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EXISTENCE RESULTS AND CONTINUITY DEPENDENCE OF SOLUTIONS FOR FRACTIONAL EQUATIONS

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Abstract. Using two fractional-order integral inequalities we investigate the existence and uniqueness of solutions of the fractional nonlinear Volterra integral equation and the fractional nonlinear integrodifferential equation in Banach space C_ξ, using an adequate norm, $||\cdot||_{\xi,\infty}$. Besides, we study the solution estimate and investigate their continuous dependence.

1. Introduction

Fractional differential equations are very interesting and important for engineering, physics, chemistry, biology, and medicine, among others, for their ability to model and describe phenomena of nature [23, 27, 36, 37, 38, 39]. In addition, they are also extremely important for mathematics, in particular, for the fractional analysis, since it allows to study the existence and uniqueness of a class of local and non-local solutions, impulsive problems in Banach space, non-local mild solutions, boundary value problems, and many others, especially where a differential integral and integrodifferential equations emerge [1, 5, 6, 8, 9, 13, 15, 18, 19, 20, 21, 22, 25, 28, 31, 34, 40].

On the other hand, iterative and numerical methods in which solutions of fractional integral equations can be approximated turn into important and interesting studies for this area [16, 17, 33]. Also, it has been investigated the existence of solutions of the integral fractional equation by Schauder fixed point and by singular nonlinear Volterra integral equation [3, 19]. It is also worth mentioning the importance of discussing the existence, as well as the attractivity of solutions of integral fractional equations in Fréchet spaces, besides the asymptotic behavior [2, 20]. The study of existence and uniqueness of solutions for problems involving fractional integral equations has become an advantageous field of fractional calculus, since it allowed to unify several areas such as in mathematics, in particular, the study of solutions of integral equations via fuzzy set theory [4, 14, 32]. In this sense, some researchers have decided to study solutions of fuzzy fractional integral equations, which have been important for the advancement in this area [4, 14, 32].

Although it has many results addressing the discussion of existence, uniqueness, and stability of fractional integrodifferential equations solutions, it is still an area that is

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growing. Some works related to existence and uniqueness of solutions of boundary and impulse problems in Banach space are highlighted here, as well as inverse problems in Sobolev space [5, 7, 10, 11, 12, 13, 22, 24, 26, 30, 44]. Besides, the study of existence and uniqueness of mild solutions in Sobolev space and impulse equations with boundary value can be seen in [29]. The variety of problems, which are investigated in the literature, is not enough to cover the vast number of papers related to this subject.

This paper considers nonlinear fractional Volterra integral and integrodifferential equations

$$x(t) = f\left(t, x(t), \int_a^t \mathscr{W}_\psi^\alpha (t, s, x(s))\, ds\right) \tag{1}$$

and

$$\begin{cases} {}^H\mathbb{D}_{a^+}^{\alpha,\beta,\psi} x(t) = f\left(t, x(t), \int_a^t \mathscr{W}_\psi^\alpha (t, s, x(s))\, ds\right) \\ I_{a^+}^{1-\gamma,\psi} x(a) \ = x_0 \end{cases} \tag{2}$$

receptively, where ${}^H\mathbb{D}_{a^+}^{\alpha,\beta,\psi}(\cdot)$ is the ψ-Hilfer fractional derivative, $I_{a^+}^{1-\gamma,\psi} x(\cdot)$ is the Riemann-Liouville fractional integral with respect to another function, with $0 < \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta(1-\alpha)$, for $-\infty < a \leqslant t < +\infty$, being x, f, k real vectors with n components such that $x \in C_\xi(I, \mathbb{R}^n)$, $k \in C_\xi(I^2 \times \mathbb{R}^n, \mathbb{R}^n)$ for $a \leqslant s \leqslant t < +\infty$, $f \in C_\xi(I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ $(I = [a, \infty))$, and to simplify the notation

$$\mathscr{W}_\psi^\alpha (t, s, x(s)) := \frac{N_\psi^\alpha (t,s)\, k(t, s, x(s))}{\Gamma(\alpha)}$$

with $N_\psi^\alpha (t,s) = \psi'(s)\, (\psi(t) - \psi(s))^{\alpha-1}$ and $\psi'(s) = \dfrac{d}{ds}\psi(s)$ denoting ordinary derivative.

The main purpose of this paper is to discuss the existence, uniqueness, solution estimate and continuous dependence of solutions of the nonlinear fractional integral equation (1) and the nonlinear fractional integrodifferential equation (2) in the sense of the ψ-Hilfer fractional derivative in Banach space employing of two suitable lemmas.

The paper is organized as follows: section 2 presents definitions of Riemann-Liouville fractional integral with respect to another function and ψ-Hilfer fractional derivative, as well as results relating both (fractional integral and derivative) and the calculation of the fractional integral of the Mittag-Leffler function. A norm and the fundamental metric for the elaboration of this paper are introduced. Two results involving the metric (complete space) and the norm (Banach space), as well as discussing particular cases are presented. Besides, two lemmas and a corollary involving inequalities that are important for the development of this paper are given. Section 3 investigates the existence and uniqueness of solutions of the nonlinear fractional Volterra integral equation and nonlinear fractional integrodifferential equation, as well as the study of the solution estimate. Section 4 intends to study the continuous dependence of solutions of the nonlinear fractional Volterra integral equations and the nonlinear fractional integrodifferential equation. Concluding remarks close the paper.

2. Preliminaries

In this section, we introduce some preliminary results that be useful in the next sections. Specifically, we recover some results involving the Riemann-Liouville fractional integral with respect to another function; ψ-Hilfer fractional derivative and the classical one-parameter Mittag-Leffler function.

Let $\mathbb{R}^+ = [0,+\infty)$ and $\mathbb{R}^n$ be the n-dimensional Euclidean space endowed with a norm $\|\cdot\|$. For $a,b \in \mathbb{R}$ with $a<b$, the space $C(J,\mathbb{R}^n)$ $(J=[a,b])$ is equipped with the norm

$$\|x\|_\infty := \sup_{t\in[a,b]} \|x(t)\|, \quad \forall x \in C([a,b],\mathbb{R}^n).$$

Let $\alpha > 0$ and $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t)$ for all $t \in J$ and $x \in L^p(J,\mathbb{R})$, $1<p<\infty$. The Riemann-Liouville fractional integral with respect to another function ψ of function x on J, is defined by [41, 42, 43]

$$I_{a^+}^{\alpha,\psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1} x(s)\,ds \tag{3}$$

where $\Gamma(\cdot)$ is a gamma function.

On the other hand, let $n-1<\alpha<n$ with $n\in\mathbb{N}$, J be an interval such that $-\infty \leqslant a < b \leqslant +\infty$ and let $x,\psi \in C^n(J,\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The ψ-Hilfer fractional derivative denoted by ${}^H\mathbb{D}_{a^+}^{\alpha,\beta,\psi}(\cdot)$ of a function x of order α and type β $(0 \leqslant \beta \leqslant 1)$, is defined by [41, 42, 43]

$${}^H\mathbb{D}_{a^+}^{\alpha,\beta,\psi} x(t) = I_{a^+}^{\beta(n-\alpha),\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha),\psi} x(t). \tag{4}$$

The ψ-Hilfer fractional derivative of an n-dimensional vector function denoted by $x(t) = (x_1(t),\cdots,x_n(t))^T$, with the superscript T indicates transposition, is defined component wise as

$${}^H\mathbb{D}_{a^+}^{\alpha,\beta;\psi} x(t) := \left({}^H\mathbb{D}_{a^+}^{\alpha,\beta;\psi} x_1(t),\cdots,{}^H\mathbb{D}_{a^+}^{\alpha,\beta;\psi} x_n(t)\right)^T.$$

THEOREM 1. *Let* $x \in C^1(J,\mathbb{R})$; $0<\alpha\leqslant 1$ *and* $0\leqslant\beta\leqslant 1$, *we have*

$${}^H\mathbb{D}_{a^+}^{\alpha,\beta;\psi} I_{a^+}^{\alpha,\psi} x(t) = x(t).$$

Proof. See [41].

THEOREM 2. *If* $x \in C^n(J,\mathbb{R})$, $0<\alpha\leqslant 1$ *and* $0\leqslant\beta\leqslant 1$, *then*

$$I_{a^+}^{\alpha,\psi}\, {}^H\mathbb{D}_{a^+}^{\alpha,\beta;\psi} x(t) = x(t) - \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a+}^{(1-\beta)(1-\alpha);\psi} x(a).$$

Proof. See [41].

Let $\xi > 0$ be a constant and consider the special space $C_\xi(I,\mathbb{R}^n)$ the set of all continuous functions $x \in C(I,\mathbb{R}^n)$ such that

$$\sup_{t\in I} \frac{\|x(t)\|}{\mathbb{E}_\alpha\left[\xi(\psi(t)-\psi(a))^\alpha\right]} < \infty$$

where $\mathbb{E}_\alpha : \mathbb{R} \to \mathbb{R}$ is the one-parameter Mittag-Leffler function given by

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \qquad \forall z \in \mathbb{R},$$

and $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$.

We consider the linear space $C_\xi(I,\mathbb{R}^n)$ with a suitable metric namely

$$d_{\xi,\infty}(x,y) := \sup_{t\in I} \frac{\|x(t)-y(t)\|}{\mathbb{E}_\alpha\left[\xi(\psi(t)-\psi(a))^\alpha\right]} < \infty \tag{5}$$

with a norm defined by

$$\|x\|_{\xi,\infty} = \sup_{t\in I} \frac{\|x(t)\|}{\mathbb{E}_\alpha\left[\xi(\psi(t)-\psi(a))^\alpha\right]}. \tag{6}$$

Note that the metric and norm as seen in (5) and (6) are in fact extensions of a class of metrics and norms, i.e., taking $\psi(t) = t$ in (5) and (6), we have [18]:

$$d_{\xi,\infty}(x,y) := \sup_{t\in I} \frac{\|x(t)-y(t)\|}{\mathbb{E}_\alpha[\xi(t-a)^\alpha]} < \infty \tag{7}$$

with a norm defined by

$$\|x\|_{\xi,\infty} = \sup_{t\in I} \frac{\|x(t)\|}{\mathbb{E}_\alpha[\xi(t-a)^\alpha]}. \tag{8}$$

On the other hand, taking $\psi(t) = t$ and taking the limit $\alpha \to 1$ in (5) and (6), we have [35]

$$d_{\xi,\infty}(x,y) := \sup_{t\in I} \frac{\|x(t)-y(t)\|}{\exp[\xi(t-a)]} < \infty \tag{9}$$

whose norm is

$$\|x\|_{\xi,\infty} = \sup_{t\in I} \frac{\|x(t)\|}{\exp[\xi(t-a)]}. \tag{10}$$

The equations (5) and (6) as above are variants of metric and norm. When we get the particular case, we note that the respective metrics and norms are variations of norms and metrics of Bielecki [35]. It may be noted that it is possible to get other variances, since the freedom of the ψ function and the limits of α and β, allows us a great advantage to consider the best metric and norm, in which we want to work.

LEMMA 1. *Given* $\xi > 0$, $n-1 < \alpha < n$ *with* $n \in \mathbb{N}$. *Consider the real function* $f(t) = \mathbb{E}_\alpha\left[\xi(\psi(t)-\psi(a))^\alpha\right]$ *where* $\mathbb{E}_\alpha(\cdot)$ *is an one-parameter Mittag-Leffler function with* $Re(\alpha) > 0$. *Then*

$$I_{a^+}^{\alpha,\psi} f(t) = \frac{1}{\xi}\left(\mathbb{E}_\alpha\left[\xi(\psi(t)-\psi(a))^\alpha\right]-1\right). \tag{11}$$

Proof. In fact, employing of the definition of the one-parameter Mittag-Leffler function, we have

$$\begin{aligned}
I_{a^+}^{\alpha,\psi}\left(\mathbb{E}_\alpha\left[\xi(\psi(t)-\psi(a))^\alpha\right]\right) &= \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(k\alpha+1)} I_{a+}^{\alpha;\psi}(\psi(t)-\psi(a))^{k\alpha} \\
&= \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(k\alpha+1+\alpha)} (\psi(t)-\psi(a))^{k\alpha+\alpha} \\
&= \frac{1}{\xi}\sum_{k=1}^{\infty} \frac{\xi^k}{\Gamma(k\alpha+1)} (\psi(t)-\psi(a))^{k\alpha} \\
&= \frac{1}{\xi}\left(\mathbb{E}_\alpha\left[\xi(\psi(t)-\psi(a))^\alpha\right]-1\right).
\end{aligned}$$

LEMMA 2. *If* $\xi > 0$ *is a constant, then*

1. $d_{\xi,\infty}$ *is a metric;*
2. $\left(C_\xi(I,\mathbb{R}^n), d_{\xi,\infty}\right)$ *is a complete metric space.*

Proof.

1. Assume that $\varepsilon > 0$, $\xi > 0$ then, we have $\mathbb{E}_\xi\left[\xi(\psi(t)-\psi(a))^\alpha\right] > 0$, $\forall t \in I$. As $\|x(t)-y(t)\|$ is a norm, easily check the properties of metrics.

2. Let $\{x_i(t)\}$ a Cauchy sequence, i.e., for each $\varepsilon > 0$, there exists a positive integer N_ε such that

$$\frac{\left\|x_i(t)-x_j(t)\right\|}{\mathbb{E}_\xi\left[\xi(\psi(t)-\psi(a))^\alpha\right]} < \varepsilon,\ \forall i,j > N_\varepsilon, \forall t \in I.$$

It follows that the sequence $\{x_i(t)\}$ is uniformly convergent. The limit of a uniformly convergent sequence of continuous functions is a continuous function. Taking $j \to \infty$ we have

$$\frac{\|x_i(t)-x(t)\|}{\mathbb{E}_\xi\left[\xi(\psi(t)-\psi(a))^\alpha\right]} < \varepsilon,\ \forall i > N_\varepsilon, \forall t \in I$$

and thus the Cauchy sequence $\{x_i(t)\}$ converges in the metric $d_{\xi,\infty}$ of $C_\xi(I,\mathbb{R}^n)$. Thus, $C_\xi\left((I,\mathbb{R}^n), d_{\xi,\infty}\right)$ is a complete metric space.

LEMMA 3. [18] *If* $\xi > 0$ *is a constant, then*

1. $\|\cdot\|_{\xi,\infty}$ *is a norm;*

2. $\left(C_{\xi}(I,\mathbb{R}^n), \|\cdot\|_{\xi,\infty}\right)$ *is a Banach space.*

Proof.

1. From item 1 of Lemma 2 it follows that $\|\cdot\|_{\xi,\infty}$ is a norm. We prove that there exists positive constants m and M such that

$$m\|x\|_0 \leqslant \|x\|_{\xi,\infty} \leqslant M\|x\|_0. \tag{12}$$

Since $\xi > 0$, we have $\mathbb{E}_{\xi}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right] > 0$, for $t \in I$. Hence

$$\xi\mathbb{E}_{\xi}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right] > 0,\ t \in I,$$

so that $\dfrac{1}{\mathbb{E}_{\xi}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right]}$ is strictly decreasing for $t \in I$. We also have $\mathbb{E}_{\xi}[\xi(\psi(a)-\psi(a))^{\alpha}] = 1$. Combining the above ideas, we have

$$\frac{\|x\|_0}{\mathbb{E}_{\xi}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right]} \leqslant \|x\|_{\xi,\infty} \leqslant 1\,\|x\|_0.$$

So that (12), holds with $m = \dfrac{1}{\mathbb{E}_{\xi}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right]}$ and $M = 1$. Then the norm $\|\cdot\|_{\xi,\infty}$ and $\|\cdot\|_0$ are equivalent.

2. From item 2 of Lemma 2 and item 1 above, Lemma follows.

We introduce the notation to facilitate the development of this paper:

1. $\mathscr{W}_{\psi}^{\alpha}(t,s,x(s)) := \dfrac{N_{\psi}^{\alpha}(t,s)\,k(t,s,x(s))}{\Gamma)(\alpha)}$ with $N_{\psi}^{\alpha}(t,s) = \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}$ and $\psi'(s) = \dfrac{d}{ds}\psi(s)$;

2. $\mathscr{W}_{\psi}^{\alpha}(t,s,0) := \dfrac{N_{\psi}^{\alpha}(t,s)\,k(t,s,0)}{\Gamma(\alpha)}$;

3. $\overline{\mathscr{W}}_{\psi}^{\alpha}(t,s,x(s)) := \dfrac{N_{\psi}^{\alpha}(t,s)\,\overline{k}(t,s,x(s))}{\Gamma(\alpha)}$;

4. $\Psi^{\gamma}(t,a) := \dfrac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}$.

The proof of Lemma 4 below will be omitted here, however, it follows the same steps as in Gronwall inequality Theorem 3 and Corollary 3.10 [40].

LEMMA 4. *Let* $u(t), v(t), g(t) \in C(I, \mathbb{R}_+)$, $r(t,\sigma) \in C(D, \mathbb{R}_+)$, *where* $D = \{(t,\tau) \in I^2; a \leqslant \tau \leqslant +\infty\}$ *and* $c \geqslant 0$ *is a constant and* u, v *are nonnegative functions and* g *nonnegative and nondecreasing function. If*

$$u(t) \leqslant v(t) + g(t)\int_a^t N_\psi^\alpha(t,\tau)\, r(t,\tau)\, u(\tau)\, d\tau \tag{13}$$

then

$$u(t) \leqslant v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{((g(t)\Gamma(\alpha))^k}{\Gamma(\alpha k)} N_\psi^{\alpha k}(t,\tau)\, r(t,\tau) v(\tau), d\tau.$$

COROLLARY 1. *Under the hypothesis of* Lemma 4, *let* r, v *be two nondecreasing functions on* I. *Then, we have*

$$u(t) \leqslant v(t)\mathbb{E}_\alpha[g(t) r(t,t)\Gamma(\alpha)(\psi(t) - \psi(a))^\alpha] \tag{14}$$

where $\mathbb{E}_\alpha(\cdot)$ *is an one-parameter Mittag-Leffler function.*

Proof. In fact, as v is nondecreasing function, so, for all $\tau \in [a,t]$, we have $v(\tau) \leqslant v(t)$ and we can write

$$\begin{aligned} u(t) &\leqslant v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau)[\psi(t) - \psi(\tau)]^{\alpha k - 1} r(t,\tau) v(\tau)\, d\tau \\ &= v(t)\left[1 + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau)[\psi(t) - \psi(\tau)]^{\alpha k - 1} r(t,\tau) v(\tau)\, d\tau\right] \\ &\leqslant v(t)\left[1 + \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \frac{[\psi(t) - \psi(a)]^{\alpha k}}{k\alpha} r(t,t)\right] \\ &= v(t)\,\mathbb{E}_\alpha\left(g(t) r(t,t)\Gamma(\alpha)[\psi(t) - \psi(a)]^\alpha\right). \end{aligned}$$

It is important to note that, Lemma 4, is a generalization of the Gronwall inequality, when we take $r(t,\tau) = 1$ [40].

LEMMA 5. *Let* $u(t), p(t), \widetilde{g}(t) \in C(I, \mathbb{R}_+)$, $r(t,\sigma) \in C(D, \mathbb{R}_+)$, *where* D *is as in* Lemma 4 *and* $\widetilde{g}(t) \geqslant 0$ *and* $u(t), p(t)$ *are nonnegative functions and* $\widetilde{g}(t)$ *nonnegative and nondecreasing function. If*

$$u(t) \leqslant \widetilde{g}(t) + \int_a^t N_\psi^\alpha(t,\tau)\, p(\tau)\left[u(\tau) + \int_a^\tau N_\psi^\alpha(t,\tau)\, r(\tau,\sigma)\, u(\sigma)\, d\sigma\right] d\tau$$

for $t \in I$, *then*

$$u(t) \leqslant \widetilde{g}(t)\mathbb{E}_\alpha[p(t)\Gamma(\alpha)\mathbb{E}_\alpha(r(t,t)\Gamma(\alpha)(\psi(t) - \psi(a))^\alpha)(\psi(t) - \psi(a))^\alpha],$$

where $\mathbb{E}_\alpha(\cdot)$ *is an one-parameter Mittag-Leffler function.*

Proof. Taking $v(t) = p(t)u(t)$ (with $p(t)$ and $u(t)$ are nonnegative functions) and $g(t) = p(t)$ nondecreasing function, substituting in (13), we have

$$u(t) \leqslant p(t)u(t) + p(t)\int_a^t N_\psi^\alpha(t,\tau)\, r(t,\tau)\, u(\tau)\, d\tau. \tag{15}$$

Applying the integral $\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} d\tau$ and summing $\widetilde{g}(t)$ on both sides of (15), we get

$$\begin{aligned} &\widetilde{g}(t) + \int_a^t N_\psi^\alpha(t,\tau)\, u(\tau)\, d\tau \\ &\leqslant \widetilde{g}(t) + \int_a^t N_\psi^\alpha(t,\tau)\left[p(\tau)u(\tau) + p(\tau)\int_a^\tau N_\psi^\alpha(s,\sigma)\, r(\tau,\sigma)\, u(\sigma)\, d\sigma\right] d\tau. \end{aligned}$$

Therefore, we conclude that

$$u(t) \leqslant \widetilde{g}(t) + \int_a^t N_\psi^\alpha(t,\tau)\left[p(\tau)u(\tau) + p(\tau)\int_a^\tau N_\psi^\alpha(s,\sigma)\, r(\tau,\sigma)\, u(\sigma)\, d\sigma\right] d\tau. \tag{16}$$

Note that, the (16) is exactly the hypotheses of this Lemma.

On the other hand, we perform the same procedure as in Lemma 4, i.e., (13). Then, taking $v(t) = p(t)u(t)$ (with $p(t)$ and $u(t)$ nonnegative functions) and $g(t) = p(t)$ nondecreasing function in (13), we have

$$u(t) \leqslant p(t)u(t)\,\mathbb{E}_\alpha\left[p(t)\, r(t,t)\,\Gamma(\alpha)(\psi(t)-\psi(a))^\alpha\right]. \tag{17}$$

Applying the integral $\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} d\tau$ and summing $\widetilde{g}(t)$ on both sides of (17), we get

$$u(t) \leqslant \widetilde{g}(t)\left(1 + \int_a^t N_\psi^\alpha(t,\tau)\, p(\tau)\mathbb{E}_\alpha\left[p(t)\, r(\tau,\tau)\,\Gamma(\alpha)(\psi(\tau)-\psi(a))^\alpha\right] u(\tau)\, d\tau\right).$$

Using Lemma 4, we have

$$u(t) \leqslant \widetilde{g}(t)\mathbb{E}_\alpha[p(t)\Gamma(\alpha)\mathbb{E}_\alpha\left(r(t,t)\Gamma(\alpha)(\psi(t)-\psi(a))^\alpha\right)(\psi(t)-\psi(a))^\alpha].$$

Thus, we conclude the proof.

3. Existence, uniqueness and estimates of solutions

In this section, we are going to present our main results concerning the existence and uniqueness of solutions of (1) and (2).

THEOREM 3. *Let $L > 0,\ \xi > 0,\ M \geqslant 0,\ \delta > 1$ be constants with $\xi = L_\delta$ (constant), $t \in I$ and $u, \overline{u}, v, \overline{v} \in C_\xi(I, \mathbb{R}^n)$. Suppose the functions f, k in (1) satisfying the conditions*

$$\|f(t,u,v) - f(t,\overline{u},\overline{v})\| \leqslant M\left(\|u-\overline{u}\| + \|v-\overline{v}\|\right) \tag{18}$$

and

$$\|k(t,s,u)-k(t,s,\overline{u})\| \leqslant L\|u-\overline{u}\| \tag{19}$$

and

$$d_1 = \sup_{t\in I}\frac{1}{\mathbb{E}_\alpha[\xi\,(\psi(t)-\psi(a))^\alpha]}\left\|f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right)\right\| < \infty \tag{20}$$

with $\mathscr{W}_\psi^\alpha\,(t,s,0) = \dfrac{N_\psi^\alpha\,(t,s)\,K\,(t,s,0)}{\Gamma(\alpha)}$.

If $M(1+1/\delta) < 1$, *then the integral* (1) *has a unique solution* $x \in C_\xi\,(I,\mathbb{R}^n)$.

Proof. First, we can rewrite the nonlinear fractional Volterra integral equation (see (1)) of the following form

$$\begin{aligned} x(t) = {} & f\left(t,x(t),\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,x(s))\,ds\right) - f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right) \\ & + f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right) \end{aligned} \tag{21}$$

for $t \in I$.

For $x \in C_\xi\,(I,\mathbb{R}^n)$, we define the operator T by

$$\begin{aligned} (Tx)(t) = {} & f\left(t,x(t),\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,x(s))\,ds\right) - f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right) \\ & + f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right). \end{aligned} \tag{22}$$

Using (22) and hypotheses, we get

$$\|Tx\|_{\xi,\infty} = \sup_{t\in I}\frac{\|(Tx)(t)\|}{\mathbb{E}_\alpha\left[\xi\,(\psi(t)-\psi(a))^\alpha\right]}. \tag{23}$$

Hence (23) can be written as

$$\begin{aligned} \|Tx\|_{\xi,\infty} = {} & \sup_{t\in I}\frac{1}{\mathbb{E}_\alpha\left[\xi\,(\psi(t)-\psi(a))^\alpha\right]}\left\| f\left(t,x(t),\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,x(s))\,ds\right)\right. \\ & \left. - f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right) + f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right)\right\| \\ \leqslant {} & \sup_{t\in I}\frac{1}{\mathbb{E}_\alpha\left[\xi\,(\psi(t)-\psi(a))^\alpha\right]}\left\| f\left(t,x(t),\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,x(s))\,ds\right)\right. \\ & \left. - f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right)\right\| \\ & + \sup_{t\in I}\frac{1}{\mathbb{E}_\alpha\left[\xi\,(\psi(t)-\psi(a))^\alpha\right]}\left\| f\left(t,0,\frac{1}{\Gamma(\alpha}\int_a^t \mathscr{W}_\psi^\alpha\,(t,s,0)\,ds\right)\right\| \end{aligned}$$

$$
\begin{aligned}
&= d_1 + \sup_{t\in I} \frac{1}{\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right]} \left\| f\left(t, x(t), \int_a^t \mathscr{W}_\psi^\alpha\left(t,s,x(s)\right) ds\right)\right. \\
&\quad \left. - f\left(t, 0, \int_a^t \mathscr{W}_\psi^\alpha\left(t,s,0\right) ds\right)\right\| \\
&\leqslant d_1 + \sup_{t\in I} \frac{M}{\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right]} \left\{\|x(t)\| + \left\| \int_a^t \mathscr{W}_\psi^\alpha\left(t,s,x(s)\right) ds\right.\right. \\
&\quad \left.\left. - \int_a^t \mathscr{W}_\psi^\alpha\left(t,s,0\right) ds\right\|\right\} \\
&\leqslant d_1 + \sup_{t\in I} \frac{M}{\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right]} \left\{\|x(t)\| + \frac{1}{\Gamma(\alpha)} \int_a^t N_\psi^\alpha(t,s) L \|x(s)\|\, ds\right\}.
\end{aligned}
\tag{24}
$$

Manipulating (24) we can write

$$
\begin{aligned}
(Tx)(t) &\leqslant d_1 + M \sup_{t\in I} \frac{\|x(t)\|}{\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right]} + ML \sup_{t\in I} \frac{\|x(t)\|}{\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right]} \\
&\quad \times \sup_{t\in I} \frac{1}{\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right]} \\
&\quad \times \left\{\frac{1}{\Gamma(\alpha)} \int_a^t N_\psi^\alpha(t,s) \mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right] ds\right\} \\
&= d_1 + M + ML\|x\|_{\xi,\infty} \sup_{t\in I} \frac{1}{\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right]} \\
&\quad \times \left\{\frac{1}{\xi}\left(\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right] - 1\right)\right\} \\
&= d_1 + M\|x\|_{\xi,\infty} \left[1 + \frac{L}{\xi} \sup_{t\in I} \left(1 - \frac{1}{\mathbb{E}_\alpha\left[\xi\left(\psi(t)-\psi(a)\right)^\alpha\right]}\right)\right].
\end{aligned}
\tag{25}
$$

Since $\mathbb{E}_\alpha(\cdot)$ is a monotone increasing function on the real line, we have

$$
\|Tx\|_{\xi,\infty} \leqslant d_1 + M\|x\|_{\xi,\infty}\left(1 + \frac{L}{\xi}\right) = d_1 + \|x\|_{\xi,\infty} M\left(1 + \frac{1}{\delta}\right) < \infty. \tag{26}
$$

Therefore, the operator T maps $C_\xi(I, \mathbb{R}^n)$ into itself, i.e.,

$$
T\left(\left(C_\xi\left(I, \mathbb{R}^n\right), \|(\cdot)\|_{\xi,\infty}\right)\right) \subset \left(C_\xi\left(I, \mathbb{R}^n\right), \|(\cdot)\|_{\xi,\infty}\right). \tag{27}
$$

Now, we need to prove the operator T is a contraction. Let $u,v \in C_{\xi}(I,\mathbb{R}^n)$, then, by (22) and hypotheses, we get

$$
\begin{aligned}
& d_{\xi,\infty}(Tu,Tv) \\
&= \sup_{t\in I} \frac{\|(Tu)(t)-(Tv)(t)\|}{\mathbb{E}_{\alpha}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right]} \\
&= \sup_{t\in I} \frac{\left\| f\left(t,u(t),\int_a^t \mathscr{W}_{\psi}^{\alpha}(t,s,u(s))\,ds\right) - f\left(t,v(t),\int_a^t \mathscr{W}_{\psi}^{\alpha}(t,s,v(s))\,ds\right)\right\|}{\mathbb{E}_{\alpha}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right]}.
\end{aligned} \tag{28}
$$

As above, manipulating (28), we can write

$$
d_{\xi,\infty}(Tu,Tv) = M d_{\xi,\infty}\left[1+\frac{L}{\xi}\sup_{t\in I}\left(1-\frac{1}{\mathbb{E}_{\alpha}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right]}\right)\right]. \tag{29}
$$

Since $\mathbb{E}_{\alpha}(\cdot)$ is a monotone increasing function on the real line, we have

$$
d_{\xi,\infty}(Tu,Tv) \leqslant M d_{\xi,\infty}(u,v)\left(1+\frac{L}{\xi}\right) = M\left(1+\frac{1}{\delta}\right)d_{\xi,\infty}(u,v). \tag{30}
$$

By hypotheses, $M(1+1/\delta)<1$, then by Banach fixed point theorem, the operator, T, has a unique fixed point in $C_{\xi}(I,\mathbb{R}^n)$. Thus, we conclude that, the fixed point of T is a solution of (1).

THEOREM 4. *Let L, ξ, M, δ be as in* Theorem 3. *Suppose the functions f and k in* (2) *satisfying the conditions given in* (18) *and* (19) *and the relation*

$$
d_2 = \sup_{t\in I}\frac{1}{\mathbb{E}_{\alpha}\left[\xi(\psi(t)-\psi(a))^{\alpha}\right]}\left\|\Psi^{\gamma}(t,a)x_0 - I_{a^+}^{\alpha,\psi} f\left(s,0,\frac{1}{\Gamma(\alpha)}\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,0)\,d\sigma\right)\right\|.
$$

If $\dfrac{M}{\xi}\left(1+\dfrac{1}{\delta}\right)<1$, *then the nonlinear fractional integrodifferential equation* (2) *has a unique solution* $x \in C_{\xi}(I,\mathbb{R}^n)$.

Proof. We only present the idea of the proof, following the same steps as in the above theorem. First, we prove that (2) is equivalent to the following nonlinear integral equation

$$
x(t) = \Psi^{\gamma}(t,a)x_0 + I_{a^+}^{\alpha,\psi} f\left(s,x(s),\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,x(\sigma))\,d\sigma\right). \tag{31}
$$

In fact, applying the fractional derivative ${}^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi}(\cdot)$ on both sides of (31) and using Theorem 1, we get

$$\begin{aligned} &{}^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi}x(t) \\ &= {}^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi}\left[\Psi^{\gamma}(t,a)x_0\right] + {}^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi}\left[I_{a^+}^{\alpha,\psi} f\left(s,x(s),\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,x(\sigma))\,d\sigma\right)\right] \\ &= f\left(t,x(t),\int_a^t \mathscr{W}_{\psi}^{\alpha}(t,s,x(s))\,ds\right) \end{aligned} \tag{32}$$

where

$${}^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi}\left[\Psi^{\gamma}(t,a)x_0\right] = 0.$$

Applying the fractional integral $I_{a^+}^{\alpha,\psi}(\cdot)$ on both sides of (32) and using Theorem 2, we get

$$x(t) = \Psi^{\gamma}(t,a)x_0 + I_{a^+}^{\alpha,\psi} f\left(s,x(s),\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,x(\sigma))\,d\sigma\right).$$

Let $x \in C_{\xi}(I,\mathbb{R}^n)$ and consider the following operator S, given by

$$\begin{aligned} (Sx)(t) = {} & \Psi^{\gamma}(t,a)x_0 + I_{a^+}^{\alpha,\psi} f\left(s,x(s),\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,x(\sigma))\,d\sigma\right) \\ & - I_{a^+}^{\alpha,\psi} f\left(s,0,\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,0)\,d\sigma\right) + I_{a^+}^{\alpha,\psi} f\left(s,0,\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,0)\,d\sigma\right) \end{aligned}$$

for $t \in I$.

The proof of $S\left((C_{\xi}(I,\mathbb{R}^n),\|(\cdot)\|_{\xi,\infty}\right) \subset \left(C_{\xi}((I,\mathbb{R}^n),\|(\cdot)\|_{\xi,\infty}\right)$ and that S is a contraction, are realized with small and appropriated modifications from the proof of the Theorem 3.

Further, we investigate the estimate of the solution of the nonlinear fractional Volterra integral equation and the nonlinear fractional integrodifferential equation. Then, we first investigated the nonlinear fractional Volterra integral equation.

THEOREM 5. *Suppose the functions f,k in (1) satisfying the conditions*

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant N(|u-\overline{u}| + |v-\overline{v}|) \tag{33}$$

and

$$|k(t,\sigma,u) - k(t,\sigma,v)| \leqslant r(t,\sigma)|u-v| \tag{34}$$

where $0 \leqslant N < 1$ is a constant, $r(t,\sigma) \in C(D,\mathbb{R}_+)$, in which $D = \{(t,\tau) \in I^2 : a \leqslant \sigma \leqslant t < \infty\}$ and $u,\overline{u},v,\overline{v} \in C_{\xi}(I,\mathbb{R}^n)$.

Let

$$C_1 = \sup_{t\in I}\left| f\left(t,0,\int_a^t \mathscr{W}_{\psi}^{\alpha}(t,\sigma,0)\,d\sigma\right)\right| < \infty.$$

If $x(t)$, $t \in I$, is any solution of (1), then

$$|x(t)| \leqslant \left(\frac{C_1}{1-N}\right) \mathbb{E}_\alpha \left[\frac{N}{1-N} r(t,t)(\psi(t)-\psi(a))^\alpha\right]$$

for $t \in I$ and $\mathbb{E}_\alpha(\cdot)$ is an one-parameter Mittag-Leffler function.

Proof. Using the fact that the solution $x(t)$ of (1) is equivalent to (21) and the hypotheses, we have

$$\begin{aligned}
|x(t)| &\leqslant \left|f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,0)\,d\sigma\right)\right| + \left|f\left(t,x(t),\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,x(\sigma))\,d\sigma\right)\right. \\
&\quad \left. - f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,0)\,d\sigma\right)\right| \\
&\leqslant \sup_{t\in I}\left|f\left(t,0,\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,0)\,d\sigma\right)\right| \\
&\quad + N\left\{|x(t)| + \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)|k(t,\sigma,x(\sigma)-k(t,\sigma,0)|\right\} \\
&\leqslant C_1 + N\ |x(t)| + \frac{N}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma) r(t,\sigma)|x(\sigma)|\,d\sigma. \qquad (35)
\end{aligned}$$

From (35) and the fact that $0 \leqslant N < 1$, we get

$$|x(t)| \leqslant \frac{C_1}{1-N} + \frac{N}{1-N}\int_a^t \psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1} r(t,\sigma)|x(\sigma)|\,d\sigma. \qquad (36)$$

Using Corollary 1, we conclude that

$$|x(t)| \leqslant \left(\frac{C_1}{1-N}\right) \mathbb{E}_\alpha \left[\frac{N}{1-N} r(t,t)(\psi(t)-\psi(a))^\alpha\right]$$

where $\mathbb{E}_\alpha(\cdot)$ is an one-parameter Mittag-Leffler function.

Now, we present an estimate of the solution of fractional integrodifferential equation (2).

THEOREM 6. *Suppose the function f in (2) satisfying the condition*

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant p(t)(|u-\overline{u}| + |v-\overline{v}|) \qquad (37)$$

where $p \in C(I,\mathbb{R}_+)$, $u,\overline{u},v,\overline{v} \in C_\xi(I,\mathbb{R}^n)$ and the function k in (2) satisfying the condition (34).

Let

$$C_2 = \sup_{t\in I}\left|\Psi^\gamma(t,a)x_0 + I_{a^+}^{\alpha,\psi} f\left(s,0,\int_a^s \mathscr{W}_\psi^\alpha(s,\sigma,0)\,d\sigma\right)\right| < \infty.$$

If $x(t)$, $t \in I$, is any solution of (2), then

$$|x(t)| \leqslant C_2 \mathbb{E}_\alpha\left\{p(t)\Gamma(\alpha)\mathbb{E}_\alpha\left[r(t,t)\Gamma(\alpha)(\psi(t)-\psi(a))^\alpha\right](\psi(t)-\psi(a))^\alpha\right\}.$$

Proof. Since $x(t)$ is a solution of (2) and using the hypotheses and Lemma 5, we have

$$\begin{aligned}
|x(t)| = & \left|\Psi^{\gamma}(t,a)x_0 + I_{a^+}^{\alpha,\psi} f\left(s,x(s),\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,x(\sigma))\,d\sigma\right)\right. \\
& \left. - I_{a^+}^{\alpha,\psi} f\left(s,0,\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,0)\,d\sigma\right) + I_{a^+}^{\alpha,\psi} f\left(s,0,\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,0)\,d\sigma\right)\right| \\
\leqslant & \left|\Psi^{\gamma}(t,a)x_0 + I_{a^+}^{\alpha,\psi} f\left(s,0,\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,0)\,d\sigma\right)\right| \\
& + \left|\begin{array}{l} I_{a^+}^{\alpha,\psi} f\left(s,x(s),\int_a^s \mathscr{W}_{\psi}^{\alpha}(s,\sigma,x(\sigma))\,d\sigma\right) \\ \quad -I_{a^+}^{\alpha,\psi} f\left(s,0,\int_a^s \mathscr{W}_{\psi}^{\alpha}(t,\sigma,0)\,d\sigma\right) \end{array}\right| \\
\leqslant & \, C_2 + I_{a^+}^{\alpha,\psi}\left\{p(s)\left(|x(s)| + \frac{1}{\Gamma(\alpha)}\int_a^s N_{\psi}^{\alpha}(s,\sigma) r(s,\sigma)|x(\sigma)|\,d\sigma\right)\right\} \\
\leqslant & \, C_2 \mathbb{E}_{\alpha}\left\{p(t)\Gamma(\alpha)\mathbb{E}_{\alpha}\left[r(t,t)\Gamma(\alpha)(\psi(t)-\psi(a))^{\alpha}\right](\psi(t)-\psi(a))^{\alpha}\right\}.
\end{aligned}$$

4. Continuous dependence

In this section, we present results regarding the continuous dependence of solutions of (1) and (2).

Consider (1) and (2) and the corresponding equations

$$y(t) = \overline{f}\left(t,y(t),\int_a^t \overline{\mathscr{W}}_{\psi}^{\alpha}(t,\sigma,y(\sigma))\,d\sigma\right) \tag{38}$$

and

$$\begin{cases} {}^{H}\mathbb{D}_{a^+}^{\alpha,\beta;\psi} y(t) = \overline{f}\left(t,y(t),\int_a^t \overline{\mathscr{W}}_{\psi}^{\alpha}(t,\sigma,y(\sigma))\,d\sigma\right) \\ I_{a^+}^{1-\gamma;\psi} y(a) \ = y_0 \end{cases} \tag{39}$$

for $t \in I$, where $\overline{k} \in C_{\xi}\left(I^2 \times \mathbb{R}^n, \mathbb{R}^n\right)$ for $a \leqslant s \leqslant t < \infty$, $\overline{f} \in C_{\xi}(I \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$.

THEOREM 7. *Let $u, \overline{u}, v, \overline{v} \in C_{\xi}(I, \mathbb{R}^n)$ and suppose the functions f, k in* (1) *satisfying the conditions*

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant N(|u-\overline{u}| + |v-\overline{v}|) \tag{40}$$

and

$$|\overline{k}(t,\sigma,u) - \overline{k}(t,\sigma,v)| \leqslant r(t,\sigma)|u-v|, \tag{41}$$

and

$$\left|f\left(t,y(t),\int_a^t \mathscr{W}_{\psi}^{\alpha}(t,\sigma,y(\sigma))\,d\sigma\right) - \overline{f}\left(t,y(t),\int_a^t \overline{\mathscr{W}}_{\psi}^{\alpha}(t,\sigma,y(\sigma))\,d\sigma\right)\right| \leqslant \varepsilon_1 \tag{42}$$

where f,k and $\overline{f},\overline{k}$ are the functions involved in (1) and (38), $\varepsilon_1 > 0$ is an arbitrary small constant and $y(t)$ is a solution of (38). Then, the solution $x(t)$, $t \in I$, of (1) depends continuously on the functions involved on the right hand side of (1).

Proof. Let $x(t)$ and $y(t)$ the solutions of (1) and (38), respectively. Using hypotheses, we have

$$\begin{aligned}
& u(t) \\
&= |x(t)-y(t)| \\
&= \left| \begin{array}{l} f\left(t,x(t),\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,x(\sigma))\,d\sigma\right) - f\left(t,y(t),\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,y(\sigma))\,d\sigma\right) \\ +f\left(t,y(t),\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,y(\sigma))\,d\sigma\right) - f\left(t,y(t),\int_a^t \overline{\mathscr{W}}_\psi^\alpha(t,\sigma,y(\sigma))\,d\sigma\right) \end{array} \right| \\
&\leqslant \varepsilon_1 + \left| f\left(t,x(t),\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,x(\sigma))\,d\sigma\right) - f\left(t,y(t),\int_a^t \mathscr{W}_\psi^\alpha(t,\sigma,x(\sigma))\,d\sigma\right) \right| \\
&\leqslant \varepsilon_1 + N\left\{ |x(t)-y(t)| + \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)|k(t,\sigma,x(\sigma)) - k(t,\sigma,y(\sigma))|\,d\sigma \right\} \\
&\leqslant \varepsilon_1 + N\left\{ |x(t)-y(t)| + \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)r(t,\sigma)|x(\sigma)-y(\sigma)|\,d\sigma \right\} \\
&= \varepsilon_1 + N\left\{ u(t) + \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)r(t,\sigma)u(\sigma)\,d\sigma \right\}. \qquad (43)
\end{aligned}$$

Note that, by (43) and using the assumption $0 \leqslant N < 1$, we get

$$u(t) \leqslant \frac{\varepsilon_1}{1-N} + \frac{N}{1-N}\left\{ \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)r(t,\sigma)u(\sigma)\,d\sigma \right\}. \qquad (44)$$

Now, employing of the Corollary 1, we rewrite (44) in the following form

$$|x(t)-y(t)| \leqslant \left(\frac{\varepsilon_1}{1-N}\right)\mathbb{E}_\alpha\left[\frac{N}{1-N}r(t,t)(\psi(t)-\psi(\sigma))^\alpha\right], \qquad (45)$$

where $\mathbb{E}_\alpha(\cdot)$ is an one-parameter Mittag-Leffler function.

From (45) it follows that the solution of (1) depends continuously on the functions involved on the right hand side of (1).

THEOREM 8. *Suppose the functions f and k in (2) satisfying the conditions (33) and (34). Furthermore, suppose that*

$$|\Psi^\gamma(t,a)x_0 - \Psi^\gamma(t,a)y_0| + I_{a^+}^{\alpha,\psi}\left(\left| \begin{array}{l} f\left(s,y(s),\int_a^s \mathscr{W}_\psi^\alpha(s,\sigma,y(\sigma))\,d\sigma\right) \\ -\overline{f}\left(s,y(s),\int_a^s \overline{\mathscr{W}}_\psi^\alpha(s,\sigma,y(\sigma))\,d\sigma\right) \end{array} \right| \right) \leqslant \varepsilon_2$$

where f,k and $\overline{f},\overline{k}$ are functions involved in (2) and (39), $\varepsilon_2 > 0$ is an arbitrary small constant and $y(t)$ is a solution of (39). Then, the solution $x(t)$, $t \in I$ of (2) depends continuously on the functions in right hand side of (2).

Proof. Let $x(t)$ and $y(t)$ the solutions of (2) and (39) and using the hypotheses we have

$$\begin{aligned}u(t) &= |x(t)-y(t)| \\ &= \left|\Psi^{\gamma}(t,a)x_0 + I_{a^+}^{\alpha,\psi} f\left(s, x(s), \int_a^s W_{\psi}^{\alpha}(s,\sigma,x(\sigma))\,d\sigma\right)\right. \\ &\qquad \left. -\Psi^{\gamma}(t,a)y_0 - I_{a^+}^{\alpha,\psi}\overline{f}\left(s,y(s),\int_a^s W_{\psi}^{\alpha}(s,\sigma,y(\sigma))\,d\sigma\right)\right|.\end{aligned}$$

As above, we add and subtract an adequate term, using ε_2 and Lemma 5, we have

$$\begin{aligned}u(t) &\leqslant \varepsilon_2 + I_{a^+}^{\alpha,\psi}\left\{p(s)\left(\begin{array}{c}|x(s)-y(s)| + \dfrac{1}{\Gamma(\alpha)}\times \\ \times \displaystyle\int_a^s N_{\psi}^{\alpha}(s,\sigma)\,(k(s,\sigma,x(\sigma)) - k(s,\sigma,y(\sigma)))\,d\sigma\end{array}\right)\right\} \\ &\leqslant \varepsilon_2 + I_{a^+}^{\alpha,\psi}\left\{p(s)\left(u(s) + \frac{1}{\Gamma(\alpha)}\int_a^s N_{\psi}^{\alpha}(s,\sigma) r(t,\sigma) u(\sigma)\,d\sigma\right)\right\} \\ &= \varepsilon_2 + I_{a^+}^{\alpha,\psi}\left\{p(s)u(s) + \frac{p(s)}{\Gamma(\alpha)}\int_a^s N_{\psi}^{\alpha}(s,\sigma) r(t,\sigma) u(\sigma)\,d\sigma\right\} \\ &= \varepsilon_2 \mathbb{E}_{\alpha}\left\{p(t)\Gamma(\alpha)\mathbb{E}_{\alpha}\left[r(t,t)\Gamma(\alpha)(\psi(t)-\psi(a))^{\alpha}\right](\psi(t)-\psi(a))^{\alpha}\right\}.\end{aligned}$$

Now, we consider the following system involving a nonlinear fractional Volterra integral equation and a nonlinear fractional Volterra integrodifferential equation

$$z(t) = h\left(t, z(t), \frac{1}{\Gamma(\alpha)}\int_a^t N_{\psi}^{\alpha}(t,\sigma) g(t,\sigma,z(\sigma))\,d\sigma, \mu\right); \tag{46}$$

$$z(t) = h\left(t, z(t), \frac{1}{\Gamma(\alpha)}\int_a^t N_{\psi}^{\alpha}(t,\sigma) g(t,\sigma,z(\sigma))\,d\sigma, \mu_0\right); \tag{47}$$

$$\begin{cases} {}^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi} z(t) = h\left(t, z(t), \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t N_{\psi}^{\alpha}(t,\sigma) g(t,\sigma,z(\sigma))\,d\sigma, \mu\right) \\ I_{a^+}^{1-\gamma,\psi} z(a) \ = z_0 \end{cases} \tag{48}$$

and

$$\begin{cases} {}^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi} z(t) = h\left(t, z(t), \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t N_{\psi}^{\alpha}(t,\sigma) g(t,\sigma,z(\sigma))\,d\sigma, \mu_0\right) \\ I_{a^+}^{1-\gamma,\psi} z(a) \ = z_0 \end{cases} \tag{49}$$

for $t \in I$, where $g \in C(I^2 \times \mathbb{R}^n, \mathbb{R}^n)$, $a \leqslant \sigma \leqslant t < \infty$ and $h \in C(I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$.

THEOREM 9. *Let $u, \overline{u}, v, \overline{v} \in C_{\xi}(I, \mathbb{R}^n)$ and suppose the functions h, g in (46) and (47) satisfying the conditions*

$$|h(t,u,v,\mu) - h(t,\overline{u},\overline{v},\mu)| \leqslant \overline{N}(|u-\overline{u}| + |v-\overline{v}|); \tag{50}$$

$$|h(t,u,v,\mu) - h(t,u,v,\mu_0)| \leqslant q(t)|\mu - \mu_0| \tag{51}$$

and

$$|g(t,\sigma,u)-g(t,\sigma,v)| \leqslant \overline{r}(t,\sigma)|u-v| \tag{52}$$

where $0 \leqslant \overline{N} < 1$ *is a constant,* $q \in C(I,\mathbb{R}_+)$ *such that* $q(t) \leqslant Q < \infty$*,* Q *is a constant and* $\overline{r}(t,\sigma) \in C(D,\mathbb{R}_+)$ *in which* D *is defined as in* Lemma 4*. Let* $z_1(t)$ *and* $z_2(t)$ *be the solutions of* (46) *and* (47)*, respectively.*

Then,

$$|z_1(t)-z_2(t)| \leqslant Q\frac{|\mu-\mu_0|}{1-\overline{N}}\mathbb{E}_\alpha\left[\frac{\overline{N}}{1-\overline{N}}r(t,t)(\psi(t)-\psi(a))^\alpha\right] \tag{53}$$

where $\mathbb{E}_\alpha(\cdot)$ *is an one-parameter Mittag-Leffler function.*

Proof. Let $x(t)$ and $y(t)$ the solutions of (46) and (47), for $t \in I$, and the hypotheses, we have

$$\begin{aligned} z(t) &= |z_1(t)-z_2(t)| \\ &= \Bigg| h\left(t, z_1(t), \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)g(t,\sigma,z_1(\sigma))\,d\sigma, \mu\right) \\ &\quad -h\left(t, z_2(t), \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)g(t,\sigma,z_2(\sigma))\,d\sigma, \mu\right) \\ &\quad +h\left(t, z_2(t), \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)g(t,\sigma,z_2(\sigma))\,d\sigma, \mu\right) \\ &\quad +h\left(t, z_2(t), \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)g(t,\sigma,z_2(\sigma))\,d\sigma, \mu_0\right)\Bigg|. \end{aligned}$$

Proceeding as in Theorem 8, we can write

$$\begin{aligned} z(t) &\leqslant \overline{N}\left\{|z_1(t)-z_2(t)| + \frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha|g(t,\sigma,z_1(\sigma))-g(t,\sigma,z_2(\sigma))|\,d\sigma\right\} \\ &\quad +q(t)|\mu-\mu_0| \\ &= \overline{N}\left\{z(t)+\frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha\overline{r}(t,\sigma)z(\sigma)\,d\sigma\right\}+Q|\mu-\mu_0|. \end{aligned} \tag{54}$$

As $0 \leqslant \overline{N} < 1$, (54) can be rewritten as follows

$$z(t) \leqslant \frac{Q|\mu-\mu_0|}{1-\overline{N}} + \frac{\overline{N}}{1-\overline{N}}\frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,\sigma)\overline{r}(t,\sigma)z(\sigma)\,d\sigma. \tag{55}$$

Using Corollary 1 in (55), we conclude that

$$|z_1(t)-z_2(t)| \leqslant \frac{Q|\mu-\mu_0|}{1-\overline{N}}\mathbb{E}_\alpha\left[\frac{\overline{N}}{1-\overline{N}}r(t,t)(\psi(t)-\psi(a))^\alpha\right]$$

where $\mathbb{E}_\alpha(\cdot)$ is an one-parameter Mittag-Leffler function.

THEOREM 10. *Suppose the functions h,g in (48) and (49), satisfying the conditions (50)-(52) with $\overline{p}(t)$ in the place of $\overline{N}$ in (50), where $\overline{p}\in C(I,\mathbb{R}_+)$ and the function $q(t)$ in (51) be such that*

$$\frac{1}{\Gamma(\alpha)}\int_a^t N_\psi^\alpha(t,s)q(s)\,ds \leqslant \overline{Q} < \infty$$

where $\overline{Q}$ is a constant. Let $z_1(t)$ and $z_2(t)$ be the solutions of (48) and (49). Then,

$$|z_1(t)-z_2(t)| \leqslant Q|\mu-\mu_0|\mathbb{E}_\alpha\Big\{\overline{p}(t)\Gamma(\alpha)\mathbb{E}_\alpha\left(\overline{r}(t,t)\Gamma(\alpha)(\psi(t)-\psi(a))^\alpha\right) \times(\psi(t)-\psi(a))^\alpha\Big\}.$$

Proof. As before in Theorem 9.

5. Concluding remarks

We concluded this paper with the aims achieved, i.e., we carried out a brief study on existence, uniqueness, estimate, and continuous dependence of solutions of nonlinear fractional Volterra integral equation, (1), and the nonlinear fractional integrodifferential equation, (2). Furthermore, we introduced the metric, (5) and the norm, (6), as well as Lemma 4, Lemma 5 and Corollary 1, which are fundamental to get our main results. In this sense, we contributed to the growth of the fractional calculus, particularly in the case of fractional differential equations and fractional integral equations, especially involving a recent and general formulation of the fractional derivative, the so-called ψ-Hilfer fractional derivative. However, as seen in the text, there are many types of differential equations, integral equations and so problems that should be investigated. We emphasize that one of the problems that deserve special mention comes from the impulsive equations, which will be an object of studies whose results will be published in future work.

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