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EXISTENCE OF POSITIVE SOLUTION FOR A CLASS OF NONLOCAL PROBLEM WITH STRONG SINGULARITY AND LINEAR TERM

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Abstract. We consider a class of nonlocal problem with strong singularity and linear term. Combining with the variational method and Nehari manifold, a necessary and sufficient condition that shows the existence of positive solution is obtained.

1. Introduction

In this paper, we consider the following nonlocal problem with strong singularity and linear term

$$\begin{cases} -\left[a+b\left(\int_\Omega |\nabla u|^2 dx\right)^m\right]\Delta u = \dfrac{f(x)}{u^\gamma}+g(x)u, & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N(N\geqslant 3)$ is a smooth bounded domain, $a,b\geqslant 0$ with $a+b>0$, and $m>0$, $\gamma>1$, $f\in L^1(\Omega)$ is nonzero and nonnegative, $g\in L^\infty(\Omega)$ is nonnegative.

When $a\geqslant 0$ and $b>0$, problem (1.1) is always called the singular Kirchhoff-type problem. Kirchhoff-type problems are often referred to as being nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2 dx)^m\Delta u$ which implies that the equation in problem (1.1) is no longer a pointwise identity. When $m=1$, problem (1.1) reduces to the stationary version of Kirchhoff equation which was presented by Kirchhoff [8] in 1883. After the work by Lions [16], people have paid much attention to Kirchhoff-type equations and a lot of classical results have been obtained.

When $a=1,b=0$, problem (1.1) degenerates to classical singular elliptic equation:

$$\begin{cases} -\Delta u = \dfrac{f(x)}{u^\gamma}+g(x)u^q, & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{1.2}$$

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where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a smooth bounded domain, $q > 0$, f and g are nonnegative functions with some certain conditions. Many results have been obtained the existence, uniqueness and multiplicity of positive solutions for problem (1.2) with different γ-values, for examples, [1], [3]-[7], [9]-[15], [18]-[21], [24]-[26]. Particularly, [3]-[7], [9]-[15], [19], [20], [24]-[26] considered the case of $0 < \gamma < 1$; [7] considered the case of $\gamma = 1$. While the case of problem (1.2) with $\gamma > 1$ and $g(x) \equiv 0$ was first considered by Lazer and Mckenna [9], they obtained a unique H_0^1-solution if and only if $\gamma < 3$ for $f \in C^\alpha(\overline{\Omega})$, $f > 0$, but no solution when $\gamma \geqslant 3$. Sun [18] obtained at least one H_0^1-solution of problem (1.2) with $\gamma > 1, 0 < q < 1$ by f and γ satisfying the condition:

$$\int_\Omega f(x)|u_0|^{1-\gamma}dx < \infty, \; u_0 \in H_0^1(\Omega), \tag{1.3}$$

which was optimal for H_0^1-solution of strongly singular problems. Subsequently, Sun and Zhang [21] proved that problem (1.2) with $\gamma > 1, g(x) \equiv 0$ has a unique H_0^1-solution. Then considered that the compatible condition of (1.3) in [18] was also valid and revealed the 3's role of [9], that is, they provided an extension of the classical Lazer-Mckenna obstruction.

Recently, [22] considered the existence of a H_0^1-solution to the strong singular Kirchhoff-type equation:

$$\begin{cases} -\left[a + b\left(\int_\Omega |\nabla u|^2 dx\right)\right]\Delta u = \dfrac{f(x)}{u^\gamma} + g(x)u^q, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ and $\gamma > 1, 0 < q < 1, a, b > 0$. They proved that the necessary and sufficient condition for the existence of positive solution of problem (1.4), was also satisfied with the condition of (1.3). [10]-[14] and [17] studied the case of $0 < \gamma < 1$, and [23] considered the case of $\gamma = 1$.

Inspired by [18], [21] and [22], we study problem (1.1) and prove that it has a unique H_0^1-solution, and generalize the nonlocal term $(\int_\Omega |\nabla u|^2 dx)^m \Delta u$ for all $m > 0$.

For all $u \in H_0^1(\Omega)$, the energy functional corresponding to problem (1.1) is given by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}\|u\|^{2m+2} - \frac{1}{1-\gamma}\int_\Omega f(x)|u|^{1-\gamma}dx - \frac{1}{2}\int_\Omega g(x)|u|^2 dx,$$

where $H_0^1(\Omega)$ is a Sobolev space equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$. Since $\gamma > 1$, it should be noted that the energy functional I is not well defined on $H_0^1(\Omega)$. Note that u is called a weak solution of problem (1.1) if $u \in H_0^1(\Omega)$ such that

$$\left(a + b\|u\|^{2m}\right)\int_\Omega (\nabla u, \nabla\varphi)dx - \int_\Omega \frac{f(x)}{u^\gamma}\varphi dx - \int_\Omega g(x)u\varphi dx = 0,$$

for all $\varphi \in H_0^1(\Omega)$. Our main results can be described as follows.

THEOREM 1. *Assume that* $f \in L^1(\Omega)$ *is positive, and* $g \in L^\infty(\Omega)$ *is nonnegative,* $a,b \geqslant 0$ *with* $a+b>0$, *and* $m>0, \gamma>1$. *Then problem* (1.1) *admits a solution* $u_* \in H_0^1(\Omega)$ *if and only if there exists* $u_0 \in H_0^1(\Omega)$ *satisfying* (1.3).

REMARK 1. To our best knowledge, problem (1.1) has not been studied up to now. By comparing with [22], we obtain the same result for the case of $q=1$ and all $m>0$. Moreover, we verify that special case of $a=0$, $b>0$ is also valid. It also extends [18] to the case of $q=1$.

2. Proof of Theorem 1

In order to obtain the solution of problem (1.1), we define the following two constrained sets:

$$N_1 := \left\{ u \in H_0^1(\Omega) : a\|u\|^2 + b\|u\|^{2m+2} - \int_\Omega f(x)|u|^{1-\gamma}dx - \int_\Omega g(x)|u|^2 dx \geqslant 0 \right\},$$

$$N_2 := \left\{ u \in H_0^1(\Omega) : a\|u\|^2 + b\|u\|^{2m+2} - \int_\Omega f(x)|u|^{1-\gamma}dx - \int_\Omega g(x)|u|^2 dx = 0 \right\}.$$

Before proving Theorem 1, we give the following useful lemmas.

LEMMA 1. *If* $\gamma>1$ *and* (1.3) *holds, then* N_1 *and* N_2 *are nonempty.*

Proof. According to (1.3), for any $t>0$ and $u \in H_0^1(\Omega)$ with

$$\int_\Omega f(x)|u|^{1-\gamma}dx < +\infty,$$

we define the function $\Psi \in C(\mathbb{R}^+, \mathbb{R})$ by

$$\Psi(t) := a\|u\|^2 + bt^{2m}\|u\|^{2m+2} - t^{-\gamma-1}\int_\Omega f(x)|u|^{1-\gamma}dx, \forall t>0.$$

We get,

$$\Psi'(t) := 2mbt^{2m-1}\|u\|^{2m+2} + (\gamma+1)t^{-\gamma-2}\int_\Omega f(x)|u|^{1-\gamma}dx > 0,$$

Ψ is increasing on $t>0$, with $\lim_{t\to+\infty}\Psi(t)=+\infty$ and $\lim_{t\to 0^+}\Psi(t)=-\infty$. Consequently, there exists a unique $t^+ = t^+(u) > 0$ such that

$$\Psi(t^+) = \int_\Omega g(x)|u|^2 dx \text{ and } \Psi'(t^+) > 0.$$

Moreover,

$$\frac{dI(tu)}{dt} = at\|u\|^2 + bt^{2m+1}\|u\|^{2m+2} - t^{-\gamma}\int_\Omega f(x)|u|^{1-\gamma}dx - t\int_\Omega g(x)|u|^2 dx$$

$$= t\left[a\|u\|^2 + bt^{2m}\|u\|^{2m+2} - t^{-\gamma-1}\int_\Omega f(x)|u|^{1-\gamma}dx - \int_\Omega g(x)|u|^2dx\right]$$
$$= t\left[\Psi(t) - \int_\Omega g(x)|u|^2dx\right],$$

obviously, $I(tu)$ is decreasing on $0 < t < t^+$ and increasing on $t > t^+$. For any $t > 0$, one gets

$$I(tu) \geqslant I(t^+(u)u), \tag{2.1}$$

that is, $I(t^+(u)u) = \min_{t>0} I(tu)$. Then, we can easy obtain that $t^+(u)u \in N_2$. Therefore, $N_2(\subset N_1)$ and N_1 are not empty. This completes the proof of Lemma 1. □

LEMMA 2. *N_1 is an unbounded closed set in $H_0^1(\Omega)$.*

Proof. Obviously, since $tu \in N_1$ for all $t > t^+(u)$, N_1 is unbounded in $H_0^1(\Omega)$. We only need prove that N_1 is closed. Assume that $\{u_n\} \subset N_1$ such that $u_n \to u$ strongly in $H_0^1(\Omega)$, we need prove $u \in N_1$. Since $\{u_n\} \subset N_1$, one has

$$a\|u_n\|^2 + b\|u_n\|^{2m+2} - \int_\Omega f(x)|u_n|^{1-\gamma}dx - \int_\Omega g(x)|u_n|^2dx \geqslant 0.$$

Consequently, it follows from Fatou's Lemma and $u_n \to u$ that

$$a\|u\|^2 + b\|u\|^{2m+2} - \int_\Omega f(x)|u|^{1-\gamma}dx - \int_\Omega g(x)|u|^2dx \geqslant 0.$$

Therefore, $u \in N_1$. This completes the proof of Lemma 2. □

LEMMA 3. *For any $u \in N_2$ and $\varphi \in H_0^1(\Omega), \varphi > 0$, there exists $\varepsilon > 0$ and a continuous function $t = t(s) > 0 (s \in \mathbb{R})$ for every $|s| < \varepsilon$ satisfies $t(0) = 1, t(s)(u + s\varphi) \in N_2, \forall s \in \mathbb{R}, |s| < \varepsilon$.*

Proof. For arbitrary $u \in N_2$, we define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(t,s) = a\int_\Omega |\nabla(u+s\varphi)|^2dx + bt^{2m}\left[\int_\Omega |\nabla(u+s\varphi)|^2dx\right]^{m+1}$$
$$- t^{-\gamma-1}\int_\Omega f(x)|u+s\varphi|^{1-\gamma}dx - \int_\Omega g(x)|u+s\varphi|^2dx,$$

then,

$$F_t(t,s) = 2mbt^{2m-1}\left[\int_\Omega |\nabla(u+s\varphi)|^2dx\right]^{m+1} + (\gamma+1)t^{-\gamma-2}\int_\Omega f(x)|u+s\varphi|^{1-\gamma}dx.$$

Since $u \in N_2$, we know $F(1,0) = 0$ and

$$F_t(1,0) = 2mb\|u\|^{2m+2} + (\gamma+1)\int_\Omega f(x)|u|^{1-\gamma}dx > 0, \text{ (since } u \neq 0),$$

applying implicit function theorem to function F at $(1,0)$ point, there is one available $\varepsilon > 0$ and a continuous function $t = t(s) > 0, s \in \mathbb{R}$ such that $t(0) = 1, t(s)(u + s\varphi) \in N_2$ for $|s| < \varepsilon$. This completes the proof of Lemma 3. □

Now, we give the proof of Theorem 1.

Proof of Theorem 1. Obviously, the necessity is true. Now, we only need prove the sufficiency. Since N_1 is a closed set in $H_0^1(\Omega)$ by Lemma 2, we can apply the Ekeland variational principle to the minimizing problem $\inf\limits_{u\in N_1} I(u)$, there exists a sequence $\{u_n\} \subset N_1$ satisfying the following properties

(i) $I(u_n) < \inf_{u\in N_1} I(u) + \frac{1}{n}$,

(ii) $I(u_n) \leqslant I(v) + \frac{1}{n}\|u_n - v\|, \ \forall v \in N_1$.

Since $I(u) = I(|u|)$, we can assume from the beginning that $u_n \geqslant 0$ in Ω. According to $u_n \in N_1$, one has

$$\int_\Omega f(x)|u_n|^{1-\gamma} dx < \infty,$$

which implies that $u_n(x) > 0$ a.e. in Ω. Moreover, $I(u)$ is coercive on N_1 and therefore $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Consequently, there exist a subsequence (still denoted by $\{u_n\}$) and $u_* \in H_0^1(\Omega)$ with $u_* \geqslant 0$, one gets

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_*, & \text{strongly in } L^2(\Omega), \\ u_n(x) \to u_*(x), & \text{a.e. in } \Omega, \end{cases}$$

as $n \to +\infty$. Since $u_n \in N_1$, by Fatou's Lemma, we obtain

$$\int_\Omega f(x) u_*^{1-\gamma} dx < \infty,$$

which implies that $u_* > 0$ a.e. in Ω. We shall prove that $u_* \in N_2$ such that $I(u_*) = \inf\limits_{u\in N_1} I(u)$. Now, we divide the following two cases to prove it.

Case one. Assume that $\{u_n\} \subset N_1 \backslash N_2$ for all n large enough.

Let $\varphi \in H_0^1(\Omega)$ with $\varphi \geqslant 0$, since $u_n \in N_1 \backslash N_2$ and $\gamma > 1$, for any $t \geqslant 0$, one obtains

$$\begin{aligned} a\|u_n\|^2 + b\|u_n\|^{2m+2} - \int_\Omega g(x) u_n^2 dx &> \int_\Omega f(x) u_n^{1-\gamma} dx \\ &\geqslant \int_\Omega f(x)(u_n + t\varphi)^{1-\gamma} dx. \end{aligned}$$

Consequently, by the continuity, choosing $t > 0$ sufficiently small such that

$$a\|u_n + t\varphi\|^2 + b\|u_n + t\varphi\|^{2m+2} - \int_\Omega g(x)(u_n + t\varphi)^2 dx > \int_\Omega f(x)(u_n + t\varphi)^{1-\gamma} dx,$$

which implies that $u_n+t\varphi\in N_1$ for $t>0$ small enough. Thus, according to (i) and (ii), we have

$$\begin{aligned}&\frac{\|t\varphi\|}{n}+\frac{a}{2}(\|u_n+t\varphi\|^2-\|u_n\|^2)+\frac{b}{2m+2}(\|u_n+t\varphi\|^{2m+2}-\|u_n\|^{2m+2})\\&-\frac{1}{2}\int_\Omega g(x)[(u_n+t\varphi)^2-u_n^2]dx\geqslant\frac{1}{1-\gamma}\int_\Omega f(x)[(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}]dx.\end{aligned}$$

Consequently, dividing the above inequality by $t>0$ and passing to the infimum limit as $t\to 0,$ it follows from Fatou's Lemma that

$$\begin{aligned}&\frac{\|\varphi\|}{n}+\left(a+b\|u_n\|^{2m}\right)\int_\Omega(\nabla u_n,\nabla\varphi)dx-\int_\Omega g(x)u_n\varphi dx\\&\geqslant\int_\Omega\liminf_{t\to 0}\frac{f(x)}{1-\gamma}\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}dx\\&=\int_\Omega f(x)u_n^{-\gamma}\varphi dx,(\text{since } u_n(x)>0 \text{ a.e. in } \Omega).\end{aligned}\tag{2.2}$$

Consequently, letting $n\to+\infty,$ by Fatou's Lemma again, it follows from (2.2) that

$$\int_\Omega f(x)u_*^{-\gamma}\varphi dx<\infty,$$

for every $\varphi\in H_0^1(\Omega)$ with $\varphi\geqslant 0.$ Moreover, choosing $\varphi=u_*$ in the above inequality, one has $\int_\Omega f(x)u_*^{1-\gamma}dx<\infty.$ According to the argument of Lemma 2.1, there exists a unique positive constant $t^+(u_*)$ such that $I(t^+(u_*)u_*)=\min_{t>0}I(tu_*).$ Then, by the weakly lower semi-continuity of norm and Fatou's Lemma, we obtain

$$\begin{aligned}\inf_{u\in N_1}I(u)&=\lim_{n\to\infty}I(u_n)\\&=\liminf_{n\to\infty}\left[\frac{a}{2}\|u_n\|^2+\frac{b}{2m+2}\|u_n\|^{2m+2}+\frac{1}{\gamma-1}\int_\Omega f(x)u_n^{1-\gamma}dx\right.\\&\quad\left.-\frac{1}{2}\int_\Omega g(x)u_n^2dx\right]\\&=\liminf_{n\to\infty}\left[\frac{a}{2}\|u_n\|^2+\frac{b}{2m+2}\|u_n\|^{2m+2}+\frac{1}{\gamma-1}\int_\Omega f(x)u_n^{1-\gamma}dx\right]\\&\quad-\frac{1}{2}\int_\Omega g(x)u_*^2dx\\&\geqslant\liminf_{n\to\infty}\frac{a}{2}\|u_n\|^2+\liminf_{n\to\infty}\frac{b}{2m+2}\|u_n\|^{2m+2}\\&\quad+\liminf_{n\to\infty}\left[\frac{1}{\gamma-1}\int_\Omega f(x)u_n^{1-\gamma}dx\right]-\frac{1}{2}\int_\Omega g(x)u_*^2dx\\&\geqslant\frac{a}{2}\|u_*\|^2+\frac{b}{2m+2}\|u_*\|^{2m+2}+\frac{1}{\gamma-1}\int_\Omega f(x)u_*^{1-\gamma}dx-\frac{1}{2}\int_\Omega g(x)u_*^2dx\\&=I(u_*)\end{aligned}$$

$$
\begin{aligned}
&\geqslant I(t^{+}(u_*)u_*)\\
&\geqslant \inf_{u\in N_2} I(u)\\
&\geqslant \inf_{u\in N_1} I(u),
\end{aligned}
$$

combining with $I(tu_*) \geqslant I(t^{+}(u_*)u_*)$, we obtain $t^{+}(u_*)=1$. Thus, one gets

$$u_* \in N_2, \quad \inf_{u\in N_1} I(u) = I(u_*). \tag{2.3}$$

Since $\|u_n\|$ is bounded, we may assume that $\|u_n\| \to A$. It follows from the weakly lower semi-continuity of the norm that $\|u_*\| \leqslant \liminf_{n\to\infty}\|u_n\| = \lim_{n\to\infty}\|u_n\| = A$. If $\|u_*\| = A$, which leads to $u_n \to u_*$ strongly in $H_0^1(\Omega)$. Otherwise, $\|u_*\| < A$, we have

$$
\begin{aligned}
I(u_*) &= \inf_{N_1} I\\
&= \lim_{n\to\infty} I(u_n)\\
&= \lim_{n\to\infty}\left[\frac{a}{2}\|u_n\|^2 + \frac{b}{2m+2}\|u_n\|^{2m+2} + \frac{1}{\gamma-1}\int_\Omega f(x)u_n^{1-\gamma}dx - \frac{1}{2}\int_\Omega g(x)u_n^2 dx\right]\\
&= \frac{a}{2}A^2 + \frac{b}{2m+2}A^{2m+2} + \lim_{n\to\infty}\frac{1}{\gamma-1}\int_\Omega f(x)u_n^{1-\gamma}dx - \frac{1}{2}\int_\Omega g(x)u_*^2 dx\\
&= \frac{a}{2}A^2 + \frac{b}{2m+2}A^{2m+2} + \liminf_{n\to\infty}\frac{1}{\gamma-1}\int_\Omega f(x)u_n^{1-\gamma}dx - \frac{1}{2}\int_\Omega g(x)u_*^2 dx\\
&\geqslant \frac{a}{2}A^2 + \frac{b}{2m+2}A^{2m+2} + \frac{1}{\gamma-1}\int_\Omega f(x)u_*^{1-\gamma}dx - \frac{1}{2}\int_\Omega g(x)u_*^2 dx\\
&\geqslant \frac{a}{2}\|u_*\|^2 + \frac{b}{2m+2}\|u_*\|^{2m+2} + \frac{1}{\gamma-1}\int_\Omega f(x)u_*^{1-\gamma}dx - \frac{1}{2}\int_\Omega g(x)u_*^2 dx\\
&= I(u_*)
\end{aligned}
$$

a contradiction. Therefore, one has $\|u_n\| \to \|u_*\|$. Combining with $u_n \rightharpoonup u_*$, we obtain that $u_n \to u_*$ in $H_0^1(\Omega)$. Moreover, for every $\varphi \in H_0^1(\Omega)$ with $\varphi \geqslant 0$, it follows from (2.2) and Fatou's Lemma that

$$\left(a + b\|u_*\|^{2m}\right)\int_\Omega (\nabla u_*, \nabla\varphi)dx - \int_\Omega g(x)u_*\varphi dx \geqslant \int_\Omega f(x)u_*^{-\gamma}\varphi dx. \tag{2.4}$$

Case two. There exists a subsequence $\{u_n\} \subset N_2$, still denoted by $\{u_n\}$. Let $\varphi \in H_0^1(\Omega)$ with $\varphi \geqslant 0$. Since $\gamma > 1$, for any $t > 0$, one has

$$\int_\Omega f(x)(u_n + t\varphi)^{1-\gamma}dx \leqslant \int_\Omega f(x)u_n^{1-\gamma}dx < \infty.$$

By Lemma 3, choosing $u = u_n \in N_2$ and $\varphi \in H_0^1(\Omega)$ with $\varphi \geqslant 0$, and $s > 0$ sufficiently small, we can obtain a sequence of continuous functions $t_n = t_n(s)$ such that $t_n(0) = 1$ and $t_n(s)(u_n + s\varphi) \in N_2$. For $t_n(s)(u_n + s\varphi) \in N_2$ and $u_n \in N_2$, we obtain

$$0 = at_n^2(s)\|u_n + s\varphi\|^2 + bt_n^{2m+2}(s)\|u_n + s\varphi\|^{2m+2} - t_n^{1-\gamma}(s)\int_\Omega f(x)(u_n + s\varphi)^{1-\gamma}dx$$

$$-t_n^2(s)\int_\Omega g(x)(u_n+s\varphi)^2dx,$$

and

$$0=a\|u_n\|^2+b\|u_n\|^{2m+2}-\int_\Omega f(x)u_n^{1-\gamma}dx-\int_\Omega g(x)u_n^2dx.$$

Therefore, one has

$$\begin{aligned}0=&a[t_n^2(s)-1]\|u_n+s\varphi\|^2+a(\|u_n+s\varphi\|^2-\|u_n\|^2)\\&+b[t_n^{2m+2}(s)-1]\|u_n+s\varphi\|^{2m+2}+b(\|u_n+s\varphi\|^{2m+2}-\|u_n\|^{2m+2})\\&-[t_n^{1-\gamma}(s)-1]\int_\Omega f(x)(u_n+s\varphi)^{1-\gamma}dx-\int_\Omega f(x)[(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}]dx\\&-[t_n^2(s)-1]\int_\Omega g(x)(u_n+s\varphi)^2dx-\int_\Omega g(x)[(u_n+s\varphi)^2-u_n^2]dx,\end{aligned}$$

and dividing the above equality by $s,$ one gets

$$\begin{aligned}0=&\left\{a[t_n(s)+1]\|u_n+s\varphi\|^2+b\frac{t_n^{2m+2}(s)-1}{t_n(s)-1}\|u_n+s\varphi\|^{2m+2}\right.\\&-\frac{t_n^{1-\gamma}(s)-1}{t_n(s)-1}\int_\Omega f(x)(u_n+s\varphi)^{1-\gamma}dx\\&\left.-[t_n(s)+1]\int_\Omega g(x)(u_n+s\varphi)^2dx\right\}\frac{t_n(s)-1}{s}+\frac{a}{s}(\|u_n+s\varphi\|^2-\|u_n\|^2)\\&+\frac{b}{s}(\|u_n+s\varphi\|^{2m+2}-\|u_n\|^{2m+2})-\frac{1}{s}\int_\Omega f(x)[(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}]dx\\&-\frac{1}{s}\int_\Omega g(x)[(u_n+s\varphi)^2-u_n^2]dx.\end{aligned}$$

Let

$$A_n(s)=\frac{t_n(s)-1}{s},$$

and letting $s\to 0^+,$ it follows that

$$\begin{aligned}0\geqslant& A_n(s)\left\{2a\|u_n\|^2+(2m+2)b\|u_n\|^{2m+2}+(\gamma-1)\int_\Omega f(x)u_n^{1-\gamma}dx\right.\\&\left.-2\int_\Omega g(x)u_n^2dx\right\}+\left[2a+(2m+2)b\|u_n\|^{2m}\right]\int_\Omega(\nabla u_n,\nabla\varphi)dx\\&-2\int_\Omega g(x)u_n\varphi dx\\=&A_n(s)\left\{2mb\|u_n\|^{2m+2}+(\gamma+1)\int_\Omega f(x)u_n^{1-\gamma}dx\right\}\\&+\left[2a+(2m+2)b\|u_n\|^{2m}\right]\int_\Omega(\nabla u_n,\nabla\varphi)dx-2\int_\Omega g(x)u_n\varphi dx.\end{aligned}$$

Since $\{u_n\} \subset N_2(\subset N_1)$ is bounded in $H_0^1(\Omega)$, consequently, by Lemma 2, one has

$$A_n(s) \leqslant C_1 \left(\limsup_{s\to 0^+} A_n(s) \leqslant C_1 \right), \tag{2.5}$$

for suitable constant $C_1 > 0$.

Next, by the subadditivity of the norm, we get

$$\|u_n - t_n(s)(u_n + s\varphi)\| \leqslant |1 - t_n(s)| \|u_n\| + s t_n(s) \|\varphi\|.$$

Then, it follows from (ii) that

$$\begin{aligned}
&\frac{|1-t_n(s)|}{s}\frac{\|u_n\|}{n} + t_n(s)\frac{\|\varphi\|}{n} \\
\geqslant &\frac{I(u_n) - I[t_n(s)(u_n+s\varphi)]}{s} \\
\geqslant &\frac{1}{1-\gamma}\left\{ \frac{a(1+\gamma)}{2}[t_n(s)+1]\|u_n+s\varphi\|^2 + \frac{b(3+\gamma)}{2m+2}\frac{t_n^{2m+2}(s)-1}{t_n(s)-1}\|u_n+s\varphi\|^{2m+2} \right. \\
&\left. - \frac{1+\gamma}{2}[t_n(s)+1]\int_\Omega g(x)(u_n+s\varphi)^2 dx \right\} \frac{t_n(s)-1}{s} \\
&+ \frac{a(1+\gamma)}{2s(1-\gamma)}(\|u_n+s\varphi\|^2 - \|u_n\|^2) \\
&+ \frac{b(3+\gamma)}{(2m+2)s(1-\gamma)}(\|u_n+s\varphi\|^{2m+2} - \|u_n\|^{2m+2}) \\
&- \frac{1+\gamma}{2s(1-\gamma)}\int_\Omega g(x)[(u_n+s\varphi)^2 - u_n^2]dx.
\end{aligned} \tag{2.6}$$

Let

$$\begin{aligned}
K_{1,n}(s) = &\frac{a(1+\gamma)}{2}[t_n(s)+1]\|u_n+s\varphi\|^2 \\
&+ \frac{b(3+\gamma)}{2m+2}\frac{t_n^{2m+2}(s)-1}{t_n(s)-1}\|u_n+s\varphi\|^{2m+2} \\
&- \frac{1+\gamma}{2}[t_n(s)+1]\int_\Omega g(x)(u_n+s\varphi)^2 dx, \\
K_{2,n}(s) = &\frac{a(1+\gamma)}{2s(1-\gamma)}(\|u_n+s\varphi\|^2 - \|u_n\|^2) \\
&+ \frac{b(3+\gamma)}{(2m+2)s(1-\gamma)}(\|u_n+s\varphi\|^{2m+2} - \|u_n\|^{2m+2}) \\
&- \frac{1+\gamma}{2s(1-\gamma)}\int_\Omega g(x)[(u_n+s\varphi)^2 - u_n^2]dx,
\end{aligned}$$

then letting $s \to 0^+$, we obtain

$$\lim_{s\to 0^+} K_{1,n}(s) = a(1+\gamma)\|u_n\|^2 + b(3+\gamma)\|u_n\|^{2m+2}$$

$$
\begin{aligned}
&-(1+\gamma)\int_\Omega g(x)u_n^2dx,\\
&=2b\|u_n\|^{2m+2}+(1+\gamma)\int_\Omega f(x)|u_n|^{1-\gamma}dx\\
&=K_{1,n}>0,\\
\lim_{s\to 0^+}K_{2,n}(s)=&\left(a\frac{1+\gamma}{1-\gamma}+b\frac{3+\gamma}{1-\gamma}\|u_n\|^{2m+2}\right)\int_\Omega(\nabla u_n,\nabla\varphi)dx\\
&-\frac{1+\gamma}{1-\gamma}\int_\Omega g(x)u_n\varphi dx\\
&=K_{2,n}.
\end{aligned}
$$

Consequently, it follows from (2.6) that

$$|A_n(s)|\frac{\|u_n\|}{n}+t_n(s)\frac{\|\varphi\|}{n}\geqslant\frac{1}{1-\gamma}K_{1,n}(s)A_n(s)+K_{2,n}(s).$$

If $A_n(s)\geqslant 0,$ we get

$$A_n(s)\geqslant\frac{K_{2,n}(s)-t_n(s)\frac{\|\varphi\|}{n}}{\frac{1}{\gamma-1}K_{1,n}(s)+\frac{\|u_n\|}{n}},$$

if $A_n(s)<0,$ we get

$$A_n(s)\geqslant\frac{K_{2,n}(s)-t_n(s)\frac{\|\varphi\|}{n}}{\frac{1}{\gamma-1}K_{1,n}(s)-\frac{\|u_n\|}{n}},$$

thus,

$$A_n(s)\geqslant\frac{K_{2,n}(s)-t_n(s)\frac{\|\varphi\|}{n}}{\frac{1}{\gamma-1}K_{1,n}(s)-\frac{\|u_n\|}{n}}.$$

From the boundness of $\{u_n\}$, there exists a constant $C_2>0$ such that $|K_{1,n}|\leqslant C_2$, also exists a constant $C_3>0$ such that $|K_{2,n}|\leqslant C_3$, and $A_n(s)$ for all n large, we obtain

$$\liminf_{s\to 0^+}A_n(s)\geqslant\frac{K_{2,n}(s)}{\frac{1}{\gamma-1}K_{1,n}(s)-\frac{\|u_n\|}{n}}\geqslant\frac{K_{2,n}(s)}{\frac{1}{\gamma-1}C_2-\frac{\|u_n\|}{n}}\geqslant\frac{K_{2,n}(s)}{\frac{1}{\gamma-1}C_2}\geqslant\frac{-C_3}{\frac{1}{\gamma-1}C_2},\tag{2.7}$$

that is $\liminf_{s\to 0^+}A_n(s)\geqslant C_4$. Finally, according to the uniformly boundness of $A_n(s)$ follows from (2.5) and (2.7), when n sufficiently large, it follows that there exists a constant $C_5>0$ such that

$$\limsup_{s\to 0^+}|A_n(s)|\leqslant C_5.\tag{2.8}$$

Now, we prove that (2.3) and (2.4) are true in Case two. By the subadditivity of norm again and (ii), we have

$$\frac{1}{n}\left[\frac{|t_n(s)-1|}{s}\|u_n\|+t_n(s)\|\varphi\|\right]$$

$$
\begin{aligned}
&\geqslant \frac{1}{n}\|t_n(s)(u_n+s\varphi)-u_n\|\frac{1}{s}\\
&\geqslant \frac{I(u_n)-I[t_n(s)(u_n+s\varphi)]}{s}\\
&\geqslant \Big\{-\frac{a}{2}[t_n(s)+1]\|u_n+s\varphi\|^2-\frac{b}{2m+2}\frac{t_n^{2m+2}(s)-1}{t_n(s)-1}\|u_n+s\varphi\|^{2m+2}\\
&\quad+\frac{1}{1-\gamma}\frac{t_n^{1-\gamma}(s)-1}{t_n(s)-1}\int_\Omega f(x)(u_n+s\varphi)^{1-\gamma}dx \\
&\quad+\frac{1}{2}[t_n(s)+1]\int_\Omega g(x)(u_n+s\varphi)^2dx\Big\}A_n(s)-\frac{a}{2s}(\|u_n+s\varphi\|^2-\|u_n\|^2)\\
&\quad-\frac{b}{(2m+2)s}(\|u_n+s\varphi\|^{2m+2}-\|u_n\|^{2m+2})\\
&\quad+\frac{1}{s(1-\gamma)}\int_\Omega f(x)[(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}]dx+\frac{1}{2s}\int_\Omega g(x)[(u_n+s\varphi)^2-u_n^2]dx.
\end{aligned}
\tag{2.9}
$$

We denote

$$
\begin{aligned}
K_{3,n}(s)=&\frac{a}{2}[t_n(s)+1]\|u_n+s\varphi\|^2+\frac{b}{2m+2}\frac{t_n^{2m+2}(s)-1}{t_n(s)-1}\|u_n+s\varphi\|^{2m+2}\\
&-\frac{1}{1-\gamma}\frac{t_n^{1-\gamma}(s)-1}{t_n(s)-1}\int_\Omega f(x)(u_n+s\varphi)^{1-\gamma}dx\\
&-\frac{1}{2}[t_n(s)+1]\int_\Omega g(x)(u_n+s\varphi)^2dx,\\
K_{4,n}(s)=&-\frac{a}{2s}(\|u_n+s\varphi\|^2-\|u_n\|^2)-\frac{b}{(2m+2)s}(\|u_n+s\varphi\|^{2m+2}-\|u_n\|^{2m+2})\\
&+\frac{1}{2s}\int_\Omega g(x)[(u_n+s\varphi)^2-u_n^2]dx.
\end{aligned}
$$

Consequently, letting $s\to 0^+$, one obtains

$$
\begin{aligned}
&\lim_{s\to 0^+}K_{3,n}(s)=a\|u_n\|^2+b\|u_n\|^{2m+2}-\int_\Omega f(x)|u_n|^{1-\gamma}dx-\int_\Omega g(x)|u_n|^2dx=0,\\
&\lim_{s\to 0^+}K_{4,n}(s)=-\left(a+b\|u_n\|^{2m}\right)\int_\Omega(\nabla u_n,\nabla\varphi)dx+\int_\Omega g(x)u_n\varphi dx.
\end{aligned}
$$

Then, it follows from (2.9) that

$$
\frac{1}{1-\gamma}\int_\Omega f(x)\frac{(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}}{s}dx\leqslant K_{3,n}(s)A_n(s)-K_{4,n}(s)+\frac{A_n(s)\|u_n\|+t_n(s)\|\varphi\|}{n}.
\tag{2.10}
$$

Since $(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}\geqslant 0,$ for every $x\in\Omega,$ thanks to Fatou's Lemma, one has

$$
\int_\Omega f(x)u_n^{-\gamma}\varphi dx\leqslant \liminf_{s\to 0^+}\frac{1}{1-\gamma}\int_\Omega f(x)\frac{(u_n+s\varphi)^{1-\gamma}-u_n^{1-\gamma}}{s}dx.
\tag{2.11}
$$

For n sufficiently large, by (2.10) and (2.11), one gets

$$\int_\Omega f(x)u_n^{-\gamma}\varphi dx \leqslant \left(a+b\|u_n\|^{2m}\right)\int_\Omega (\nabla u_n,\nabla\varphi)dx - \int_\Omega g(x)u_n\varphi dx + \frac{A_n(s)\|u_n\|+t_n(s)\|\varphi\|}{n},$$

which implies that

$$\liminf_{n\to\infty}\int_\Omega f(x)u_n^{-\gamma}\varphi dx \leqslant \left(a+b\liminf_{n\to\infty}\|u_n\|^{2m}\right)\int_\Omega (\nabla u_*,\nabla\varphi)dx - \int_\Omega g(x)u_*\varphi dx.$$

By using Fatou's Lemma again, one obtains

$$\int_\Omega f(x)u_*^{-\gamma}\varphi dx \leqslant \left(a+b\liminf_{n\to\infty}\|u_n\|^{2m}\right)\int_\Omega (\nabla u_*,\nabla\varphi)dx - \int_\Omega g(x)u_*\varphi dx. \quad (2.12)$$

On the one hand, choosing $\varphi = u_*$ in (2.12), one has

$$\left(a+b\liminf_{n\to\infty}\|u_n\|^{2m}\right)\|u_*\|^2 \geqslant \int_\Omega g(x)u_*^2 dx + \int_\Omega f(x)u_*^{1-\gamma}dx. \quad (2.13)$$

On the other hand, since $u_n \in N_2$ and $u_n \to u_*$ in $L^2(\Omega)$, we obtain

$$\lim_{n\to\infty}(a\|u_n\|^2 + b\|u_n\|^{2m+2}) = \int_\Omega g(x)u_*^2 dx + \int_\Omega f(x)u_*^{1-\gamma}dx.$$

Further, by the weak lower semi-continuity of the norm, we get

$$\begin{aligned}\left(a+b\liminf_{n\to\infty}\|u_n\|^{2m}\right)\|u_*\|^2 &\leqslant \left(a+b\limsup_{n\to\infty}\|u_n\|^{2m}\right)\limsup_{n\to\infty}\|u_n\|^2\\ &= \int_\Omega g(x)u_*^2 dx + \int_\Omega f(x)u_*^{1-\gamma}dx. \quad (2.14)\end{aligned}$$

By (2.13) and (2.14), one has

$$\liminf_{n\to\infty}\|u_n\| = \limsup_{n\to\infty}\|u_n\| = \|u_*\|. \quad (2.15)$$

Combining with $u_n \rightharpoonup u_*$ and (2.15), one gets $u_n \to u_*$ in $H_0^1(\Omega)$. Thus, we can easy obtain that (2.3) holds. Moreover, by (2.12), one has (2.4) is true.

Finally, we are ready to prove that u_* is a weak solution of problem (1.1), that is, we only need prove that (2.4) holds for any $\varphi \in H_0^1(\Omega)$. For any $\varphi \in H_0^1(\Omega)$, since $u_* \in N_2$, we can replace φ with $(u_*+t\varphi)^+$ in (2.4) for $t>0$ small enough, and dividing by t in (2.4), one has

$$0 \leqslant \frac{1}{t}\left(a+b\|u_*\|^{2m}\right)\int_\Omega (\nabla u_*,\nabla(u_*+t\varphi)^+)dx - \frac{1}{t}\int_\Omega f(x)u_*^{-\gamma}(u_*+t\varphi)^+dx - \frac{1}{t}\int_\Omega g(x)u_*(u_*+t\varphi)^+dx$$

$$
\begin{aligned}
&= \frac{1}{t}\left(a+b\|u_*\|^{2m}\right)\int_\Omega (\nabla u_*, \nabla(u_*+t\varphi))dx - \frac{1}{t}\int_\Omega f(x)u_*^{-\gamma}(u_*+t\varphi)dx \\
&\quad - \frac{1}{t}\int_\Omega g(x)u_*(u_*+t\varphi)dx - \frac{1}{t}\left(a+b\|u_*\|^{2m}\right)\int_{\{x\in\Omega|u_*+t\varphi<0\}} (\nabla u_*, \nabla(u_*+t\varphi))dx \\
&\quad + \frac{1}{t}\int_{\{x\in\Omega|u_*+t\varphi<0\}} [f(x)u_*^{-\gamma}(u_*+t\varphi) + g(x)u_*(u_*+t\varphi)]dx \\
&\leqslant \frac{1}{t}\left[a\|u_*\|^2 + b\|u_*\|^{2m+2} - \int_\Omega f(x)u_*^{1-\gamma}dx - g(x)u_*^2 dx\right] \\
&\quad + \left(a+b\|u_*\|^{2m}\right)\int_\Omega (\nabla u_*, \nabla\varphi)dx - \int_\Omega [f(x)u_*^{-\gamma}\varphi + g(x)u_*\varphi]dx \\
&\quad - \left(a+b\|u_*\|^{2m}\right)\int_{\{x\in\Omega|u_*+t\varphi<0\}} (\nabla u_*, \nabla\varphi)dx \\
&= \left(a+b\|u_*\|^{2m}\right)\int_\Omega (\nabla u_*, \nabla\varphi)dx - \int_\Omega [f(x)u_*^{-\gamma}\varphi + g(x)u_*\varphi]dx \\
&\quad - \left(a+b\|u_*\|^{2m}\right)\int_{\{x\in\Omega|u_*+t\varphi<0\}} (\nabla u_*, \nabla\varphi)dx,
\end{aligned}
$$

where the last inequality is used $u_* \in N_2$. Since meas $\{x \in \Omega | u_* + t\varphi < 0\} \to 0$ as $t \to 0^+$ and the arbitrariness of $\varphi \in H_0^1(\Omega)$, that is, for every $\varphi \in H_0^1(\Omega)$,

$$
\left(a+b\|u_*\|^{2m}\right)\int_\Omega (\nabla u_*, \nabla\varphi)dx - \int_\Omega f(x)u_*^{-\gamma}\varphi dx - \int_\Omega g(x)u_*\varphi dx = 0.
$$

Thus, u_* is indeed a positive solution of problem (1.1). This completes the proof of Theorem 1. □

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