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UNIQUE SOLVABILITY OF FRACTIONAL QUADRATIC NONLINEAR INTEGRAL EQUATIONS

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Abstract. In this paper we study the existence of monotonic solutions of fractional nonlinear quadratic integral equations in the space of Lebesgue integrable functions on $[0,\tau]$. The uniqueness of the solution is also discussed. In addition an example is given to illustrate our abstract results.

1. Introduction

In [19], the author studied the existence of a unique bounded continuous and non-negative solution of the equation

$$x(t) = k\left(h(t) + \int_0^t A(t-s)x(s)ds\right)\cdot\left(g(t) + \int_0^t B(t-s)x(s)ds\right), \tag{1.1}$$

and this equation arises in the spread of an infectious disease that does not induce permanent immunity (see, for example [3, 20]). In [28], a new integral inequality was used to study the boundedness, the asymptotic behavior and the growth of the solutions of (1.1) and in [1, 29], some integral inequalities are used to study the boundedness and the asymptotic behavior of continuous solutions of (1.1). The author in [27] studied the existence and uniqueness of continuous solutions of the general integral equation

$$x(t) = \prod_{i=1}^{m}\left(g_i(t) + \int_a^t K_i(t,s,x(s))ds\right),\ \ t\in[a,b],$$

where K_i is Lipschitz for $i = 1,\cdots,m$ and in [6], the authors used the measure of noncompactness to discuss the solvability of the integral equations

$$x(t) = u(t,x(t)) + \left(h(t) + \int_0^t k_1(t,s)f_1(s,x(s))ds\right)\cdot\left(g(t) + \int_0^t k_2(t,s)f_2(s,x(s))ds\right)$$

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in the Banach space of real functions being integrable on [0,1].

In this paper, we discuss the existence of monotonic solutions in the space $L_1[0,\tau]$ (the space of Lebesgue integrable functions on $[0,\tau]$), for the fractional nonlinear quadratic integral equations, namely

$$x(t)=[h_1(t)+g(t)\cdot(Tx)(t)]\cdot\left[h_2(t)+\frac{|x(t)|^{\frac{1}{p}}}{\Gamma(\alpha)}\int_0^t\frac{f(s,x(s))}{(t-s)^{1-\alpha}}ds\right], \tag{1.2}$$

where $T(x)$ is a general operator and $p>1$. Note that (1.2) contains as particular cases many integral and functional-integral equations which arise in real world problems in mechanics, economics, and physics (see [5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 25, 26]). To establish existence we apply the fixed point theorem due to Darbo associated with the measure of weak noncompactness. Under suitable assumptions we study the uniqueness of the solution of (1.2) as well. Finally, we present an example to illustrate our abstract results.

2. Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+$ be the interval $[0,\infty)$, $J=[0,\tau]$ be a fixed interval and let $L_p(J)$, $1\leqslant p<\infty$ be the space of Lebesgue integrable functions with the norm $\|x\|_{L_p(J)}=(\int_J|x(s)|^p\,ds)^{\frac{1}{p}}$. We will write L_1, L_p and L_q instead of $L_1(J), L_p(J)$ and $L_q(J)$.

Let $S=S(J)$ denote the set of measurable (in Lebesgue sense) functions on J and let *meas* stand for the Lebesgue measure on J. Identifying the functions equal almost everywhere the set S furnished with the metric

$$d(x,y)=\inf_{a>0}\left[a+meas(\{s:|x(s)-y(s)|\geqslant a\})\right],$$

becomes a complete metric space. Moreover, the convergence in measure on J is equivalent to the convergence with respect to the metric d (Proposition 2.14 in [30]). The compactness in such a space is called "compactness in measure".

THEOREM 1. *Let X be a bounded subset of L_1 and suppose that there is a family of measurable subsets $(\Omega_c)_{0\leqslant c\leqslant 1}$ of the interval J such that $meas(\Omega_c)=c$ for every $c\in J$ and for $x\in X$*

$$x(t_1)\geqslant x(t_2);\ t_1\in\Omega_c,\ t_2\notin\Omega_c.$$

Then the set X is compact in measure in L_1.

Now we present the concept of measure of noncompactness. Assume that $(E,\|\cdot\|)$ is an arbitrary Banach space with zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta,r)$. Denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ ($\mathcal{N}_E^W$) its subfamily consisting of all relatively (weakly relatively) compact sets. The symbols $\bar{X}$ and $\bar{X}^W$ stand for the closure and the weak closure of a set X, respectively and the symbol Conv$X$ will denote the convex closed hull of a set X.

DEFINITION 1. [4] A mapping $\mu : \mathscr{M}_E \to \mathbb{R}_+$ is said to be a regular measure of noncompactness in E if it satisfies the following conditions:

(i) $\mu(X) = 0 \iff X \in \mathscr{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.

(iii) $\mu(\bar{X}) = \mu(\text{Conv}X) = \mu(X)$.

(iv) $\mu(\lambda X) = |\lambda|\mu(X),\ \lambda \in \mathbb{R}$.

(v) $\mu(X+Y) \leqslant \mu(X) + \mu(Y)$.

(vi) $\mu(X \bigcup Y) = \max\{\mu(X), \mu(Y)\}$.

(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An example of such a mapping is the following:

DEFINITION 2. [4] Let X be a nonempty and bounded subset of E. The Hausdorff measure of noncompactness $\chi(X)$ is defined as

$$\chi(X) = \inf\{r > 0 : \text{ there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + B_r\}.$$

DEFINITION 3. [5] A mapping $\mu : \mathscr{M}_E \to \mathbb{R}_+$ is said to be a regular measure of weak noncompactness in E if it satisfies conditions $(ii) - (vi)$ of Definition 1 and the following two conditions (being counterparts of (i) and (vii)) hold:

(i') $\mu(X) = 0 \iff X \in \mathscr{N}_E^W$.

(vii') If X_n is a sequence of nonempty, bounded, weakly closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Consider a nonempty and bounded subset X of the space L_1. For any $\varepsilon > 0$, let c be a measure of equiintegrability of the set X (the so-called Sadovskii functional [2, p. 39]) i.e.

$$c(X) = \lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left[\int_D |x(t)|\,dt : D \subset J,\ meas(D) \leqslant \varepsilon\right]\right\}\right\}. \tag{2.1}$$

It forms a regular measure of noncompactness if restricted to the family of subsets being compact in measure (cf. [18]).

Next, we discuss some properties of operators acting on different function spaces.

DEFINITION 4. [2] Assume that a function $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in J$. Then to every function $x(t)$ being measurable on J we may assign the function

$$F_f(x)(t) = f(t, x(t)),\ t \in J.$$

The operator F_f is called the superposition (Nemytskii) operator generated by the function f.

THEOREM 2. [2] *Suppose f satisfies the Carathéodory conditions. The superposition operator F_f generated by the function f maps continuously the space L_p into L_q $(p,q \geqslant 1)$ if and only if*

$$|f(t,x)| \leqslant a(t) + b|x|^{\frac{p}{q}}, \tag{2.2}$$

for all $t \in J$ and $x \in \mathbb{R}$, where $a \in L_q$ and $b \geqslant 0$.

Let us recall some properties of operators preserving monotonicity properties of functions.

LEMMA 1. [10] *Suppose the function $t \to f(t,x)$ is a.e. nondecreasing on a finite interval J for each $x \in \mathbb{R}$ and the function $x \to f(t,x)$ is a.e. nondecreasing on $\mathbb{R}$ for any $t \in J$. Then the superposition operator F_f generated by f transforms functions being a.e. nondecreasing on J into functions having the same property.*

LEMMA 2. [21, Lemma 17.5] *Assume that a function $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator F_f maps a sequence of functions convergent in measure into a sequence of functions convergent in measure.*

For the integral operator of the form $(Ku)(t) = \int_J k(t,s)u(s)\, ds$ we have the following theorem due to Krzyż ([22, Theorem 6.2]):

THEOREM 3. *The operator K preserve the monotonicity of functions if and only if*

$$\int_0^l k(t_1,s)\, ds \geqslant \int_0^l k(t_2,s)\, ds$$

for $t_1 < t_2$, $t_1,t_2 \in J$ and for any $l \in J$.

DEFINITION 5. [23] Let $f \in L_1$, and $\alpha \in \mathbb{R}_+$. The Riemman-Liouville (R-L) fractional integral of the function f of order α is defined as

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,\ \alpha > 0,\ t > 0,$$

where $\Gamma(\alpha)$ is the Euler's gamma function.

We state here some results concerning the above mentioned operators, that are relevant to our work (cf. [23, 24]).

PROPOSITION 1. *For $\alpha \in \mathbb{R}_+$, we have*

(a) *The operator I^α maps L_p into itself continuously.*

(b) *I^α maps the nonnegative and a.e. nondecreasing functions into functions of the same type.*

In our approach we will need the following fixed point theorem due to Darbo [4].

THEOREM 4. *Let C be a nonempty, bounded, closed, and convex subset of E and let $H : C \to C$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists $k \in [0,1)$ such that*

$$\mu(H(X)) \leqslant k\mu(X),$$

for any nonempty subset X of C. Then H has at least one fixed point in the set C.

3. Main results

First, we rewrite equation (1.2) in the form

$$x = (Hx) = (Ax) \cdot (Bx), \tag{3.1}$$

where

$$(Ax)(t) = h_1(t) + g(t) \cdot (Tx)(t),$$

$$(Bx)(t) = h_2(t) + |x(t)|^{\frac{1}{p}} I^{\alpha} F_f(x)(t),$$

(Tx) is a general operator, F_f is the superposition operator as in Definition 4 and I^{α} is as in Definition 5.

Let $\frac{1}{p} + \frac{1}{q} = 1,\ p > 1$ and $\alpha > \frac{1}{p}$. We make the following assumptions:

(i) $g, h_1, h_2 : J \to \mathbb{R}_+$ are a.e. nondecreasing functions, where g is bounded function with $M = \sup_{t\in J} |g(t)|$, $h_1 \in L_q$ and $h_2 \in L_p(J)$, $J = [0, \tau]$.

(ii) The operator $T : L_1 \to L_q$ is continuous and maps a.e. nondecreasing functions into functions of the same type. Moreover, $Tx \geqslant 0$ a.e. for $x \in L_1$ and there exist a function $a_1 \in L_q$ and a nonnegative constant b_1 such that

$$|(Tx)(t)| \leqslant a_1(t) + b_1|x(t)|^{\frac{1}{q}} \text{ a. e. } t \in J,\ x \in L_1(J). \tag{3.2}$$

(iii) Assume that the function $f : J \times \mathbb{R} \to \mathbb{R}$, satisfies the Carathéodory conditions. Moreover, $f(t,x) \geqslant 0$ for $(t,x) \in J \times \mathbb{R}$ and f is assumed to be nondecreasing with respect to both variable t and x separately. Moreover, there are a nonnegative constant b_2 and a nonnegative function $a_2 \in L_p$ such that

$$|f(t,x)| \leqslant a_2(t) + b_2|x|^{\frac{1}{p}} \text{ a. e. } t \in J,\ x \in \mathbb{R}. \tag{3.3}$$

(iv) Assume there exists a number $r > 0$ with

$$Mb_1\gamma\left(\|a_2\|_{L_p} + b_2 r^{\frac{1}{p}} \right) < 1$$

and

$$\left(\|h_1\|_{L_q}+M\|a_1\|_{L_q}\right)\|h_2\|_{L_p}+Mb_1\|h_2\|_{L_p}r^{\frac{1}{q}}+\left(\|h_1\|_{L_q}+M\|a_1\|_{L_q}\right)\gamma\|a_2\|_{L_p}r^{\frac{1}{p}}$$
$$+\gamma b_2\left(\|h_1\|_{L_q}+M\|a_1\|_{L_q}\right)\cdot r^{\frac{2}{p}}+M\gamma b_1 b_2\cdot r^{1+\frac{1}{p}}+Mb_1\gamma\|a_2\|_{L_p}\cdot r\leqslant r,$$

where $\gamma=\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}$.

REMARK 1. Assumption (iv) takes the form $C+Dr^{\frac{1}{q}}+Er^{\frac{1}{p}}+Gr^{\frac{2}{p}}+Ir^{\frac{1}{p}+1}+Kr\leqslant r$. For example for $r=1$ we would need $C+D+E+G+I+K\leqslant 1$.

REMARK 2. Assume $x\in L_1$ and $z\in L_p$. Then we have

$$\left\||x(t)|^{\frac{1}{p}}I^{\alpha}z(t)\right\|_{L_p}\leqslant\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|z\|_{L_p}\|x\|_{L_1}^{\frac{1}{p}}.$$

Indeed, we have

$$\begin{aligned}\left\||x(t)|^{\frac{1}{p}}I^{\alpha}z(t)\right\|_{L_p}&=\left\||x(t)|^{\frac{1}{p}}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)\,ds\right\|_{L_p}\\&\leqslant\left\|\frac{|x(t)|^{\frac{1}{p}}}{\Gamma(\alpha)}\left(\int_0^t(t-s)^{q(\alpha-1)}\,ds\right)^{\frac{1}{q}}\|z\|_{L_p}\right\|_{L_p}\\&=\left\|\frac{|x(t)|^{\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{t^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}}\|z\|_{L_p}\right\|_{L_p}\\&=\left\|\frac{|x(t)|^{\frac{1}{p}}}{\Gamma(\alpha)}t^{\frac{\alpha p-1}{p}}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|z\|_{L_p}\right\|_{L_p}\\&\leqslant\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|z\|_{L_p}\left\||x|^{\frac{1}{p}}\right\|_{L_p}\\&\leqslant\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|z\|_{L_p}\|x\|_{L_1}^{\frac{1}{p}},\end{aligned}$$

where we used $q=\frac{p}{p-1}$. Similarly if $D\subset J$, we have

$$\left\||x(t)|^{\frac{1}{p}}I^{\alpha}z(t)\right\|_{L_p(D)}\leqslant\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|z\|_{L_p}\|x\|_{L_1(D)}^{\frac{1}{p}}.$$

THEOREM 5. *Let assumptions $(i)-(iv)$ be satisfied. Then (1.2) has at least one integrable solution a.e. nondecreasing on J.*

Proof. From assumptions (iii) and Theorem 2 we have that F_f maps L_1 into L_p continuously and from Proposition 1 and Remark 2, we have that the operator B maps L_1 into L_p continuously. From assumptions (i) and (ii), we deduce that the operator A maps L_1 into L_q and is continuous. Finally, the Hölder inequality implies that the operator H maps L_1 into itself continuously.

Using equation (3.1) and Remark 2 with assumptions $(i)-(iii)$, we have for $x \in L_1$ that

$$
\begin{aligned}
&\|Hx\|_{L_1} \\
&\leqslant \|(Ax)\cdot(Bx)\|_{L_1} \\
&\leqslant \|Ax\|_{L_q}\,\|Bx\|_{L_p} \\
&\leqslant \|h_1 + g\cdot(Tx)\|_{L_q}\,\left\|h_2 + |x|^{\frac{1}{p}} I^{\alpha} F_f(x)\right\|_{L_p} \\
&\leqslant \left(\|h_1\|_{L_q} + M\|a_1 + b_1|x|^{\frac{1}{q}}\|_{L_q}\right) \\
&\qquad \times \left(\|h_2\|_{L_p} + \left\| |x(t)|^{\frac{1}{p}} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\,ds \right\|_{L_p}\right) \\
&\leqslant \left(\|h_1\|_{L_q} + M\|a_1\|_{L_q} + Mb_1 \left(\int_0^{\tau} |(x(t))^{\frac{1}{q}}|^q dt\right)^{\frac{1}{q}}\right) \\
&\qquad \times \left(\|h_2\|_{L_p} + \frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}} \|f(\cdot,x(\cdot))\|_{L_p} \|x\|_{L_1}^{\frac{1}{p}}\right) \\
&\leqslant \left(\|h_1\|_{L_q} + M\|a_1\|_{L_q} + Mb_1 \left(\int_0^{\tau} |x(t)|\,dt\right)^{\frac{1}{q}}\right) \\
&\qquad \times \left(\|h_2\|_{L_p} + \gamma\|x\|_{L_1}^{\frac{1}{p}} \left\| a_2 + b_2|x|^{\frac{1}{p}} \right\|_{L_p}\right) \\
&\leqslant \left(\|h_1\|_{L_q} + M\|a_1\|_{L_q} + Mb_1\|x\|_{L_1}^{\frac{1}{q}}\right)\left(\|h_2\|_{L_p} + \gamma\|x\|_{L_1}^{\frac{1}{p}}\left[\|a_2\|_{L_p} + b_2\|x\|_{L_1}^{\frac{1}{p}}\right]\right).
\end{aligned}
$$

Thus $H : L_1 \to L_1$. Let r be as in assumption (iv) and let $x \in B_r$, where $B_r = \{m \in L_1 : \|m\|_{L_1} \leqslant r\}$. Then

$$
\|Hx\|_{L_1} \leqslant \left(\|h_1\|_{L_q} + M\|a_1\|_{L_q} + Mb_1\cdot r^{\frac{1}{q}}\right)\left(\|h_2\|_{L_p} + \gamma\cdot r^{\frac{1}{p}}\left[\|a_2\|_{L_p} + b_2\cdot r^{\frac{1}{p}}\right]\right) \leqslant r.
$$

Thus $H : B_r \to B_r$ (and is continuous).

Further, let Q_r is a subset of B_r which has the functions a.e. nondecreasing on J. A standard argument (see for example [24]) guarantees that this set is nonempty, bounded (by r), convex and closed in L_1. In view of Theorem 1 the set Q_r is compact in measure.

Now, we will show that H preserves the monotonicity of functions. Take $x \in Q_r$. Then $x(t)$ is a.e. nondecreasing on J and consequently f is also of the same type from assumption (iii). In addition, I^{α} is a.e. nondecreasing on J from Proposition

1. Moreover, $(Tx)(t), (Ax)(t)$ and $(Bx)(t)$ are also of the same type. Thus we can deduce that $(Hx) = (Ax)(Bx)$ is also a.e. nondecreasing on J. Then $H : Q_r \to Q_r$ and is continuous.

Now we assume that X is a nonempty subset of Q_r and the constant $\varepsilon > 0$ is arbitrary, but fixed. Then for an arbitrary $x \in X$ and for a set $D \subset J$ with measD $\leqslant \varepsilon$ we obtain

$$\begin{aligned}
\|Hx\|_{L_1(D)} &= \int_D |(Hx)(t)|dt \leqslant \|(Ax)\cdot(Bx)\|_{L_1(D)} \\
&\leqslant \|(Ax)\|_{L_q(D)} \cdot \|(Bx)\|_{L_p(D)} \\
&\leqslant \|h_1 + g\cdot(Tx)\|_{L_q(D)} \cdot \left\| h_2 + |x|^{\frac{1}{p}} I^{\alpha} F_f(x) \right\|_{L_p(D)} \\
&\leqslant \left(\|h_1\|_{L_q(D)} + M\|a_1 + b_1|x|^{\frac{1}{q}}\|_{L_q(D)} \right) \\
&\quad \times \left(\|h_2\|_{L_p(D)} + \left\| |x(t)|^{\frac{1}{p}} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \right\|_{L_p(D)} \right) \\
&\leqslant \left(\|h_1\|_{L_q(D)} + M\|a_1\|_{L_q(D)} + b_1 M \left(\int_D |x(t)|dt \right)^{\frac{1}{q}} \right) \\
&\quad \times \left(\|h_2\|_{L_p(D)} + \frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1} \right)^{1-\frac{1}{p}} \|f(\cdot,x(\cdot))\|_{L_p} \|x\|_{L_1(D)}^{\frac{1}{p}} \right) \\
&\leqslant \left(\|h_1\|_{L_q(D)} + M\|a_1\|_{L_q(D)} + b_1 M \|x\|_{L_1(D)}^{\frac{1}{q}} \right) \\
&\quad \times \left(\|h_2\|_{L_p(D)} + \gamma \left(\|a_2\|_{L_p} + b_2 \|x\|_{L_1}^{\frac{1}{p}} \right) \|x\|_{L_1(D)}^{\frac{1}{p}} \right) \\
&\leqslant \left(\|h_1\|_{L_q(D)} + M\|a_1\|_{L_q(D)} + b_1 M \|x\|_{L_1(D)}^{\frac{1}{q}} \right) \\
&\quad \times \left(\|h_2\|_{L_p(D)} + \gamma \left(\|a_2\|_{L_p} + b_2 r^{\frac{1}{p}} \right) \|x\|_{L_1(D)}^{\frac{1}{p}} \right).
\end{aligned}$$

Since $h_1, a_1 \in L_q$ and $h_2 \in L_p$, we have the equalities

$$\lim_{\varepsilon\to 0}\{\sup_{x\in X}\{\sup[\ \|h_1\|_{L_q(D)} + M\|a_1\|_{L_q(D)} :\ D \subset J,\ meas(D) \leqslant \varepsilon]\}\} = 0$$

and

$$\lim_{\varepsilon\to 0}\{\sup_{x\in X}\{\sup[\ \|h_2\|_{L_p(D)} :\ D \subset J,\ meas(D) \leqslant \varepsilon]\}\} = 0.$$

From formula (2.1), we get

$$c(H(X)) \leqslant M b_1 \gamma \left(\|a_2\|_{L_p} + b_2 r^{\frac{1}{p}} \right) \cdot c(X).$$

Recall that $M b_1 \gamma \left(\|a_2\|_{L_p} + b_2 r^{\frac{1}{p}} \right) < 1$ and the inequality obtained above together with the properties of the operator H on the set Q_r (see also the description before Definition 4) allow us to apply Theorem 4 which completes the proof. □

Next, we discuss the uniqueness of solutions.

THEOREM 6. *Let the assumptions of Theorem 5 be satisfied with replacing (3.2) and (3.3) by the following assumptions*

(v)

$$|T(x)(0)| \leqslant a_1(t), \;\; |T(x)-T(y)| \leqslant b_1|x-y|^{\frac{1}{q}}$$

and

$$|f(t,0)| \leqslant a_2(t), \;\; |f(t,x)-f(t,y)| \leqslant b_2|x-y|^{\frac{1}{p}}.$$

(vi) If for any constant $W \geqslant 0$*, we have*

$$\begin{aligned} W \leqslant\ & Mb_1\|h_2\|_{L_p}W^{\frac{1}{q}} + b_2\gamma r^{\frac{1}{p}}\|h_1\|_{L_q}W^{\frac{1}{p}} + \gamma\|h_1\|_{L_q}\big(\|a_2\|_{L_p}+b_2r^{\frac{1}{p}}\big)W^{\frac{1}{p}} \\ & + Mb_2\gamma r^{\frac{1}{p}}\big(\|a_1\|_{L_q}+b_1r^{\frac{1}{q}}\big)W^{\frac{1}{p}} \\ & + M\gamma\big(\|a_1\|_{L_q}+b_1r^{\frac{1}{q}}\big)\big(\|a_2\|_{L_p}+b_2\|y\|_{L_1}^{\frac{1}{p}}\big)W^{\frac{1}{p}} \\ & + Mb_1\gamma r^{\frac{1}{p}}\big(\|a_2\|_{L_p}+b_2r^{\frac{1}{p}}\big)W^{\frac{1}{q}}, \text{ then } W=0. \end{aligned}$$

Then (1.2) has a unique solution in Q_r *where* r *is given in assumption (iv).*

Proof. From assumption (v), we have

$$\begin{aligned} ||f(t,x)|-|f(t,0)|| &\leqslant |f(t,x)-f(t,0)| \leqslant b_2|x|^{\frac{1}{p}} \\ &\Rightarrow |f(t,x)| \leqslant |f(t,0)|+b_2|x|^{\frac{1}{p}} \leqslant a_2(t)+b_2|x|^{\frac{1}{p}}. \end{aligned}$$

Similarly, we have $|(Tx)| \leqslant a_1(t)+b_1|x|^{\frac{1}{q}}$. Then all assumptions of Theorem 5 are satisfied, and therefore (1.2) has at least one solution $x \in Q_r$.

Now, let x and y be two solutions of (1.2) in Q_r. Then

$$\begin{aligned} & |x(t)-y(t)| \\ = & \Bigg|[h_1(t)+g(t)\cdot(Tx)(t)]\left(h_2(t)+|x(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds\right) \\ & -[h_1(t)+g(t)\cdot(Ty)(t)]\left(h_2(t)+|y(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds\right)\Bigg| \\ \leqslant & |g(t)|\cdot|h_2(t)|\cdot|(Tx)(t)-(Ty)(t)| \\ & +|h_1(t)||x(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s)) - f(s,y(s))|ds \\ & +|h_1(t)|\Big||x(t)|^{\frac{1}{p}}-|y(t)|^{\frac{1}{p}}\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|ds \end{aligned}$$

$$
\begin{aligned}
&+|g(t)||(Tx)(t)||x(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,|f(s,x(s))\ -\ f(s,y(s))|\,ds\\
&+|g(t)||(Tx)(t)|\left||x(t)|^{\frac{1}{p}}-|y(t)|^{\frac{1}{p}}\right|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|ds\\
&+|g(t)|\left|(Tx)(t)\ -\ (Ty)(t)\right||y(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|ds\\
\leqslant\ & M\cdot|h_2(t)|\cdot b_1|x(t)-y(t)|^{\frac{1}{q}}+|h_1(t)||x(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}b_2|x(s)\ -\ y(s)|^{\frac{1}{p}}ds\\
&+|h_1(t)|\left|x(t)-y(t)\right|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(a_2(s)+b_2|y(s)|^{\frac{1}{p}}\big)ds\\
&+M\big(a_1(t)+b_1|x(t)|^{\frac{1}{q}}\big)|x(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}b_2|x(s)\ -\ y(s)|^{\frac{1}{p}}ds\\
&+M\big(a_1(t)+b_1|x(t)|^{\frac{1}{q}}\big)\left|x(t)-y(t)\right|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(a_2(s)+b_2|y(s)|^{\frac{1}{p}}\big)ds\\
&+Mb_1\left|x(t)\ -\ y(t)\right|^{\frac{1}{q}}|y(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(a_2(s)+b_2|y(s)|^{\frac{1}{p}}\big)ds,
\end{aligned}
$$

since $\left||x|^{\frac{1}{p}}-|y|^{\frac{1}{p}}\right|\leqslant|x-y|^{\frac{1}{p}}$. Therefore,

$$
\begin{aligned}
&\|x-y\|_{L_1}\\
\leqslant\ & Mb_1\|h_2\|_{L_p}\big\||x-y|^{\frac{1}{q}}\big\|_{L_q}+b_2\|h_1\|_{L_q}\left\||x(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|^{\frac{1}{p}}ds\right\|_{L_p}\\
&+\|h_1\|_{L_q}\left\|\left|x(t)-y(t)\right|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(a_2(s)+b_2|y(s)|^{\frac{1}{p}}\big)ds\right\|_{L_p}\\
&+Mb_2\big\|a_1+b_1|x|^{\frac{1}{q}}\big\|_{L_q}\left\||x(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)\ -\ y(s)|^{\frac{1}{p}}ds\right\|_{L_p}\\
&+M\big\|a_1+b_1|x|^{\frac{1}{q}}\big\|_{L_q}\left\|\left|x(t)-y(t)\right|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(a_2(s)+b_2|y(s)|^{\frac{1}{p}}\big)ds\right\|_{L_p}\\
&+Mb_1\big\||x(t)\ -\ y(t)|^{\frac{1}{q}}\big\|_{L_q}\left\||y(t)|^{\frac{1}{p}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(a_2(s)+b_2|y(s)|^{\frac{1}{p}}\big)ds\right\|_{L_p}\\
\leqslant\ & Mb_1\|h_2\|_{L_p}\|x-y\|_{L_1}^{\frac{1}{q}}+b_2\|h_1\|_{L_q}\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\big\||x-y|^{\frac{1}{p}}\big\|_{L_p}\|x\|_{L_1}^{\frac{1}{p}}\\
&+\|h_1\|_{L_q}\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|a_2+b_2|y|^{\frac{1}{p}}\|_{L_p}\|x-y\|_{L_1}^{\frac{1}{p}}\\
&+Mb_2\big(\|a_1\|_{L_q}+b_1\|x\|_{L_1}^{\frac{1}{q}}\big)\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\big\||x\ -\ y|^{\frac{1}{p}}\big\|_{L_p}\|x\|_{L_1}^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&+M\left(\|a_1\|_{L_q}+b_1\|x\|_{L_1}^{\frac{1}{q}}\right)\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|a_2+b_2|y|^{\frac{1}{p}}\|_{L_p}\|x-y\|_{L_1}^{\frac{1}{p}}\\
&+Mb_1\|x\ -\ y\|_{L_1}^{\frac{1}{q}}\frac{\tau^{\frac{\alpha p-1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|a_2+b_2|y|^{\frac{1}{p}}\|_{L_p}\|y\|_{L_1}^{\frac{1}{p}}\\
\leqslant\ & Mb_1\|h_2\|_{L_p}\|x-y\|_{L_1}^{\frac{1}{q}}+b_2\gamma r^{\frac{1}{p}}\|h_1\|_{L_q}\|x-y\|_{L_1}^{\frac{1}{p}}\\
&+\gamma\|h_1\|_{L_q}\left(\|a_2\|_{L_p}+b_2\|y\|_{L_1}^{\frac{1}{p}}\right)\|x-y\|_{L_1}^{\frac{1}{p}}+Mb_2\gamma r^{\frac{1}{p}}\left(\|a_1\|_{L_q}+b_1r^{\frac{1}{q}}\right)\|x\ -\ y\|_{L_1}^{\frac{1}{p}}\\
&+M\gamma\left(\|a_1\|_{L_q}+b_1r^{\frac{1}{q}}\right)\left(\|a_2\|_{L_p}+b_2\|y\|_{L_1}^{\frac{1}{p}}\right)\|x-y\|_{L_1}^{\frac{1}{p}}\\
&+Mb_1\gamma r^{\frac{1}{p}}\left(\|a_2\|_{L_p}+b_2\|y\|_{L_1}^{\frac{1}{p}}\right)\|x\ -\ y\|_{L_1}^{\frac{1}{q}}\\
\leqslant\ & Mb_1\|h_2\|_{L_p}\|x-y\|_{L_1}^{\frac{1}{q}}+b_2\gamma r^{\frac{1}{p}}\|h_1\|_{L_q}\|x-y\|_{L_1}^{\frac{1}{p}}\\
&+\gamma\|h_1\|_{L_q}\left(\|a_2\|_{L_p}+b_2r^{\frac{1}{p}}\right)\|x-y\|_{L_1}^{\frac{1}{p}}+Mb_2\gamma r^{\frac{1}{p}}\left(\|a_1\|_{L_q}+b_1r^{\frac{1}{q}}\right)\|x\ -\ y\|_{L_1}^{\frac{1}{p}}\\
&+M\gamma\left(\|a_1\|_{L_q}+b_1r^{\frac{1}{q}}\right)\left(\|a_2\|_{L_p}+b_2r^{\frac{1}{p}}\right)\|x-y\|_{L_1}^{\frac{1}{p}}\\
&+Mb_1\gamma r^{\frac{1}{p}}\left(\|a_2\|_{L_p}+b_2r^{\frac{1}{p}}\right)\|x\ -\ y\|_{L_1}^{\frac{1}{q}}.
\end{aligned}
$$

From the above inequality and assumption (vi), we deduce that $x=y$ (a.e), which completes the proof. □

EXAMPLE 1. For $t\in[0,1]$, consider the following integral equation

$$
x(t)=\left[t^{10}+t^5\left(t^3+\frac{1}{200}|x(t)|^{\frac{1}{2}}\right)\right]\left[\sqrt{t}e^{2t^2}+\frac{|x(t)|^{\frac{1}{2}}}{\Gamma(\frac{2}{3})}\int_0^t\frac{\sqrt{\ln\left(1+|x(t)|^2\right)}}{\sqrt[3]{t-s}}\,ds\right]. \tag{3.4}
$$

Let $p=q=2$. Then one can easily check that:

1. $h_1(t)=t^{10}\in L_2[0,1]$ and $\|h_1\|_{L_2[0,1]}=\frac{1}{\sqrt{21}}$.
2. $h_2(t)=\sqrt{t}e^{2t^2}\in L_2[0,1]$ and $\|h_2\|_{L_2[0,1]}=\frac{\sqrt{2}}{4}\sqrt{e^4-1}$.
3. $g(t)=t^5$ and $M=\sup_{0\leqslant t\leqslant 1}t^5=1$.
4. $|(Tx)(t)|\leqslant t^3+\frac{1}{200}|x|^{\frac{1}{2}}$, then $a_1(t)=t^3, b_1=\frac{1}{200}$.
5. $f(t,x)=\sqrt{\ln\left(1+(x(s))^2\right)}$ and $|f(t,x)|\leqslant|x|^{\frac{1}{2}}$, then $a_2(t)=0, b_2=1$.
6. Let $r=1$ and note that

$$
Mb_1\gamma\left(\|a_2\|_{L_2}+b_2r^{\frac{1}{p}}\right)=\frac{\sqrt{3}}{200\Gamma(\frac{2}{3})}<1
$$

and

$$\left(||h_1||_{L_2} + M\|a_1\|_{L_2} + Mb_1\right)\left(||h_2||_{L_2} + b_2\gamma\right)$$
$$= \left(\frac{1}{\sqrt{21}} + \frac{1}{\sqrt{7}} + \frac{1}{200}\right)\left(\frac{\sqrt{2}}{4}\sqrt{e^4 - 1} + \frac{\sqrt{3}}{\Gamma(\frac{2}{3})}\right) \leqslant 1.$$

Therefore, assumption (iv) holds.

Hence, using Theorem 5, we deduce that (3.4) has at least one integrable solution a.e. nondecreasing in $[0,1]$.

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