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ON SINGULAR ELLIPTIC EQUATION WITH SINGULAR NONLINEARITIES, HARDY–SOBOLEV CRITICAL EXPONENT AND WEIGHTS

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Abstract. This article is devoted to the existence and multiplicity to the following singular elliptic equation with singular nonlinearities, Hardy-Sobolev critical exponent and weights:

$$\begin{cases} -\Delta u-\mu \frac{u}{|x|^2}=\frac{|u|^{p-2}u}{|x|^s}+\lambda \frac{u}{|x|^\alpha}|u|^{-\beta}, & x\in\Omega,\\ u>0 & x\in\Omega,\\ u=0 & x\in\partial\Omega.\end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geqslant 3)$, $0\in\Omega$, $\lambda>0$, $0\leqslant\mu<\bar{\mu}_N:=(N-2)^2/4$, $p=2^*(s)=2(N-s)/(N-2)$ with $0<s<2$ is the critical Hardy-Sobolev critical exponent, $0\leqslant\alpha<N(p-1+\beta)/p$, $0<\beta<1$ and $2<p\leqslant 2^*:=2N/(N-2)$ is the critical Sobolev exponent.

By using the Nehari manifold and mountain pass theorem, the existence of at least four distinct solutions is obtained.

1. Introduction

The main purpose of this article is to investigate the existence of nontrivial nonnegative solutions of the following problem (1.1) with Dirichlet boundary value conditions ≡

$$(1.1)\qquad \begin{cases} -\Delta u-\mu \frac{u}{|x|^2}=\frac{|u|^{p-2}u}{|x|^s}+\lambda \frac{u}{|x|^\alpha}|u|^{-\beta}, & x\in\Omega,\\ u=0 & x\in\partial\Omega.\end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geqslant 3)$, $0\in\Omega$, $\lambda>0$, $0\leqslant\mu<\bar{\mu}_N:=(N-2)^2/4$, $p=2^*(s)=2(N-s)/(N-2)$ we have $2<p\leqslant 2^*:=2N/(N-2)$ when $0\leqslant s<2$, with $2^*(s)$ is the critical Hardy-Sobolev critical exponent and 2^* is the critical Sobolev exponent, $0\leqslant\alpha<N(p-1+\beta)/p$, $0<\beta<1$.

In recent years, many auteurs have paid much attention to the following singular elliptic problem,

$$(1.2)\qquad \begin{cases} -\Delta u-\mu|x|^{-2}u=h|u|^{p-2}u+\lambda f(x,u), & \text{in } \Omega\\ u=0 & \partial\Omega,\end{cases}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geqslant 3)$, $0 \in \Omega$, $\lambda > 0$, $0 \leqslant \mu < \bar{\mu}_N := (N-2)^2/4$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent for $h \equiv 1$, see $[1,5,6,7,10,13,15,18]$ and references therein, after the work of Brezis and Nirenberg $[4]$. When $\mu = 0$ and $s = 0$, problem (1.2) becomes the well-known Brezis and Nirenberg problem, and is studied extensively in $[18]$. When $\mu \neq 0$, the problem has its singularity at 0 and attracts much attention. Ding and Tang in $[9]$ studied the existence of positive solutions with $N \geqslant 3$, $0 \leqslant s < 2$ and $f(x,u)$ satisfying (AR) condition in the case $\lambda = 1$. Kang in $[14]$ showed the existence of positive solutions replacing $f(x,u)$ by $|u|^{q-2}u$ with $q > 2$ for $0 \leqslant s < 2$. The quasilinear form of (1.2) is discussed in $[12]$. Some results are already available for (1.1). Wang and Zhou $[19]$ proved that there exist at least two solutions for (1.1) with, $0 < \mu \leqslant \bar{\mu}_N = (N-2)^2/4$, Bouchekif and Matallah $[2]$ showed the existence of two solutions of (1.1) under certain conditions on a weighted function h, when $0 < \mu \leqslant \bar{\mu}_N$, $\lambda \in (0, \Lambda_*)$ with Λ_* a positive constant.

The novelty in this article is that the function $f(x,u) := \frac{u}{|x|^\alpha}|u|^{-\beta}$ with $0 < \beta < 1$ presents a singular nonlinearity thing which will allow us to combine the perturbation with the variational methods. It should be noted that the problem studied in this work is not the fruit of the fertile imagination of a theorist, on the contrary, the problems dealt with in applied mathematics have their origins in different fields we will cite as example: heterogeneous chemical catalysis, kinetic chemical catalysis, heat induction or electrical induction, non-Newtonian fluid theory, and viscous fluid theory. For further discussion on this subject we refer the reader to $[8]$ and $[11]$.

Before giving our main result, we state here some definitions, notation and known results.

We denote by $\mathscr{D}_0^{1,2} = \mathscr{D}_0^{1,2}(\Omega \setminus \{0\})$ and $\mathscr{H}_\mu = \mathscr{H}_0^1(\Omega \setminus \{0\})$, the closure of $C_0^\infty(\Omega \setminus \{0\})$ with respect to the norms

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2\right) dx\right)^{1/2}$$

and

$$\|u\|_\mu = \left(\int_\Omega \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2}\right) dx\right)^{1/2},$$

respectively, with $\mu < \bar{\mu}_N = ((N-2)/2)^2$ for $N \neq 2$.

By weighted Hardy inequality, it is easy to see that the norm $\|u\|_\mu$ is equivalent to $\|u\|$. More explicitly, we obtain

$$\left(1 - \left(\sqrt{\bar{\mu}_N}\right)^{-2} \mu^+\right)^{1/2} \|u\| \leqslant \|u\|_\mu \leqslant \left(1 - \left(\sqrt{\bar{\mu}_N}\right)^{-2} \mu^-\right)^{1/2} \|u\|,$$

with $\mu^+ = \max(\mu, 0)$ and $\mu^- = \min(\mu, 0)$ for all $u \in \mathscr{H}_\mu$.

We list here a few integral inequalities.

The starting point for studying (1.1), is the Hardy inequality with cylindrical weights [10]. It states that

$$\bar{\mu}_N \int_\Omega |x|^{-2} v^2 dx \leqslant \int_\Omega |\nabla v|^2 dx, \text{ for all } v \in \mathscr{H}_\mu, \tag{1.1}$$

Since our approach is variational, we define the functional J on $\mathscr{H}_\mu$ by

$$J(u) := (1/2)\|u\|_\mu^2 - (1/p)\int_\Omega |x|^{-s}|u|^p dx - (\lambda/(2-\beta))\int_\Omega |x|^{-\alpha}|u|^{2-\beta} dx, u \in \mathscr{H}_\mu$$

A point $u \in \mathscr{H}_\mu$ is a weak solution of the equation (1.1) if it satisfies

$$\begin{aligned}&\langle J'(u), \varphi\rangle \\ :=& \int_\Omega \left(\nabla u \nabla \varphi - \mu |x|^{-2} u\varphi\right) dx - \int_\Omega |x|^{-s}|u|^{p-2} u\varphi dx - \lambda \int_\Omega |x|^{-\alpha}|u|^{-\beta} u\varphi dx = 0,\end{aligned}$$

for $u \in \mathscr{H}_\mu$ and for all $\varphi \in \mathscr{H}_\mu$.

Here $\langle .,. \rangle$ denotes the product in the duality $\mathscr{H}_\mu'$, $\mathscr{H}_\mu$ $\left(\mathscr{H}_\mu' \text{ dual of } \mathscr{H}_\mu\right)$.

Let

$$S_\mu := \inf_{u \in \mathscr{H}_\mu \setminus \{0\}} \frac{\|u\|_\mu^2}{\left(\int_\Omega \frac{|u|^p}{|x|^s} dx\right)^{2/p}}$$

From [16], S_μ is achieved.

In our work, we search the critical points as the minimizers of the energy functional associated to the problem (1.1) on the constraint defined by the Nehari manifold, which are solutions of our system.

Let λ_* be positive number such that

$$\lambda_* := (S_\mu)^{\frac{p+\beta-2}{(p-2)}} \left(\frac{p-2}{(p-2+\beta)A}\right)\left[\left(\frac{\beta}{(p-2+\beta)}\right)\right]^{\frac{\beta}{p-2}},$$

where $A = \left[\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot \frac{(p-2+\beta)}{N(p-2+\beta)-\alpha p}\right]^{\frac{p-2+\beta}{p}} R_0^{\frac{N}{p}(p-1+\beta)-\alpha} > 0$, with $0 \leqslant \alpha < \frac{N(p-2+\beta)}{p}$.

Now we can state our main results.

THEOREM 1. *Assume that* $N \geqslant 3$, $0 \leqslant s < 2$, $-\infty < \mu < \bar{\mu}_N$, $0 \leqslant \alpha < N(p-2+\beta)/p$, $\beta \in (0,1)$ *and* λ *verifying* $0 < \lambda < \lambda_*$, *then the problem* (1.1) *has at least one positive solution.*

THEOREM 2. *Under the assumptions of Theorem 1, there exists* $\lambda_{**} := (S_\mu)^{\frac{\beta-2}{2}} \left(\frac{(p-2)(2-\beta)}{(p-2+\beta)A}\right)$ *such that if* λ *satisfying* $0 < \lambda < \lambda_{**}$, *then* (1.1) *has at least two positive solutions.*

THEOREM 3. *Under the assumptions of Theorem 2 then, there exists a positive real* λ^* *such that, if* λ *satisfy* $0 < \lambda < \lambda^*$, *then* (1.1) *has at least two positive solutions and at least one pair of sign-changing solutions.*

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1 and 2. In the last Section, we prove the Theorem 3.

2. Preliminaries

DEFINITION 1. Let $c \in \mathbb{R},\ E$ a Banach space and $I \in C^1(E,\mathbb{R})$.

(i) $(u_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if

$$I(u_n) = c + o_n(1) \text{ and } I^{'}(u_n) = o_n(1),$$

where $o_n(1)$ tends to 0 as n goes at infinity.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

LEMMA 1. *Let X Banach space, and $J \in C^1(X,\mathbb{R})$ verifying the Palais-Smale condition. Suppose that $J(0) = 0$ and that:*

$i)$ there exist $\rho > 0,\ r > 0$ such that if $\|u\| = \rho$, then $J(u) \geqslant r$;

$ii)$ there exist $(u_0) \in X$ such that $\|u_0\| > \rho$ and $J(u_0) \leqslant 0$;
let $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} (J(\gamma(t)))$ where

$$\Gamma = \{\gamma \in C([0,1];X) \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = u_0\},$$

then c is critical value of J such that $c \geqslant r$.

2.1. Nehari manifold

It is well known that J is of class C^1 in $\mathscr{H}_\mu$ and the solutions of (1.1) are the critical points of J which is not bounded below on $\mathscr{H}_\mu$. Consider the following Nehari manifold σ

$$\mathscr{M} = \left\{u \in \mathscr{H}_\mu \setminus \{0\} : \ \left\langle J'(u), u\right\rangle = 0\right\},$$

Thus, $u \in \mathscr{M}$ if and only if

$$\|u\|_\mu^2 - \int_\Omega |x|^{-s} |u|^p \, dx - \lambda \int_\Omega |x|^{-\alpha} |u|^{2-\beta} \, dx = 0. \tag{2.1}$$

Note that $\mathscr{M}$ contains every nontrivial solution of the problem (1.1). In order to obtain the first positive solution, we give the following important lemmas.

LEMMA 2. *J is coercive and bounded from below on $\mathscr{M}$.*

Proof. Let $R_0 > 0$ such that $\Omega \subset B(0,R_0) = \left\{x \in \mathbb{R}^N : r = |x| < R_0 \in (0,1)\right\}$. If $u \in \mathscr{M}$, then by (2.1), $p > 2, \beta \in (0,1), s \geqslant 0$ and the Hölder inequality, we obtain

$$\int_\Omega |x|^{-s} |u|^p \, dx \leqslant \left(\|u\|_\mu^p \left(S_\mu\right)^{\frac{-p}{2}} \right)^{\frac{(2-\beta)}{p}}$$

and

$$\begin{aligned}\int_\Omega |x|^{-\alpha} |u|^{2-\beta} \, dx &\leqslant \left(\int_\Omega r^{-s} |u|^p \, dx \right)^{\frac{(2-\beta)}{p}} \left(\int_\Omega r^{\left(s\frac{2-\beta}{p} - \alpha\right)\frac{p}{p-2+\beta}} \, dx \right)^{\frac{p-2+\beta}{p}} \\ &\leqslant \left(\|u\|_\mu^p \left(S_\mu\right)^{\frac{-p}{2}} \right)^{\frac{(2-\beta)}{p}} \left[\sigma_N \int_0^{R_0} r^{N-1+\left(s\frac{2-\beta}{p} - \alpha\right)\frac{p}{p-2+\beta}} \, dr \right]^{\frac{p-2+\beta}{p}},\end{aligned}$$

where $\sigma_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$ is the area of the $(N-1)$-dimensional unit sphere

In the integral to avoid a singularity at zero we must take $0 \leqslant \alpha < N(p-2+\beta)/p$ because $p-2+\beta > 0$, so we get that

$$\int_\Omega |x|^{-\alpha} |u|^{2-\beta} \, dx \leqslant \left[\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot \frac{(p-2+\beta)}{N(p-2+\beta) - \alpha p} \right]^{\frac{p-2+\beta}{p}} R_0^{\frac{N}{p}(p-1+\beta)-\alpha} \|u\|_\mu^{2-\beta} \left(S_\mu\right)^{\frac{\beta-2}{2}} \tag{2.2}$$

and we deduce that

$$J(u) = ((p-2)/2p)\|u\|_\mu^2 - \lambda((p-2+\beta)/p(2-\beta)) \int_\Omega |x|^{-\alpha} |u|^{2-\beta} \, dx \tag{2.3}$$

$$\geqslant ((p-2)/2p)\|u\|_\mu^2 - \lambda((p-2+\beta)/p(2-\beta)) A \left(S_\mu\right)^{\frac{\beta-2}{2}} \|u\|_\mu^{2-\beta}, \tag{2.4}$$

with $A = \left[\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot \frac{(p-2+\beta)}{N(p-2+\beta)-\alpha p} \right]^{\frac{p-2+\beta}{p}} R_0^{\frac{N}{p}(p-1+\beta)-\alpha} > 0$ for $0 \leqslant \alpha < N(p-2+\beta)/p$.

Thus, J is coercive and bounded from below on $\mathscr{M}$. □

Define

$$\phi(u) = \left\langle J'(u), u \right\rangle.$$

Then, for $u \in \mathscr{M}$

$$\begin{aligned}\left\langle \phi'(u), u \right\rangle &= 2\|u\|_\mu^2 - p \int_\Omega |x|^{-s} |u|^p \, dx - \lambda(2-\beta) \int_\Omega |x|^{-\alpha} |u|^{2-\beta} \, dx \\ &= \beta \|u\|_\mu^2 - (p-2+\beta) \int_\Omega |x|^{-s} |u|^p \, dx \\ &= \lambda(p-2+\beta) \int_\Omega |x|^{-\alpha} |u|^{2-\beta} \, dx - (p-2)\|u\|_\mu^2.\end{aligned}$$

Now, we split $\mathscr{M}$ in three parts:

$$\mathscr{M}^{+} = \left\{u \in \mathscr{M} : \left\langle \phi'(u), u \right\rangle > 0\right\}$$
$$\mathscr{M}^{0} = \left\{u \in \mathscr{M} : \left\langle \phi'(u), u \right\rangle = 0\right\}$$
$$\mathscr{M}^{-} = \left\{u \in \mathscr{M} : \left\langle \phi'(u), u \right\rangle < 0\right\}.$$

We have the following results.

LEMMA 3. *Suppose that u_0 is a local minimizer for J on $\mathscr{M}$. Then, if $u_0 \notin \mathscr{M}^0$, u_0 is a critical point of J.*

Proof. If u_0 is a local minimizer for J on $\mathscr{M}$, then u_0 is a solution of the optimization problem

$$\min_{\{u/\ \phi(u)=0\}} J(u).$$

Hence, there exists a Lagrange multipliers $\theta \in \mathbb{R}$ such that

$$J'(u_0) = \theta \phi'(u_0) \text{ in } \mathscr{H}'$$

Thus,

$$\left\langle J'(u_0), u_0 \right\rangle = \theta \left\langle \phi'(u_0), u_0 \right\rangle.$$

But $\langle \phi'(u_0), u_0 \rangle \neq 0$, since $u_0 \notin \mathscr{M}^0$. Hence $\theta = 0$. This completes the proof. □

LEMMA 4. *There exists a positive number λ_0 such that for all λ, verifying*

$$0 < \lambda < \lambda_*,$$

we have $\mathscr{M}^0 = \emptyset$.

Proof. Let us reason by contradiction.

Suppose $\mathscr{M}^0 \neq \emptyset$ such that $0 < \lambda < \lambda_*$. Then, by (2.5) and for $u \in \mathscr{M}^0$, we have

$$\beta \|u\|_{\mu}^{2} - (p-2+\beta) \int_{\Omega} |x|^{-s} |u|^{p} \, dx = 0$$
$$\lambda (p-2+\beta) \int_{\Omega} |x|^{-\alpha} |u|^{2-\beta} \, dx - (p-2) \|u\|_{\mu}^{2} = 0$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|u\|_{\mu} \geqslant (S_{\mu})^{p/2(p-2)} \left[\beta/(p-2+\beta)\right]^{1/(p-2)} \tag{2.5}$$

and

$$\|u\|_{\mu} \leqslant \left[\lambda \left(\frac{p-2+\beta}{p-2}\right) A\right]^{1/\beta}. \tag{2.6}$$

with $A=\left[\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot\frac{(p-2+\beta)}{N(p-2+\beta)-\alpha p}\right]^{\frac{p-2+\beta}{p}} R_0^{\frac{N}{p}(p-1+\beta)-\alpha}$.

From (2.5) and (2.6), we obtain $\lambda \geqslant \lambda_*$, which contradicts the fact that $\lambda < \lambda_*$. □

Thus $\mathcal{M}=\mathcal{M}^+\cup\mathcal{M}^-$. Define

$$c := \inf_{u\in\mathcal{M}} J(u),\ c^+ := \inf_{u\in\mathcal{M}^+} J(u) \text{ and } c^- := \inf_{u\in\mathcal{M}^-} J(u).$$

For the sequel, we need the following Lemma.

LEMMA 5. (i) *For all* λ *such that* $0<\lambda<\lambda_*$, *one has* $c\leqslant c^+<0$.
(ii) *There exists* $\lambda_{**}>0$ *such that for all* λ *such that* $0<\lambda<\lambda_{**}$, *one has*

$$c^- > C_0 = C_0\left(\lambda, S_\mu, \beta\right)$$

Proof. (i) Let $u\in\mathcal{M}^+$. By (2.5), we have

$$\left(\frac{\beta}{p-2+\beta}\right)\|u\|_\mu^2 > \int_\Omega |x|^{-s}|u|^p\,dx$$

and so

$$\begin{aligned} J(u) &= [(-\beta)/2(2-\beta)]\|u\|_\mu^2 + [(p-2+\beta)/p(2-\beta)]\int_\Omega |x|^{-s}|u|^p\,dx \\ &< \left(\frac{-\beta}{2-\beta}\right)\left(\frac{p-2}{2p}\right)\|u\|_\mu^2 < 0, \end{aligned}$$

since $\beta\in(0,1)$ and $p>2$.

We conclude that $c\leqslant c^+<0$.

(ii) Let $u\in\mathcal{M}^-$. By (2.5), we get

$$\left(\frac{\beta}{p-2+\beta}\right)\|u\|_\mu^2 < \int_\Omega |x|^{-s}|u|^p\,dx.$$

Moreover, by Sobolev embedding theorem, we have

$$\int_\Omega |x|^{-s}|u|^p\,dx \leqslant \left(S_\mu\right)^{-p/2}\|u\|_\mu^p.$$

This implies

$$\|u\|_\mu > \left(S_\mu\right)^{\frac{p}{2(p-2)}}\left[\frac{\beta}{p-2+\beta}\right]^{\frac{1}{(p-2)}},\ \text{for all } u\in\mathcal{M}^-. \tag{2.7}$$

By (2.3), we get

$$\begin{aligned} J(u) &\geqslant \|u\|_\mu^2\left(\frac{(p-2)}{2p}\right) - \lambda\left(\frac{(p-2+\beta)}{p(2-\beta)}\right)A\|u\|_\mu^{2-\beta}\left(S_\mu\right)^{\frac{\beta-2}{2}} \\ &\geqslant \left(\frac{(p-2)}{2p}\right)\left[\frac{\beta}{p-2+\beta}\right]^{\frac{2}{(p-2)}}\left(S_\mu\right)^{\frac{p}{(p-2)}} - \lambda\left(\frac{(p-2+\beta)^2}{p\beta(2-\beta)S_\mu}\right)A \end{aligned}$$

Thus, for all λ such that

$$0<\lambda<\lambda_{**}=\left(\frac{(p-2)}{2p(p-2+\beta)^2 A}\right)\left(\frac{\beta}{p-2+\beta}\right)^{\frac{2}{p-2}} S_\mu^{\frac{2-(p-1)}{(p-2)}},$$

we have $J(u)\geqslant C_0$. $\square$

PROPOSITION 1. *(see [3]) (i) For all λ such that $0<\lambda<\lambda_*$, there exists a $(PS)_{c^+}$ sequence in $\mathscr{M}^+$.*

(ii) For all λ such that $0<\lambda<\lambda_{**}$, there exists a a $(PS)_{c^-}$ sequence in $\mathscr{M}^-$. and for each $u\in\mathscr{H}$, we write

$$t_M:=t_{\max}(u)=\left[\frac{\beta\|u\|_\mu^2}{(p-2+\beta)\int_\Omega |x|^{-s}|u|^p\,dx}\right]^{1/(p-2)}>0.$$

LEMMA 6. [3] *Let λ be a real parameter such that $0<\lambda<\lambda_*$. For each $u\in\mathscr{H}$, there exist unique t^+ and t^- such that $0<t^+<t_M<t^-$,*

$(t^+u)\in\mathscr{M}^+$, $(t^-u)\in\mathscr{M}^-$

$$J\left(t^+u\right)=\inf J(tu)\ \textit{for}\ 0\leqslant t\leqslant t_M,$$

and

$$J\left(t^-u\right)=\sup J(tu)\ \textit{for}\ t\geqslant 0.$$

Proof. With minor modifications, we refer to [3]. $\square$

3. Proof of Theorems 1

Now, taking as a starting point the work of Tarantello [17], we establish the existence of a local minimum for J on $\mathscr{M}^+$.

PROPOSITION 2. *For all λ such that $0<\lambda<\lambda_*$, the functional J has a minimizer $u_0^+\in\mathscr{M}^+$ and it satisfies:*

(i) $J\left(u_0^+\right)=c=c^+$,

(ii) $\left(u_0^+\right)$ is a nontrivial solution of (1.1).

Proof. If $0<\lambda<\lambda_*$, then by Proposition 1 (i) there exists a $(u_n)_n-(PS)_{c^+}$ sequence in $\bar{B}_R\subset\mathscr{M}^+$, thus it bounded by Lemma 2. Then, there exists $u_0^+\in\mathscr{H}$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$\begin{aligned} &u_n\rightharpoonup u_0^+ \text{ weakly in } \mathscr{H} \\ &u_n\rightharpoonup u_0^+ \text{ weakly in } L^p\left(\Omega,|x|^{-s}\right) \\ &u_n\to u_0^+ \text{ a.e in } \Omega \end{aligned} \tag{3.1}$$

By 2.2 and 3.1, we have

$$\lim_{n\longrightarrow\infty}\int_\Omega |x|^{-\alpha}|u_n|^{2-\beta}dx=\int_\Omega |x|^{-\alpha}\left|u_0^+\right|^{2-\beta}dx+o(1)$$

Thus, by (3.1), u_0^+ is a weak nontrivial solution of (1.1). Now, we show that u_n converges to u_0^+ strongly in $\mathscr{H}$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\left\|u_0^+\right\|_\mu<\liminf_{n\longrightarrow\infty}\|u_n\|_\mu$ and we obtain

$$\begin{aligned}c&\leqslant J\left(u_0^+\right)\\&=((p-2)/2p)\left\|u_0^+\right\|_\mu^2-\lambda\left[(p-2+\beta)/p(2-\beta)\right]\int_\Omega |x|^{-\alpha}\left|u_0^+\right|^{2-\beta}dx\\&<\liminf_{n\longrightarrow\infty}J(u_n)=c.\end{aligned}$$

We get a contradiction. Therefore, u_n converge to u_0^+ strongly in $\mathscr{H}$. Moreover, we have $u_0^+\in\mathscr{M}^+$. If not, then by Lemma 6, there are two numbers t_0^+ and t_0^-, uniquely defined so that $\left(t_0^+u_0^+\right)\in\mathscr{M}^-$ and $\left(t^-u_0^+\right)\in\mathscr{M}^+$. In particular, we have $t_0^-<t_0^+=1$. Since

$$\frac{d}{dt}J\left(tu_0^+\right)_{\rfloor t=t_0^+}=0$$

and

$$\frac{d^2}{dt^2}J\left(tu_0^+\right)_{\rfloor t=t_0^+}>0,$$

there exists $t_0^-<t^-\leqslant t_0^+$ such that $J\left(t_0^-u_0^+\right)<J\left(t^+u_0^+\right)$. By Lemma 6, we get

$$J\left(t_0^-u_0^+\right)<J\left(t^-u_0^+\right)<J\left(t_0^+u_0^+\right)=J\left(u_0^+\right),$$

which contradicts the fact that $J\left(u_0^+\right)=c^+$. Since $J\left(u_0^+\right)=J\left(\left|u_0^+\right|\right)$ and $\left|u_0^+\right|\in\mathscr{M}^+$, then by Lemma 3, we may assume that u_0^+ is a nontrivial nonnegative solution of (1.1). By the Harnack inequality, we conclude that $u_0^+>0$, see for example [19]. □

4. Proof of Theorem 2

Next, we establish the existence of a local minimum for J on $\mathscr{M}^-$. For this, we require the following Lemma.

LEMMA 7. *For all λ such that $0<\lambda<\lambda_{**}$, the functional J has a minimizer u_0^- in $\mathscr{M}^-$ and it satisfies:*

(i) $J\left(u_0^-\right)=c^->0$,

(ii) u_0^- is a nontrivial solution of (1.1) in $\mathscr{H}$.

Proof. If $0<\lambda<\lambda_{**}$, then by Proposition 1 (ii) there exists a $(u_n)_n$, $(PS)_{c^-}$-sequence in $\mathscr{M}^-$, thus it bounded by Lemma 2. Then, there exists $u_0^-\in\mathscr{H}$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^- \text{ weakly in } \mathscr{H} \\ u_n &\rightharpoonup u_0^- \text{ weakly in } L^p\left(\Omega, |x|^{-s}\right) \\ u_n &\to u_0^- \text{ a.e in } \Omega \end{aligned}$$

This implies

$$\int_\Omega |x|^{-s} |u_n|^p\, dx \to \int_\Omega |x|^{-s} \left|u_0^-\right|^p dx, \text{as } n \text{ goes to } \infty.$$

Moreover, by (2.5) we obtain

$$\left(\frac{\beta}{p-2+\beta}\right) \|u_n\|_\mu^2 < \int_\Omega |x|^{-s} |u_n|^p\, dx, \tag{4.1}$$

By (2.5) and (4.1) there exists a positive number

$$C_1 := \left(\frac{\beta}{p-2+\beta}\right)^{\frac{p}{p-2}} (S_\mu)^{\frac{p}{p-2}},$$

such that

$$\int_\Omega |x|^{-s} |u_n|^p\, dx > C_1. \tag{4.2}$$

This implies that

$$\int_\Omega |x|^{-s} \left|u_0^-\right|^p dx \geqslant C_1.$$

Now, we prove that $(u_n)_n$ converges to u_0^- strongly in $\mathscr{H}$. Suppose otherwise. Then, either $\left\|u_0^-\right\|_\mu < \liminf_{n\longrightarrow\infty} \|u_n\|_\mu$. By Lemma 6 there is a unique t_0^- such that $\left(t_0^- u_0^-\right) \in \mathscr{M}^-$. Since

$$u_n \in \mathscr{M}^-,\ J(u_n) \geqslant J(tu_n), \text{ for all } t \geqslant 0,$$

we have

$$J\left(t_0^- u_0^-\right) < \lim_{n\longrightarrow\infty} J\left(t_0^- u_n\right) \leqslant \lim_{n\longrightarrow\infty} J(u_n) = c^-,$$

and this is a contradiction. Hence,

$$(u_n)_n \to u_0^- \text{ strongly in } \mathscr{H}.$$

Thus,

$$J(u_n) \text{ converges to } J\left(u_0^-\right) = c^- \text{ as } n \text{ tends to } +\infty.$$

Since $J\left(u_0^-\right) = J\left(\left|u_0^-\right|\right)$ and $u_0^- \in \mathscr{M}^-$, then by (4.2) and Lemma 3, we may assume that u_0^- is a nontrivial nonnegative solution of (1.1). By the maximum principle, we conclude that $u_0^- > 0$. □

Now, we complete the proof of Theorem 2. By Propositions 2 and Lemma 7, we obtain that (1.1) has two positive solutions $u_0^+ \in \mathscr{M}^+$ and $u_0^- \in \mathscr{M}^-$. Since $\mathscr{M}^+ \cap \mathscr{M}^- = \emptyset$, this implies that u_0^+ and u_0^- are distinct.

5. Proof of Theorem 3

In this section, we consider the following Nehari submanifold of $\mathscr{M}$

$$\mathscr{M}_\rho = \left\{ u \in \mathscr{H} \setminus \{0\} : \ \left\langle J'(u), u \right\rangle = 0 \text{ and } \|u\|_\mu \geqslant \rho > 0 \right\}.$$

Thus, $u \in \mathscr{M}_\rho$ if and only if

$$\|u\|_\mu^2 - \int_\Omega |x|^{-s} |u|^p \, dx - \lambda \int_\Omega |x|^{-\alpha} |u|^{2-\beta} \, dx = 0 \text{ and } \|u\|_\mu \geqslant \rho > 0.$$

Firstly, we need the following Lemmas

LEMMA 8. *Under the hypothesis of theorem 3, there exist,* $\Lambda_1 > 0$ *such that* $\mathscr{M}_\rho$ *is nonempty for any* $\lambda \in (0, \Lambda_1)$.

Proof. Fix $u_0 \in \mathscr{H} \setminus \{0\}$ and let

$$\begin{aligned} g(t) &= \left\langle J'(tu_0), tu_0 \right\rangle \\ &= t^2 \|u_0\|_{\mu,}^2 - t^p \int_\Omega |x|^{-s} |u_0|^p \, dx - t^{2-\beta} \lambda \int_\Omega |x|^{-\alpha} |u_0|^{2-\beta} \, dx. \end{aligned}$$

Clearly $g(0) = 0$ and $g(t) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. Moreover, we have

$$\begin{aligned} g(1) &= \|u_0\|_\mu^2 - \int_\Omega |x|^{-s} |u_0|^p \, dx - \lambda \int_\Omega |x|^{-\alpha} |u_0|^{2-\beta} \, dx \\ &\geqslant \|u_0\|_\mu^{2-\beta} \left[\|u_0\|_\mu^\beta - \left(S_\mu\right)^{-p/2} \|u_0\|_\mu^{(p-2+\beta)} - \lambda A \left(S_\mu\right)^{(\beta-2)/2} \right]. \end{aligned}$$

for $t \geqslant 0$ put $\varphi(t) = t^\beta - \left(S_\mu\right)^{-p/2} t^{p-2+\beta}$ then we obtain $\max\limits_{t \geqslant 0} \varphi(t) = \varphi(t_1) > 0$ since $p > 2$ with $t_1 = \left(\frac{\beta}{p-2+\beta}\right)^{\frac{p}{p-2}} \left(S_\mu\right)^{\frac{p}{2(p-2)}}$. Thus, we obtain

$$g(1) \geqslant \|u_0\|_\mu^{2-\beta} \left[\varphi(t_1) - \lambda A \left(S_\mu\right)^{(\beta-2)/2} \right] > 0,$$

if $\lambda < \left(\frac{\left(S_\mu\right)^{(2-\beta)/2}}{A} \right) \varphi(t_1) := \Lambda_1$.

Then, there exists $t_0 > 0$ such that $g(t_0) = 0$. Thus, $(t_0 u_0) \in \mathscr{M}_\rho$ and $\mathscr{M}_\rho$ is nonempty. □

LEMMA 9. *There exist* δ, Λ_{**} *positive real numbers such that* $\langle \phi'(u), u \rangle < -\delta < 0$*, for* $u \in \mathscr{M}_\rho$ *and any* λ *verifying*

$$0 < \lambda < \min(\Lambda_1, \Lambda_2).$$

Proof. Let $u \in \mathscr{M}_\rho$, then by (2.1), (2.5) and the Holder inequality, allows us to write

$$\begin{aligned} \left\langle \phi'(u), u \right\rangle &= \lambda (p-1+\beta) A \left(S_\mu\right)^{(\beta-2)/2} \|u\|_\mu^{2-\beta} - (p-2)\|u\|_\mu^2 \\ &\leqslant \|u\|_\mu^{2-\beta} \left[\lambda (p-2+\beta) A \left(S_\mu\right)^{(\beta-2)/2} - (p-2)\|u\|_\mu^\beta\right] \\ &\leqslant \|u\|_\mu^{2-\beta} \left[\lambda (p-2+\beta) A \left(S_\mu\right)^{(\beta-2)/2} - (p-2)\rho^\beta\right], \end{aligned}$$

Thus, if

$$0 < \lambda < \Lambda_2 = \left[\frac{(p-2)\rho^\beta}{(p-2+\beta)A}\right] \left(S_\mu\right)^{(\beta-2)/2},$$

and choosing $\lambda^* := \min(\Lambda_1, \Lambda_2)$ with Λ_1 defined in Lemma 8, then we obtain that

$$\left\langle \phi'(u), u \right\rangle < 0, \text{ for any } u \in \mathscr{M}_\rho. \quad \square \tag{5.1}$$

LEMMA 10. *Suppose* $0 \leqslant s < 2,\ p > 2,\ \beta \in (0,1)$ *and* $0 < \lambda < \min(\Lambda_1, \Lambda_2, \Lambda_3)$ *when*

$$\Lambda_3 = \left[\frac{p(p-2)(2-\beta)}{2p(p-2+\beta)A}\left(\frac{2-\beta}{2}\right)^{\frac{\beta}{1+\beta}}\right] \left(S_\mu\right)^{(2-p)/2}.$$

Then, there exist ε *and* η *positive constants such that*

i) we have

$$J(u) \geqslant \eta > 0 \ \textit{for}\ \|u\|_\mu = \varepsilon.$$

ii) there exists $v \in \mathscr{M}_\rho$ *when* $\|v\|_\mu > \varepsilon$, *with* $\varepsilon = \|u\|_\mu$, *such that* $J(v) \leqslant 0$.

Proof. We can suppose that the minima of J are realized by $\left(u_0^+\right)$ and u_0^-. The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have

i) By (2.5), (5.1) we get

$$\begin{aligned} J(u) &= ((p-2)/2p)\|u\|_\mu^2 - \lambda((p-2+\beta)/p(2-\beta)) \int_\Omega |x|^{-\alpha} |u|^{2-\beta} dx \\ &\geqslant ((p-2)/2p)\|u\|_\mu^2 - \lambda((p-2+\beta)/p(2-\beta)) A \left(S_\mu\right)^{\frac{\beta-2}{2}} \|u\|_\mu^{2-\beta}, \end{aligned} \tag{5.2}$$

By exploiting the function $\phi(t) = at^2 - bt^{2-\beta}$ which achiev its maximum at the point $t_1 = \left(\frac{2-\beta}{2}\right)^{\frac{2}{\beta}} \left(\frac{a}{b}\right)^{\frac{1}{p-2}}$ such that $\max_{t \geqslant 0} \phi(t) = \phi(t_1) > 0$ if

$$\lambda < \Lambda_3 = \left[\frac{p(p-2)(2-\beta)}{2p(p-2+\beta)A}\left(\frac{2-\beta}{2}\right)^{\frac{\beta}{1+\beta}}\right] \left(S_\mu\right)^{(2-p)/2}$$

and the fact that $0 \leqslant s < 2,\ p > 2, \beta \in (0,1)$ then, we obtain that

$$J(u) \geqslant \eta > 0 \text{ when } \varepsilon = \|u\|_\mu \text{ small.}$$

$ii)$ Let $t > 0$, then we have for all $\varphi \in \mathscr{M}_\rho$

$$J(t\varphi) := \frac{t^2}{2}\|\varphi\|_\mu^2 - \left(\frac{t^p}{p}\right)\int_\Omega |x|^{-s}|\varphi|^p\,dx - \lambda\left(\frac{t^{2-\beta}}{2-\beta}\right)\int_\Omega |x|^{-\alpha}|\varphi|^{2-\beta}\,dx.$$

Letting $v = t\varphi$ for t large enough, we obtain $J(v) \leqslant 0$. For t large enough we can ensure $\|v\|_\mu > \varepsilon$. $\square$

Let Γ and c defined by

$$\Gamma := \left\{\gamma : [0,1] \longrightarrow \mathscr{M}_r :\ \gamma(0) = u_0^- \text{ and } \gamma(1) = u_0^+\right\}$$

and

$$c := \inf_{\gamma \in \Pi} \max_{t \in [0,1]} (J(\gamma(t))).$$

Proof of Theorem 3. If

$$0 < \lambda < \lambda^* := \min(\Lambda_1, \Lambda_2, \Lambda_3),$$

then, by the Lemmas 2 and Proposition 1 (ii), J verifying the Palais -Smale condition in $\mathscr{M}_\rho$. Moreover, from the Lemmas 3, 9 and 10, there exists u_c such that

$$J(u_c) = c \text{ and } u_c \in \mathscr{M}_\rho.$$

Thus u_c is the third solution of our system such that $u_c \neq u_0^+$ and $u_c \neq u_0^-$. Since (1.1) is odd with respect u, we obtain that $-u_c$ is also a solution of (1.1). $\square$

CONCLUSION 1. In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem on the constraint defined by the Nehari manifold $\mathscr{M}$, which are solutions of our problem. Under some sufficient conditions on coefficients of equation of (1.1) such that $N \geqslant 3$, $0 \leqslant s < 2$, $-\infty < \mu < \bar{\mu}_N$, $0 \leqslant \alpha < N(p-1+\beta)/p$, $p > 2$ $\beta \in (0,1)$, we split $\mathscr{M}$ in two disjoint subsets $\mathscr{M}^+$ and $\mathscr{M}^-$ thus we consider the minimization problems on $\mathscr{M}^+$ and $\mathscr{M}^-$ respectively. In the Sections 3 and 4 we have proved the existence of at least two nontrivial solutions on $\mathscr{M}_\rho$ for all $0 < \lambda < \lambda^* := \min(\Lambda_1, \Lambda_2, \Lambda_3)$ if $N \geqslant 3,\ 0 \leqslant s < 2$ and $\beta \in (0,1)$.

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