Differential
Equations
& Applications
Volume 12, Number 4 (2020), 443–456
doi:10.7153/dea-2020-12-27

A REMARK ON THE LOCAL WELL–POSEDNESS FOR A COUPLED SYSTEM OF MKDV TYPE EQUATIONS IN $H^s \times H^k$

XAVIER CARVAJAL

(*Communicated by P. I. Naumkin*)

Abstract. We consider the initial value problem associated to a system consisting modified Korteweg-de Vries type equations

$$\begin{cases} \partial_t v + \partial_x^3 v + \partial_x (v w^2) = 0, & v(x,0) = \phi(x), \\ \partial_t w + \alpha \partial_x^3 w + \partial_x (v^2 w) = 0, & w(x,0) = \psi(x), \end{cases}$$

and using only bilinear estimates of the type $\|J^{\gamma} F_{b_1}^1 \cdot J^{\beta} F_{b_2}^2\|_{L_x^2 L_t^2}$, where J is the Bessel potential and $F_{b_j}^j$, $j = 1,2$ are multiplication operators, we prove the local well-posedness results for given data in low regularity Sobolev spaces $H^s(\mathbb{R}) \times H^k(\mathbb{R})$ for $\alpha \neq 0,1$. In this work we improve the previous result in [6], extending the LWP region from $|s-k| < 1/2$ to $|s-k| < 1$. This result is sharp in the region of the LWP with $s \leqslant 0$ and $k \leqslant 0$, in the sense of the trilinear estimates fails to hold.

1. Introduction

In this work we consider the initial value problem (IVP) associated to the following system of the modified Korteweg-de Vries (mKdV) type equations

$$\begin{cases} \partial_t v + \partial_x^3 v + \partial_x (v w^2) = 0, & v(x,0) = \phi(x), \\ \partial_t w + \alpha \partial_x^3 w + \partial_x (v^2 w) = 0, & w(x,0) = \psi(x), \end{cases} \tag{1.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}$; $v = v(x,t)$ and $w = w(x,t)$ are real-valued functions, and $\alpha \in \mathbb{R}$ is a constant.

We mean by local well-posedness (LWP) in H^s that for any initial data $u_0 \in H^s$, there exist $R > 0$, a time $T = T(R) > 0$ and a unique solution u, belonging to some space-time function space continuously embedded in $C([0,T];H^s)$, such that for any $t \in [0,T]$ the map $u_0 \to u(t)$ is continuous from the ball $B_R(u_0) \subseteq H^s$ into H^s. If the above T can be any large number we obtain the global well-posedness.

For $\alpha = 1$, the system (1.1) reduces to a special case of a broad class of nonlinear evolution equations considered by Ablowiz, Kaup, Newell and Segur [1] in the inverse scattering context. In this case, the well-posedness issues along with existence and

Mathematics subject classification (2010): 35Q35, 35Q53.
Keywords and phrases: Korteweg-de Vries equation, Cauchy problem, local well-posedness.

Paper DEA-12-27

stability of solitary waves for this system is widely studied in the literature. Using the technique developed by Kenig, Ponce and Vega [17], Montenegro [22] proved that the IVP (1.1) with $\alpha = 1$ is locally well-posed for given data (ϕ, ψ) in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geqslant \frac{1}{4}$. In this approach one uses the smoothing property of the linear group combined with the $L_x^p L_t^q$ Strichartz estimates and maximal function estimates. Tao [21] showed that this local result can also be proved by using the Fourier transform restriction norm space $X_{s,b}$ (see definition (1.6) below) introduced by Bourgain [5]. In this method the trilinear estimate

$$\|\partial_x(uvw)\|_{X_{s,b'}} \lesssim \|u\|_{X_{s,b}} \|v\|_{X_{s,b}} \|w\|_{X_{s,b}} \tag{1.2}$$

that is valid for $s \geqslant \frac{1}{4}$ plays a central role to apply contraction mapping principle. Author in [22] also proved global well-posedness for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geqslant 1$, using the conservation laws

$$I_1(v,w) := \int_{\mathbb{R}} (v^2 + w^2)\, dx$$

and

$$I_2(v,w) := \int_{\mathbb{R}} (v_x^2 + w_x^2 - v^2 w^2)\, dx,$$

satisfied by the flow of (1.1). This global result is further improved in [12] by proving for data with regularity $s > \frac{1}{4}$ (see also [8]). For existence and estability of solitary waves to the system (1.1) we refer to works in [2] and [22]. It is worth noting that the local well-posedness result for the system (1.1) with $\alpha = 1$ is sharp as it can be justified in two different way. First, the key trilinear estimate (1.2) fails whenever $s < \frac{1}{4}$, see [16].

Other works that studied LWP in systems with two different KdV groups were the work of Alvarez et al. [3] in Gear-Grimshaw systems and, later, the work of Oh [15] in Majda-Vielo system.

For $0 < \alpha < 1$, very less is known regarding well-posedness issues for the IVP (1.1). We note that, the approach of Kenig, Ponce, Vega [17] yields local well-posedness for $s \geqslant \frac{1}{4}$ for $\alpha \neq 1$ too. However, if one uses the Fourier transform restriction norm space the situation is quite different. In fact the case $\alpha \in (0,1)$ and $k = s$, with $s > -1/2$ in order to obtain LWP was proved in [7], later it was extended to the case $\alpha \in \mathbb{R} \setminus \{0,1\}$, $k > -1/2$, $s > -1/2$ and $|k-s| \leqslant 1/2$ in [6]. In both works the main idea was to use bilinear estimates in Bourgain spaces to obtain the results. In this work we use a approach different with the use of bilinear estimates of the type $\|J^\gamma F_{b_1}^1 \cdot J^\beta F_{b_2}^2\|_{L_x^2 L_t^2}$, see Lemma 5.

Remember that the KdV equation and the mKdV equation are particular cases of the generalized KdV equation:

$$\partial_t v + \partial_x^3 v + c\partial_x(v^{k+1}) = 0, \quad v(x,0) = \phi(x). \tag{1.3}$$

The case $k = 1$ is the Korteweg-de Vries (KdV) equation. The case $k = 2$ is the modified KdV (mKdV) equation. The case $k = 4$ is the critical KdV equation. The mKdV equation is linked with the KdV equation through the Miura transform

$$v \mapsto u := c_1 \partial_x v + c_2 v^2.$$

The Miura transform behaves roughly like a derivative, so the result for mKdV at a certain regularity is similar to that for KdV at one lower regularity.

Observe that if v is a solution of generalized KdV equation then

$$u(x,t) = \lambda^{2/k} v(\lambda x, \lambda^3 t)$$

is also is a solution of generalized KdV equation, and holds

$$\|u(t)\|_{\dot{H}^s} = \lambda^{s+2/k-1/2} \|v(\lambda^3 t)\|_{\dot{H}^s}, \quad \lambda > 0,$$

where both norms are the same if and only if

$$s = s_k = \frac{1}{2} - \frac{2}{k}. \tag{1.4}$$

This argument scaling suggests that the best LWP to the KdV equation $(k = 1)$ is $s_1 = -3/2$ and to the mKdV equation is $s_2 = -1/2$. Also if $v_1(x,t)$ and $w_1(x,t)$ are solutions of the IVP (1.1), then $v_2(x,t) = \lambda v_1(\lambda x, \lambda^3 t)$ and $w_2(x,t) = \lambda w_1(\lambda x, \lambda^3 t)$ also are solutions of the IVP (1.1) and therefore the above scaling argument suggests that the space $H^{-1/2} \times H^{-1/2}$ is the sharp space in order to obtain local well-posedness to the IVP (1.1). Therefore the model provided by the IVP (1.1) is very interesting in the sense that the end point in the LWP found in [7], attains its scaling index (see Table 1 below).

Table 1: *Summary of Results (LWP in $H^s \times H^s$ spaces)*

Equation/System	L^pL^q-Estimates	$X_{s,b}$-Spaces
KdV equation	H^s, $s > \frac{3}{4}$	H^s, $s > -\frac{3}{4}$
mKdV equation	H^s, $s \geqslant \frac{1}{4}$	H^s, $s \geqslant \frac{1}{4}$
KdV-KdV system (same groups)	$H^s \times H^s$, $s > \frac{3}{4}$	$H^s \times H^s$, $s > -\frac{3}{4}$
KdV-KdV system (different groups)	$H^s \times H^s$, $s > \frac{3}{4}$	$H^s \times H^s$, $s \geqslant 0$ (T. Oh [15])
mKdV-mKdV system (same groups)	$H^s \times H^s$, $s \geqslant \frac{1}{4}$	$H^s \times H^s$, $s \geqslant \frac{1}{4}$
mKdV-mKdV system (different groups)	$H^s \times H^s$, $s \geqslant \frac{1}{4}$	$H^s \times H^s$, $s > -\frac{1}{2}$ (Carvajal, Panthee [7])

Remember some local well-posedness results for the KdV equation:

- LWP in $H^s(\mathbb{R})$, $s > 3/4$ by Kenig, Ponce, and Vega, 1993 [17].
- LWP in $H^s(\mathbb{R})$, $s \geqslant 0$ by Bourgain, 1993 [5].
- LWP in $H^s(\mathbb{R})$, $s > -5/8$ and $s > -3/4$, by Kenig, Ponce, and Vega, 1993 and 1996 respectively [18] and [16].
- LWP in $H^{-3/4}(\mathbb{R})$, by Kishimoto and Guo, 2009, [20] and [14] respectively.

- LWP in $H^{-1}(\mathbb{R})$, by Killip and Visan, 2019 [19].

Regarding some local well-posedness results for the mKdV equation we have:

- LWP in $H^s(\mathbb{R})$, $s \geqslant 1/4$ by Kenig, Ponce, and Vega, 1993 [17].
- Weak LWP in $H^s(\mathbb{R})$, $-1/8 < s < 1/4$ by Christ, Holmer and Tataru, 2012 [11].
- LWP in modulation spaces $M^{1/4}_{2,q}(\mathbb{R})$, which contains a class of functions in $H^{-1/4}$ by Chen and Guo, 2018 [10].

We use $\widehat{f}(\xi)$ to denote the Fourier transform of $f(x)$ defined by

$$\widehat{f}(\xi) = c\int_{\mathbb{R}} e^{-ix\xi} f(x)dx$$

and $\widetilde{f}(\xi,\tau)$ or $\widehat{f}(\xi,\tau)$ to denote the Fourier transform of $f(x,t)$ defined by

$$\widetilde{f}(\xi,\tau) = c\int_{\mathbb{R}^2} e^{-i(x\xi+t\tau)} f(x,t)dxdt.$$

We also use $\widehat{f}(\xi,t)$ to denote the spatial Fourier transform of $f(x,t)$ and $\widehat{f}(x,\tau)$ to denote the time Fourier transform of $f(x,t)$.

We use H^s to denote the L^2-based Sobolev space of order $s \in \mathbb{R}$ with norm

$$\|f\|_{H^s(\mathbb{R})} = \|\langle\xi\rangle^s \widehat{f}(\xi)\|_{L^2_\xi},$$

where $\langle\xi\rangle = 1+|\xi|$. The Bessel potential is defined by $\widehat{J^s f}(\xi,t) = \langle\xi\rangle^s \widehat{f}(\xi,t)$.

The Bourgain space $X^\alpha_{s,b}$, for $s,b \in \mathbb{R}$, to be the completion of the Schwartz class $\mathscr{S}(\mathbb{R}^2)$ under the norm

$$\|f\|_{X^\alpha_{s,b}} := \|U^\alpha(t)f\|_{H^b_t(\mathbb{R};H^s_x)} = \|\langle\xi\rangle^s\langle\tau-\alpha\xi^3\rangle^b \tilde{f}(\tau,\xi)\|_{L^2_{\tau,\xi}}. \tag{1.5}$$

If $b > 1/2$, we have that $X^\alpha_{s,b} \hookrightarrow C(\mathbb{R} : H^s_x(\mathbb{R}))$ and thus for an interval $I = [-\delta,\delta]$, we can define the restricted bourgain spaces $X^{\alpha,\delta}_{s,b}$ endowed with the norm

$$\|f\|_{X^{\alpha,\delta}_{s,b}} = \inf\{\|g\|_{X^\alpha_{s,b}} ; g|_{[-\delta,\delta]} = f\}. \tag{1.6}$$

We write $X^\delta_{s,b}$ instead of $X^{1,\delta}_{s,b}$. We use c or C to denote various constants whose exact values are immaterial and may vary from one line to the next. We use $A \lesssim B$ to denote an estimate of the form $A \leqslant cB$ and $A \sim B$ if $A \leqslant cB$ and $B \leqslant cA$.

On the ill-posedness of the trilinear estimates, in [6] they proved the following results:

PROPOSITION 1. *Let* $\alpha \neq 0,1$.

(a) The trilinear estimate (2.3) *fail to hold for any* $b \in \mathbb{R}$ *whenever* $s-2k > 1$ *or* $k < -1/2$.

(b) The trilinear estimate (2.4) *fail to hold for any* $b \in \mathbb{R}$ *whenever* $k-2s > 1$ *or* $s < -1/2$.

and

PROPOSITION 2. *Let* $\alpha \neq 0,1$.

(a) The trilinear estimate (2.3) *fails to hold whenever* $s-k>2$, *for any* ε *such that* $0<\varepsilon<\frac{2}{3}(s-k-2)$.

(b) The trilinear estimate (2.4) *fails to hold whenever* $k-s>2$, *for any* ε *such that* $0<\varepsilon<\frac{2}{3}(s-k-2)$.

In this work we improve the previous result in [6], extending the LWP region from $|s-k|<1/2$ to $|s-k|<1$. Indeed we obtain the following local well-posedness result for the IVP (1.1).

THEOREM 1. *Let* $\alpha \in \mathbb{R}$, $\alpha \neq 0,1$, $s,k>-1/2$ *and* $|s-k|<1$, $s-2k<1$ *and* $k-2s<1$, *then for any* $(\phi,\psi)\in H^s(\mathbb{R})\times H^k(\mathbb{R})$, *there exist* $\delta=\delta(\|(\phi,\psi)\|_{H^s\times H^k})$ *(with* $\delta(\rho)\to\infty$ *as* $\rho\to 0$*) and a unique solution* $(v,w)\in X^{1,\delta}_{s,b}\times X^{\alpha,\delta}_{k,b}$ *to the IVP* (1.1) *in the time interval* $[0,\delta]$. *Moreover, the solution satisfies the estimate*

$$\|(v,w)\|_{X^{1,\delta}_{s,b}\times X^{\alpha,\delta}_{k,b}} \lesssim \|(\phi,\psi)\|_{H^s\times H^k},$$

where the norm $\|\cdot\|_{X^{1,\delta}_{s,b}}$ *and* $\|\cdot\|_{X^{\alpha,\delta}_{s,b}}$ *are as defined in* (1.6).

REMARK 1. We observe that the LWP given by Theorem 1 is sharp in the case $k,s\leqslant 0$, see the Proposition 1 and the Figure 1.

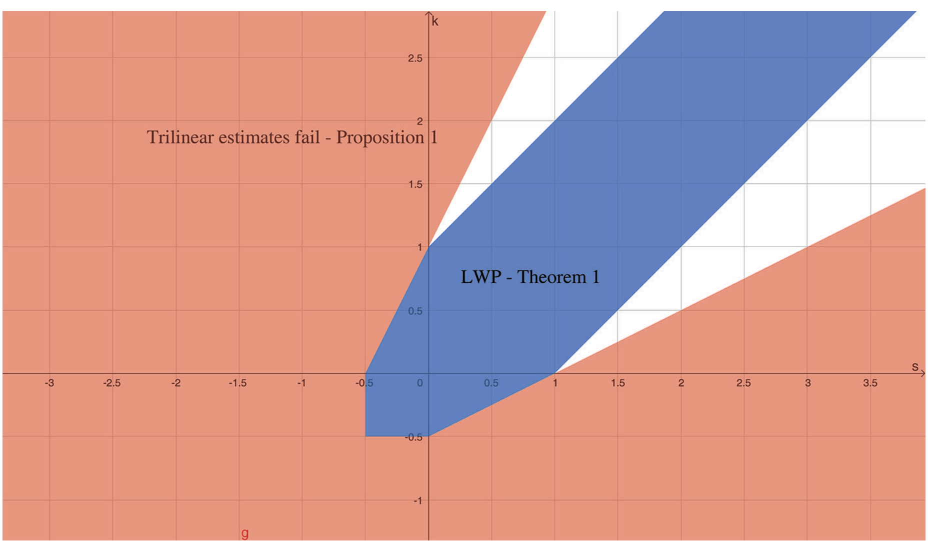

Figure 1: *In blue is the region of the LWP of Theorem* 1. *In red the region where the trilinear estimates fail to hold*

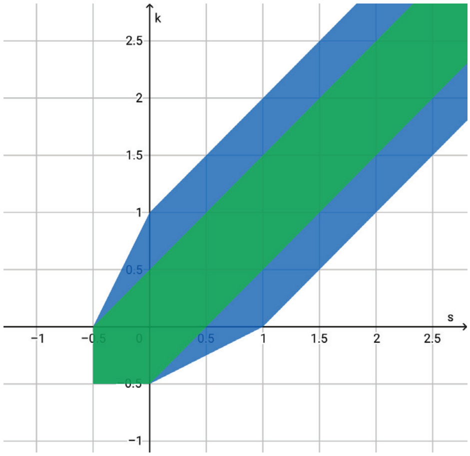

Figure 2: *In green is the region of the LWP of* [6]

The LWP in the case $0<\alpha<1$ is equivalent to the LWP in the case $\alpha>1$ by using the transformation $v(x,t):=\tilde{v}(\alpha^{-1/3}x,t)$ and $u(x,t):=\tilde{u}(\alpha^{-1/3}x,t)$ where

$$\begin{cases}\partial_t\tilde{v}+\frac{1}{\alpha}\partial_x^3\tilde{v}+\partial_x(\tilde{v}\tilde{w}^2)=0,\\ \partial_t\tilde{w}+\partial_x^3\tilde{w}+\partial_x(\tilde{v}^2\tilde{w})=0.\end{cases}$$

So we restrict ourselves to prove Theorem 1 in the case $\alpha\in(-\infty,0)\cup(1,+\infty)$.

2. Preliminar estimates

The following lemma will be useful in the proof of the trilinear estimates

LEMMA 1. *(i) If $a,b>0$ and $a+b>1$, we have*

$$\int_{\mathbb{R}}\frac{dx}{\langle x-\alpha\rangle^a\langle x-\beta\rangle^b}\lesssim\frac{1}{\langle\alpha-\beta\rangle^c},\qquad c=\min\{a,b,a+b-1\}. \tag{2.1}$$

(ii) For $l>1/3$,

$$\int_{\mathbb{R}}\frac{dx}{\langle x^3+a_2x^2+a_1x+a_0\rangle^l}\lesssim 1. \tag{2.2}$$

Proof. Proof of (2.1) can be found in [23] and proof of (2.2) in [4].
The main ingredients in the proof of Theorem 1 are the new trilinear estimates

PROPOSITION 3. *Let $\alpha\in\mathbb{R}$, $\alpha\neq 0,1$, $-1/2<s,k$.*
If $s-k<1$ and $s-2k<1$, then the following trilinear estimate

$$\|(vw_1w_2)_x\|_{X^1_{s,b'}}\lesssim\|v\|_{X^1_{s,b}}\|w_1\|_{X^\alpha_{k,b}}\|w_2\|_{X^\alpha_{k,b}} \tag{2.3}$$

hold for $b=\frac{1}{2}+\varepsilon$, $b'=-1/2+2\varepsilon$ *and* $0<\varepsilon\ll 1$.

And if $k-s<1$ *and* $k-2s<1$, *then the following trilinear estimate*

$$\|(v_1v_2w)_x\|_{X^{\alpha}_{k,b'}}\lesssim\|v_1\|_{X^1_{s,b}}\|v_2\|_{X^1_{s,b}}\|w\|_{X^{\alpha}_{k,b}} \tag{2.4}$$

hold for $b=\frac{1}{2}+\varepsilon$, $b'=-1/2+2\varepsilon$ *and* $0<\varepsilon\ll 1$.

LEMMA 2. *Let* $\alpha<1$ *or* $\alpha<0$ *and* $s>-1/2$, *then*

$$\sup_{\xi,\tau}\int_{\mathbb{R}}\frac{\langle\xi_2\rangle}{\langle\xi_1\rangle^{2s}\langle\tau-\xi_2^3-\alpha\xi_1^3\rangle^{1-4\varepsilon}}d\xi_1\lesssim 1.$$

Proof. For the case $0<\alpha<1$, see the proof of Lemma 3.2 in [7] (estimate of $\mathscr{L}_1$) and for the case $\alpha\leqslant 0$ see Lemma 3.3 in [6].

COROLLARY 1. *Let* $\iota<1$ *and* $\alpha>1$ *or* $\alpha<0$, *then*

$$\sup_{\xi,\tau}\int_{\mathbb{R}}\frac{\langle\xi_1\rangle^{\iota}\langle\xi_2\rangle}{\langle\tau-\alpha\xi_2^3-\xi_1^3\rangle^{1-4\varepsilon}}d\xi_1\lesssim 1. \tag{2.5}$$

Proof. This inequality follows from the previous Lemma, considering

$$\langle\tau-\alpha\xi_2^3-\xi_1^3\rangle\sim\left\langle\frac{\tau}{\alpha}-\frac{1}{\alpha}\xi_1^3-\xi_2^3\right\rangle.$$

LEMMA 3. *Let* $\mu<2$, $\alpha>1$ *or* $\alpha<0$ *and* $b>1/2$, *then*

$$\sup_{\xi,\tau}\int_{\mathbb{R}}\frac{\langle\xi_1\rangle^{\mu}}{\langle\tau-\xi_1^3-\alpha\xi_2^3\rangle^{2b}}d\xi_1\lesssim 1. \tag{2.6}$$

Proof. Using the inequality (2.2) we can suppose that $|\xi_1|>1$ and this implies that $|\xi_1|\sim\langle\xi_1\rangle$, thus it is sufficient to estimate the integral

$$\int_{\mathbb{R}}\frac{|\xi_1|^{\mu}}{\langle\tau-\xi_1^3-\alpha\xi_2^3\rangle^{2b}}d\xi_1. \tag{2.7}$$

Observe that in the case $\alpha<0$, if $H(\xi_1)=\tau-\xi_1^3-\alpha\xi_2^3$, then $-H'(\xi_1)=3\xi_1^2-3\alpha\xi_2^2>3\xi_1^2\gtrsim|\xi_1|^{\mu}$ and in this case we obtain the estimate (2.7). So from now on we will consider $\alpha>1$.

We will go to consider the following regions:

Case 1. If $|\xi|\lesssim 1$. In this case we have

$$\langle\xi_1\rangle\sim\langle\xi_2\rangle$$

and

$$\langle\xi_1\rangle^{\mu} = \langle\xi_1\rangle^{\mu-1}\langle\xi_1\rangle \lesssim \langle\xi_1\rangle^{\iota}\langle\xi_2\rangle,$$

where $\iota = \mu - 1 < 1$ and the estimate is consequence of the Corollary 1.

Case 2. If $\frac{c_1}{2} < |\xi| < c_1|\xi_1|$, where $c_1 = \dfrac{\alpha-1}{3\alpha}$ is such that

$$\begin{aligned}[(6\alpha\xi)\xi_1 - (3\alpha\xi^2)] &\leqslant 6\alpha|\xi||\xi_1| - 3\alpha\xi^2\\ &\leqslant (6\alpha)c_1\xi_1^2\\ &= 2(\alpha-1)\xi_1^2,\end{aligned}$$

consequently $H'(\xi_1) = 3(\alpha-1)\xi_1^2 - [(6\alpha\xi)\xi_1 - (3\alpha\xi^2)] \geqslant (\alpha-1)\xi_1^2$ and we proceed as above.

Case 3. If $|\xi| \geqslant \dfrac{1}{c_2}|\xi_1|$, where $c_2 = \dfrac{3-\lambda}{6}$, $\lambda \in (0,3)$ is such that

$$\begin{aligned}[(6\alpha\xi)\xi_1 - 3(\alpha-1)\xi_1^2)] &\leqslant (6\alpha)c_2\xi^2\\ &= (3-\lambda)\alpha\xi^2,\end{aligned}$$

and thus $H'(\xi_1) = 3\alpha\xi^2 - [(6\alpha\xi)\xi_1 - 3(\alpha-1)\xi_1^2] \geqslant 3\alpha\xi^2 - (3-\lambda)\alpha\xi^2 \geqslant \lambda\alpha\xi^2$. And in this case we have

$$\xi_1^2 \lesssim \xi^2 \lesssim H'(\xi_1).$$

Finally we consider the region

Case 4. If $c_1|\xi_1| < |\xi| < \dfrac{1}{c_2}|\xi_1|$ we get

$$\int_{\mathscr{R}} \frac{\xi_1^2}{\langle\tau - \xi_1^3 - \alpha\xi_2^3\rangle^{2b}}d\xi_1 \sim \xi^2 \int_{\mathscr{R}} \frac{1}{\langle H(\xi_1)\rangle^{2b}}d\xi_1 \lesssim 1,$$

where $\mathscr{R} = \{\xi_1;\ c_2|\xi| < |\xi_1| < \frac{1}{c_1}|\xi|\}$ and in the last inequality we use the estimate of inequality (3.7) case d) in [7] (see the estimate of $\mathscr{X}$ defined in (3.27), and also see (3.33), (3,41) and (3.47)).

COROLLARY 2. *Let* $0 < \varepsilon \ll 1$ *and* $\alpha > 1$ *or* $\alpha < 0$*, then*

$$\sup_{\xi,\tau} \int_{\mathbb{R}} \frac{\langle\xi_1\rangle^{2-20\varepsilon}}{\langle\tau - \alpha\xi_2^3 - \xi_1^3\rangle^{1-4\varepsilon}}d\xi_1 \lesssim 1. \tag{2.8}$$

Proof.
Case 1. $\langle H\rangle \leqslant \langle\xi_1\rangle^3$. In this case we have

$$\frac{\langle\xi_1\rangle^{2-20\varepsilon}}{\langle H\rangle^{1-4\varepsilon}} \leqslant \frac{\langle\xi_1\rangle^{2-2\varepsilon}}{\langle H\rangle^{1+2\varepsilon}},$$

and (2.8) follows using the Lemma 3.

Case 2. $\langle H\rangle \geqslant \langle\xi_1\rangle^3$. With this condition we obtain

$$\int_{\mathbb{R}} \frac{\langle\xi_1\rangle^{2-20\varepsilon}}{\langle H\rangle^{1-4\varepsilon}}d\xi_1 \leqslant \int_{\mathbb{R}} \frac{\langle\xi_1\rangle^{2-20\varepsilon}}{\langle\xi_1\rangle^{3-12\varepsilon}}d\xi_1 = \int_{\mathbb{R}} \frac{1}{\langle\xi_1\rangle^{1+8\varepsilon}}d\xi_1 \lesssim 1.$$

LEMMA 4. *Let* $0\leqslant\mu<2$ *and* $0\leqslant r\leqslant\mu$, $\alpha>1$ *or* $\alpha\leqslant 0$, *then*

$$\mathscr{I}_r =: \int_{\mathbb{R}} \frac{\langle\xi_1\rangle^r\langle\xi_2\rangle^{\mu-r}}{\langle\tau-\xi_1^3-\alpha\xi_2^3\rangle^{2b}}d\xi_1 \lesssim 1.$$

Proof.
Case 1. $|\xi|\lesssim 1$. In this case is $\langle\xi_1\rangle\sim\langle\xi_2\rangle$, thus using the Lemma 3 we have

$$\mathscr{I}_r \sim \int_{\mathbb{R}} \frac{\langle\xi_1\rangle^{\mu}}{\langle\tau-\xi_1^3-\alpha\xi_2^3\rangle^{2b}}d\xi_1 \lesssim 1.$$

Case 2. $|\xi|\gtrsim 1$, $|\xi|\lesssim|\xi_1|$. Here is $\langle\xi_2\rangle\lesssim\langle\xi_1\rangle$ and we estimate $\mathscr{I}_r$ as above.

Case 3. $|\xi|\gtrsim 1$, $|\xi_1|\leqslant c_2|\xi|$. Where $0<c_2=\dfrac{3-\lambda}{6}<\dfrac{1}{2}$ as in the Case 3 in Lemma 3, in this way we have $|\xi_1|\leqslant c_2|\xi_1+\xi_2|\leqslant c_2|\xi_1|+c_2|\xi_2|$ consequently $|\xi_1|\leqslant\frac{c_2}{1-c_2}|\xi_2|$, thus $\langle\xi_1\rangle\lesssim\langle\xi_2\rangle\lesssim\langle\xi\rangle$ and $H'(\xi_1)\gtrsim\xi^2$ and $\mathscr{I}_r\lesssim 1$.

Now we introduce the following operators

$$\widetilde{F_bf}(\xi,\tau)=\frac{f(\xi,\tau)}{\langle\sigma_1\rangle^b},\quad \widetilde{G_bf}(\xi,\tau)=\frac{f(\xi,\tau)}{\langle\sigma_2\rangle^b}, \tag{2.9}$$

where $\sigma_1=\tau-\xi^3$, $\sigma_2=\tau-\alpha\xi^3$.

LEMMA 5. *Let* $\alpha>1$ *or* $\alpha<0$, $b=1/2+\varepsilon$, $b'=-1/2+2\varepsilon$, $0<\varepsilon\ll 1$, $0\leqslant\gamma<1$, $0\leqslant r\leqslant\gamma$ *and* $\theta<1/2$, *then*

$$\|J^r(F_bf_1)\cdot J^{\gamma-r}(G_bf_2)\|_{L_x^2L_t^2}\lesssim\|f_1\|_{L^2}\|f_2\|_{L^2}, \tag{2.10}$$

$$\|J^{\theta}(F_{-b'}f)\cdot J^{1/2}(G_bf_1)\|_{L_x^2L_t^2}\lesssim\|f\|_{L^2}\|f_1\|_{L^2}, \tag{2.11}$$

and

$$\|J^{1-10\varepsilon}(F_{-b'}f)\cdot G_bf_1)\|_{L_x^2L_t^2}\lesssim\|f\|_{L^2}\|f_1\|_{L^2}. \tag{2.12}$$

Proof. First we will prove (2.10). Using Plancherel's identity we have

$$\begin{aligned}\|J^r(F_bf_1)\cdot J^{\gamma-r}(G_bf_2)\|^2_{L_x^2L_t^2} &\lesssim \Big\|\int_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^r}{\langle\sigma_1\rangle^b}|f_1(\xi_1,\tau_1)|\cdot\frac{\langle\xi_2\rangle^{\gamma-r}}{\langle\sigma_2\rangle^b}|f_2(\xi_2,\tau_2)|d\xi_1d\tau_1\Big\|^2_{L_\xi^2L_\tau^2}\\ &=\Big\|\int_{\mathbb{R}^2}\mathscr{K}_1(\xi_1,\tau_1,\xi_2,\tau_2)\cdot|f_1(\xi_1,\tau_1)|\cdot|f_2(\xi_2,\tau_2)|d\xi_1d\tau_1\Big\|^2_{L_\xi^2L_\tau^2}\\ &\lesssim\sup_{\xi,\tau}\|\mathscr{K}_1(\xi_1,\tau_1,\xi_2,\tau_2)\|_{L^2_{\xi_1}L^2_{\tau_1}}\|f_1\|_{L^2}\|f_2\|_{L^2},\end{aligned} \tag{2.13}$$

where $\xi = \xi_1 + \xi_2$, $\tau = \tau_1 + \tau_2$ and

$$\mathcal{K}_1(\xi_1, \tau_1, \xi_2, \tau_2) = \frac{\langle \xi_1 \rangle^r \langle \xi_2 \rangle^{\gamma - r}}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b}.$$

Now we proceed to estimate the kernel $\mathcal{K}_1$:

$$\begin{aligned} \|\mathcal{K}_1(\xi_1, \tau_1, \xi_2, \tau_2)\|^2_{L^2_{\xi_1} L^2_{\tau_1}} &= \int_{\mathbb{R}} \langle \xi_1 \rangle^{2r} \langle \xi_2 \rangle^{2\gamma - r} \left(\int_{\mathbb{R}} \frac{1}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b}} d\tau_1 \right) d\xi_1 \\ &\lesssim \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{2r} \langle \xi_2 \rangle^{2\gamma - 2r}}{\langle \tau - \xi_1^3 - \alpha \xi_2^3 \rangle^{2b}} d\xi_1 \\ &\lesssim 1. \end{aligned} \tag{2.14}$$

Similarly, in order to prove (2.11) we consider the kernel

$$\mathcal{K}_2(\xi_1, \tau_1, \xi_2, \tau_2) = \frac{\langle \xi_1 \rangle^{\theta} \langle \xi_2 \rangle^{1/2}}{\langle \sigma_1 \rangle^{-b'} \langle \sigma_2 \rangle^b}.$$

In this way

$$\begin{aligned} \|\mathcal{K}_2(\xi_1, \tau_1, \xi_2, \tau_2)\|^2_{L^2_{\xi_1} L^2_{\tau_1}} &= \int_{\mathbb{R}} \langle \xi_1 \rangle^{2\theta} \langle \xi_2 \rangle \left(\int_{\mathbb{R}} \frac{1}{\langle \sigma_1 \rangle^{-2b'} \langle \sigma_2 \rangle^{2b}} d\tau_1 \right) d\xi_1 \\ &\lesssim \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{2\theta} \langle \xi_2 \rangle}{\langle \tau - \xi_1^3 - \alpha \xi_2^3 \rangle^{-2b'}} d\xi_1 \\ &\lesssim 1, \end{aligned} \tag{2.15}$$

where in the last inequality was used the Corollary 1. Analogously we prove the inequality (2.12) using the Corollary 2.

LEMMA 6. *Let* $b_1 > 0$, $b_2 > 0$, $b_1 + b_2 > 1/2$, *and* $c = \min\{2b_1, 2b_2, 2b_1 + 2b_2 - 1\} > 1/3$, *then*

$$\|F_{b_1} f_1 \cdot G_{b_2} f_2\|_{L^2_x L^2_t} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2}. \tag{2.16}$$

Proof. In order to prove (2.16) we observe that it is enough to estimate the L^2 norm of

$$\mathcal{K}_2(\xi_1, \tau_1, \xi_2, \tau_2) = \frac{1}{\langle \sigma_1 \rangle^{b_1} \langle \sigma_2 \rangle^{b_2}},$$

in this way

$$\begin{aligned} \|\mathcal{K}_2(\xi_1, \tau_1, \xi_2, \tau_2)\|^2_{L^2_{\xi_1} L^2_{\tau_1}} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{\langle \sigma_1 \rangle^{2b_1} \langle \sigma_2 \rangle^{2b_2}} d\tau_1 \right) d\xi_1 \\ &\lesssim \int_{\mathbb{R}} \frac{1}{\langle \tau - \xi_1^3 - \alpha \xi_2^3 \rangle^{c}} d\xi_1 \\ &\lesssim 1. \end{aligned} \tag{2.17}$$

LEMMA 7. *Assume that f, f_1, f_2 and f_3 belong to Schwartz space on $\mathbb{R}^2$. Then, we have*

$$\int_{\mathbb{R}^6} \overline{\widetilde{f}(\xi,\tau)}\widetilde{f_1}(\xi_1,\tau_1)\widetilde{f_2}(\xi_2,\tau_2)\widetilde{f_3}(\xi_3,\tau_3)d\xi_1 d\tau_1 d\xi_2 d\tau_2 d\xi d\tau = \int_{\mathbb{R}^2} \overline{f} f_1 f_2 f_3(x,t)dxdt,$$

where $\xi_3 = \xi - \xi_1 - \xi_2$ and $\tau_3 = \tau - \tau_1 - \tau_2$, i.e. $\xi_1+\xi_2+\xi_3 = \xi$ and $\tau_1+\tau_2+\tau_3 = \tau$.

Proof. Using the Fourier transform of the product of two functions

$$\begin{aligned}
\widetilde{f_1 f_2 f_3}(\xi,\tau) &= \widetilde{f_1} * \widetilde{f_2 f_3}(\xi,\tau)\\
&= \int_{\mathbb{R}^2} \widetilde{f_1}(\xi_1,\tau_1)\widetilde{f_2 f_3}(\xi-\xi_1,\tau-\tau_1)d\xi_1 d\tau_1\\
&= \int_{\mathbb{R}^2} \widetilde{f_1}(\xi_1,\tau_1)\int_{\mathbb{R}^2} \widetilde{f_2}(\xi_2,\tau_2)\widetilde{f_3}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi_1 d\tau_1\\
&= \int_{\mathbb{R}^4} \widetilde{f_1}(\xi_1,\tau_1)\widetilde{f_2}(\xi_2,\tau_2)\widetilde{f_3}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_1 d\tau_1 d\xi_2 d\tau_2.
\end{aligned}$$

The above equality and the Plancherel theorem give

$$\begin{aligned}
&\int_{\mathbb{R}^2} \overline{f} f_1 f_2 f_3(x,t)dxdt\\
&= \int_{\mathbb{R}^2} \overline{\widetilde{f}(\xi,\tau)}\widetilde{f_1 f_2 f_3}(\xi,\tau)d\xi d\tau\\
&= \int_{\mathbb{R}^6} \overline{\widetilde{f}(\xi,\tau)}\widetilde{f_1}(\xi_1,\tau_1)\widetilde{f_2}(\xi_2,\tau_2)\widetilde{f_3}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_1 d\tau_1 d\xi_2 d\tau_2 d\xi d\tau.
\end{aligned}$$

3. Proof of Proposition 3

We will only prove the inequality (2.3), the proof of the estimative (2.4) is similar. Using duality and Plancherel's identity we need to prove

$$\begin{aligned}
\mathscr{X} = &\int_{\mathbb{R}^6} K(\xi,\xi_1,\xi_2,\xi_3)\widetilde{F_{-b'}f}(\xi,\tau)\widetilde{F_b f_1}(\xi_1,\tau_1)\widetilde{G_b f_2}(\xi_2,\tau_2)\\
&\times \widetilde{G_b f_3}(\xi_3,\tau_3)d\xi_1 d\xi_2 d\tau_1 d\tau_2 d\xi d\tau \\
&\lesssim \|f\|_{L^2}\Pi_{j=1}^3\|f_j\|_{L^2},
\end{aligned} \tag{3.1}$$

where $\xi = \xi_1+\xi_2+\xi_3$, $\tau = \tau_1+\tau_2+\tau_3$, and

$$K(\xi,\xi_1,\xi_2,\xi_3) = \xi\langle\xi\rangle^s\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-k}\langle\xi_3\rangle^{-k}.$$

For simplicity in the notation we define:

$$F_{-b'} := F_{-b'}f, \quad F_b^1 := F_b f_1, \quad F_b^2 := G_b f_2, \quad F_b^3 := G_b f_3.$$

And also without loss of generality, we can assume that $\widehat{\overline{f}} \geqslant 0$ and $\widehat{f_j} \geqslant 0$, $j = 1,2,3$.

We divide the proof into the following cases:

Case 1. $|\xi| \leqslant 2|\xi_1|$. Then the kernel K is bounded by

$$\begin{aligned}|K(\xi,\xi_1,\xi_2,\xi_3)| &\lesssim \langle\xi\rangle^{1/2-10\varepsilon+(1/2+s+10\varepsilon)}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-k}\langle\xi_3\rangle^{-k}\\ &\lesssim \langle\xi\rangle^{1/2-10\varepsilon}\langle\xi_1\rangle^{1/2+s+10\varepsilon-s}\langle\xi_2\rangle^{1/2}\langle\xi_3\rangle^{1/2-12\varepsilon}\\ &\lesssim \langle\xi\rangle^{1/2-10\varepsilon}\langle\xi_2\rangle^{1/2}\langle\xi_1\rangle^{1/2+10\varepsilon}\langle\xi_3\rangle^{1/2-12\varepsilon},\end{aligned} \tag{3.2}$$

where $0<\varepsilon \ll \frac{1}{12}(\frac{1}{2}+k)$. Thus using the Lemma 7, we get

$$\begin{aligned}\mathscr{X} &\lesssim \int_{\mathbb{R}^2} J^{1/2-10\varepsilon}F_{-b'}\cdot J^{1/2}F_b^2\cdot J^{1/2+10\varepsilon}F_b^1\cdot J^{1/2-12\varepsilon}F_b^3(x,t)dxdt\\ &\lesssim \|J^{1/2-10\varepsilon}F_{-b'}\cdot J^{1/2}F_b^2\|_{L_x^2L_t^2}\|J^{1/2+10\varepsilon}F_b^1\cdot J^{1/2-12\varepsilon}F_b^3\|_{L_x^2L_t^2}\\ &\lesssim \|f\|_{L^2}\Pi_{j=1}^3\|f_j\|_{L^2}.\end{aligned} \tag{3.3}$$

Case 2. $|\xi_1| \leqslant \frac{1}{2}|\xi|$. By simmetry without loss of generality we can suppose $|\xi_3| \leqslant |\xi_2|$. Observe that in this case is $|\xi| \leqslant 4|\xi_2|$ and $|\xi_1| \leqslant 4|\xi_2|$.

Subcase 2.1) $-1/2<s<0$, $k>-1/2$. In this subcase if $k>0$ we choose $0<\varepsilon<\frac{-s}{10}$ and if $k<0$ we choose $0<\varepsilon<\min\{\frac{-s}{10},\frac{1+2k}{12}\}$, then we get $\langle\xi_1\rangle^{-s} = \langle\xi_1\rangle^{10\varepsilon}\langle\xi_1\rangle^{-s-10\varepsilon} \leqslant \langle\xi_1\rangle^{10\varepsilon}\langle\xi\rangle^{-s-10\varepsilon}$ and

$$\begin{aligned}|K(\xi,\xi_1,\xi_2,\xi_3)| &\lesssim \langle\xi\rangle^{1+s}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-k}\langle\xi_3\rangle^{-k}\\ &\lesssim \langle\xi\rangle^{1-10\varepsilon}\langle\xi_1\rangle^{10\varepsilon}\langle\xi_2\rangle^{1-12\varepsilon}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathscr{X} &\lesssim \int_{\mathbb{R}^2} J^{1-10\varepsilon}F_{-b'}\cdot F_b^3\cdot J^{10\varepsilon}F_b^1\cdot J^{1-12\varepsilon}F_b^2(x,t)dxdt\\ &\lesssim \|J^{1-10\varepsilon}F_{-b'}\cdot F_b^3\|_{L_x^2L_t^2}\|J^{10\varepsilon}F_b^1\cdot J^{1-12\varepsilon}F_b^2\|_{L_x^2L_t^2}\\ &\lesssim \|f\|_{L^2}\Pi_{j=1}^3\|f_j\|_{L^2}.\end{aligned} \tag{3.5}$$

Subcase 2.2) $s \geqslant 0$, $s-k<1$, $s-2k<1$. Without loss of generality we can suppose $|\xi_3| \leqslant |\xi_2|$.

We will consider the followings subcases

Subcase 2.2 *i*) $k \geqslant 0$. If $0<\varepsilon<\frac{1-(s-k)}{12}$, we have

$$\begin{aligned}|K(\xi,\xi_1,\xi_2,\xi_3)| &\leqslant \langle\xi\rangle^{1-10\varepsilon+s+10\varepsilon}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-k}\langle\xi_3\rangle^{-k}\\ &\lesssim \langle\xi\rangle^{1-10\varepsilon}\langle\xi_2\rangle^{s-k+10\varepsilon}\langle\xi_3\rangle^{-k}\\ &\lesssim \langle\xi\rangle^{1-10\varepsilon}\langle\xi_2\rangle^{1-2\varepsilon}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\mathscr{X} &\lesssim \int_{\mathbb{R}^2} J^{1-10\varepsilon}F_{-b'}\cdot F_b^3\cdot F_b^1\cdot J^{1-2\varepsilon}F_b^2(x,t)dxdt\\ &\lesssim \|J^{1-10\varepsilon}F_{-b'}\cdot F_b^3\|_{L_x^2L_t^2}\|F_b^1\cdot J^{1-2\varepsilon}F_b^2\|_{L_x^2L_t^2}\\ &\lesssim \|f\|_{L^2}\Pi_{j=1}^3\|f_j\|_{L^2}.\end{aligned} \tag{3.7}$$

Subcase 2.2 *ii*) $k \leqslant 0$. In this subcase, if $0 < \varepsilon < \frac{1-(s-2k)}{12}$ we have

$$\begin{aligned} |K(\xi,\xi_1,\xi_2,\xi_3)| &\leqslant \langle\xi\rangle^{1-10\varepsilon+s+10\varepsilon}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-k}\langle\xi_3\rangle^{-k} \\ &\lesssim \frac{\langle\xi\rangle^{1-10\varepsilon}\langle\xi_2\rangle^{s-k+10\varepsilon}\langle\xi_3\rangle^{-k}}{\langle\xi_1\rangle^{s}} \\ &\lesssim \frac{\langle\xi\rangle^{1-10\varepsilon}\langle\xi_2\rangle^{s-2k+10\varepsilon}}{\langle\xi_1\rangle^{s}} \\ &\lesssim \langle\xi\rangle^{1-10\varepsilon}\langle\xi_2\rangle^{1-2\varepsilon}, \end{aligned} \tag{3.8}$$

as above, and we also have

$$\mathscr{X} \lesssim \|f\|_{L^2}\Pi_{j=1}^{3}\|f_j\|_{L^2}. \tag{3.9}$$

Now, the proof of Theorem follows from the trilinear estimates (2.3), (2.4) and fixed point argument, see [6], [7] for details.

REFERENCES

[1] M. Ablowitz, D. Kaup, A. Newell and H. Segur, *Nonlinear evolution equations of physical significance*, Phys. Rev. Lett. **31** (2) (1973) 125–127.

[2] E. Alarcon, J. Angulo and J. F. Montenegro, *Stability and instability of solitary waves for a nonlinear dispersive system*, Nonlinear Analysis **36** (1999) 1015–1035.

[3] B. Alvarez, X. Carvajal, *On the local well-posedness for some systems of coupled KdV equations*, Nonlinear Analysis **69**, (2008) 692–715.

[4] D. Bekiranov, T. Ogawa, G. Ponce, *Weak solvability and well-posedness of a coupled Schrödinger-Korteweg de Vries equation for capillary-gravity wave interactions*, Proceedings of the American Mathematical Society **125** (1997) 2907–2919.

[5] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II, The KdV-equation*, Geom. Funct. Anal. **3** (1993) 209–262.

[6] X. Carvajal, L. Esquivel, R. Santos, *Sharp well-posedness for a coupled system of mkdv type equations*, (2020) arXiv:2003.12619v1.

[7] X. Carvajal, M. Panthee, *Sharp well-posedness for a coupled system of mkdv type equations*, J. Evol. Equ. **19**, (2019), 1167–1197.

[8] X. Carvajal, *Sharp global well-posedness for a higher order Schrödinger equation*, J. Fourier Anal. Appl. **12** (2006) no. 1, 53–70.

[9] X. Carvajal, *Local well-posedness for a higher order nonlinear Schrödinger equation in Sobolev spaces of negative indices*, Electron. J. Differential Equations **13** (2004) 1–13.

[10] M. Chen, B. Guo, *Local Well and Ill Posedness for the Modified KdV Equations in Subcritical Modulation Spaces*, (2018) arXiv:1811.05182.

[11] M. Christ, J. Holmer, D. Tataru, *Low regularity a priori bounds for the modified Korteweg-de Vries equation*, Libertas Mathematica **32** (2012) no. 1 51–75.

[12] A. Corcho, M. Panthee, *Global well-posedness for a coupled modified KdV system*, Bull Braz Math Soc, New Series **43**, (2012) 27–57.

[13] J. Ginibre, Y. Tsutsumi, G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151** (1997), no. 2, 384–436.

[14] Z. Guo, *Global well-posedness of Korteweg-de Vries equation in $H^{-3/4}(\mathbb{R})$*, J. Math. Pures Appl. **91** (2009) 583–597.

[15] T. Oh, *Diophantine conditions in well-posedness theory of coupled KdV-type systems: local theory*, Int. Math. Res. Not. IMRN (2009), no. 18, 3516–3556.

[16] C. E. Kenig, G. Ponce, L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **92** (1996) 573–603.

[17] C. E. KENIG, G. PONCE, L. VEGA, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993) 527–620.

[18] C. E. KENIG, G. PONCE, L. VEGA, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. **71** (1993), no. 1, 1–21.

[19] R. KILLIP, M. VISAN, *KdV is well-posed in* H^{-1}, Ann. of Math. **190**, no. 1 (2019) 249–305.

[20] NOBU KISHIMOTO, *Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity*, Differential Integral Equations **22** (2009), no. 5–6, 447–464.

[21] T. TAO, *Multilinear weighted convolution of* L^2*-functions, and applications to nonlinear dispersive equations*, Amer. J. Math. **123** (2001) no. 5, 839–908.

[22] J. F. MONTENEGRO, *Sistemas de equações de evolução não lineares; Estudo local, global e estabilidade de ondas solitarias*, Ph. D. Thesis, IMPA, Rio de Janeiro (1995).

[23] H. TAKAOKA, *Well-posedness for the higher order nonlinear Schrödinger equation*, Adv. Math. Sci. Appl. **10** (2000) 149–171.

(Received June 23, 2020)

Xavier Carvajal
Instituto de Matemática
Universidade Federal do Rio de Janeiro
Ilha do Fundão, 21945-970, Rio de Janeiro, Brazil
e-mail: carvajal@im.ufrj.br