamma}+(1-x)^{3-\gamma}\right)}{\Gamma(4-\gamma)} + \frac{24\left(x^{4-\gamma}+(1-x)^{4-\gamma}\right)}{\Gamma(5-\gamma)}\right).$$

Where the FRDTM of initial conditions are

$$U_0(x) = x^2(1-x)^2, \quad U_1(x) = U_2(x) = U_3(x) = U_4(x) = 0, \quad \text{and}$$
$$U_5(x) = -4x^2(1-x)^2, \quad U_6(x) = U_7(x) = 0 \tag{6.9}$$

From (6.8) and (6.9) we will get

$$U_8(x) = U_9(x) = 0, \quad U_{10}(x) = 4x^2(1-x)^2 \ \textit{and} \ \ U_k(x) = 0 \ \textit{for} \ k \geqslant 11. \tag{6.10}$$

Using inverse transformation we will have

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)\,t^{0.2k} = x^2(1-x)^2\left(1-4t+4t^2\right) = x^2(1-x)^2(1-2t)^2. \tag{6.11}$$

Zhao et al. [2] used fractional difference and finite element methods in spatial direction to obtain numerical solution for Eq (6.7). Contrary, with our method the exact solution is achieved for this equation which demonstrate that this method is effective and reliable for fractional telegraph equation with Riesz space-fractional derivative.

EXAMPLE 3. Consider the following Riesz space fractional telegraph equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 4\frac{\partial u(x,t)}{\partial t} + 4u(x,t) - \frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma} = f(x,t), \tag{6.12}$$

where initial and boundary conditions are

$$u(0,t) = u(1,t) = 0, \qquad 0 \leqslant t \leqslant T,$$
$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = x^2(1-x)^2 e^x, \quad 0 \leqslant x \leqslant 1,$$

and the inhomogeneous term is

$$f(x,y) = x^2(1-x)^2 e^x[3\sin t + 4\cos t]$$
$$+ \frac{\sin t}{2\cos\frac{\gamma\pi}{2}} \sum_{n=0}^{\infty} \frac{1}{n!}\left\{\frac{\Gamma(n+5)}{\Gamma(n+5-\gamma)}\left(x^{n+4-\gamma}+(1-x)^{n+4-\gamma}\right)\right.$$
$$\left. +2\frac{\Gamma(n+4)}{\Gamma(n+4-\gamma)}\left(x^{n+3-\gamma}+(1-x)^{n+3-\gamma}\right) + \frac{\Gamma(n+3)}{\Gamma(n+3-\gamma)}\left(x^{n+2-\gamma}+(1-x)^{n+2-\gamma}\right)\right\}.$$

Under these assumptions, the exact solution is given by $u(x,t) = x^2(1-x)^2 e^x \sin t$. Figures 1, 2 demonstrate the surface area of exact and numerical solutions of Example 3 and Table 2 represents the absolute errors with applying this method for different values of N at $t = 1$.

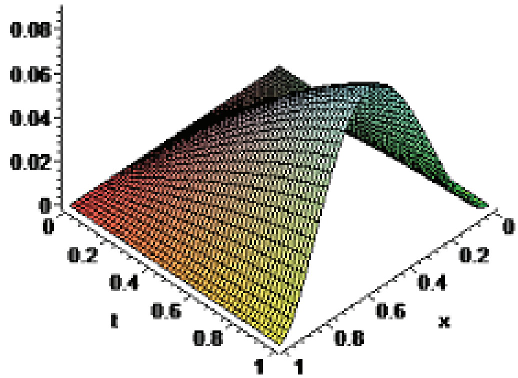

Figure 1: The exact solution surface of Example 3

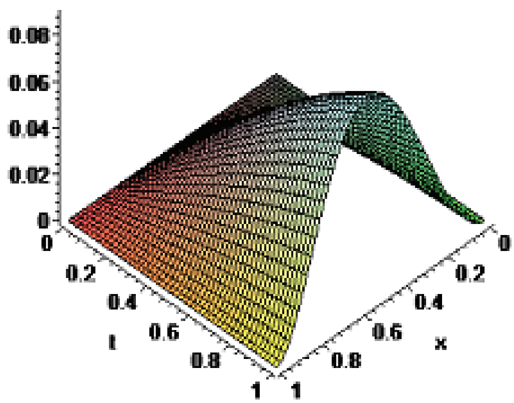

Figure 2: The numerical solution surface of Example 3 for $N = 5$

Table 2: The absolute errors $|u(x,t) - u_N(x,t)|$ by FRDTM of Example 3 at $t = 1$

x	$N = 1$	$N = 3$	$N = 5$	$N = 7$
0	0	0	0	0
0.2	$4.9568e-3$	$2.5444e-4$	$6.1185e-6$	$8.5401e-8$
0.4	$1.3622e-2$	$6.9926e-4$	$1.6814e-5$	$2.3466e-7$
0.6	$1.6638e-2$	$8.5407e-4$	$2.0537e-5$	$2.8661e-7$
0.8	$9.0320e-3$	$4.6363e-4$	$1.1148e-5$	$2.3466e-7$
1	0	0	0	0

7. Conclusion

Riesz derivative operator appears in some partial differential equations such as telegraph equation, wave equation, diffusion equation, advection-dispersion equation and other partial differential equations. These types of equations previously were solved by FRDTM without considering Riesz derivative operator. In this paper we have developed FRDTM for solving telegraph equation with Riesz space fractional derivatives. It is remarkable that when the telegraph equation has fractional order with Riesz operator then improved FRDTM is used, and in the case we have integer order telegraph equation with Riesz operator then improved RDTM will be used. Compared to the other numerical methods, the acquired results in both cases demonstrated that this method required less amount of computational work; moreover it was efficient and powerful technique. Providing convergent series solution with fast convergence rate was the main advantage of the proposed method, which the numerical examples revealed these facts.

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