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MULTIPLE SOLUTIONS TO A THIRD–ORDER THREE–POINT NONHOMOGENEOUS BOUNDARY VALUE PROBLEM AIDED BY NONLINEAR PROGRAMMING METHODS

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Abstract. In this work, we consider a third order equation of three points with non-homogeneous conditions at the border. We apply Avery Peterson's theorem, and present a theoretical result that guarantees the existence of multiple solutions to this problem under certain conditions. In addition, we present non-trivial examples and a new numerical method based on optimization is introduced.

1. Introduction

In this article we address conditions for the existence of multiple solutions to the third order limit value problem:

$$u''' + f(t,u,u') = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda, \tag{2}$$

where $\eta \in (0,1), \alpha \in [0, \frac{1}{\eta})$ are constants and $\lambda \in (0,\infty)$ is a parameter. In the literature, several studies analyze the existence of solutions with qualitative and quantitative aspects for problems similar to that defined in (1) and (2), we recommend the works: [3], [5], [4], [8], [9], [11], [14], [16], [18], [19], [20], [29], [30] and the references therein.

Boundary value problems of ordinary differential equations play an important role in many fields. Various applications of boundary value problems to different areas of applied mathematics and physics, documented in the literature; for example, the works [21] and [10] on deformation of structures, and the monograph of [27] on the effects of soil settlement are good sources of such applications.

Some specific studies have analyzed conditions for the existence of solutions to this class of problems. In [17], the authors use the Krasnoselskii's fixed point theorem

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to determine sufficient conditions for the existence of a positive solution. In this context, the authors consider that f, defined in (1), does not depend on u'. In [28], the problem is also considered in a similar way. Recently, in [22], the authors obtained some sufficient conditions for the existence of the positive solutions. Essentially, the combination of Leray-Schauder Alternative and Krasnoselskii's theorem to show the existence of a positive solution for (1)–(2). The dependence of the term u' in f is an important fact when dealing with results of the existence of multiple solutions. Adding higher order derivatives to the argument of f does not mean that we are contributing to the existence of solutions to an equation that represents a new physical model, but represents an advance from the point of view of the technique used for existence in relation to the generality of the equation, which is an important aspect of mathematical research. Considering the aforementioned problem (1)–(2), the core of the techniques explored in the literature to show the existence of multiple solutions commonly based on one of the following theorems:

i) Krasnoselskii's Fixed Point Theorem [1];

ii) Leggett-Willians Fixed Point Theorem [15];

iii) Avery-Peterson's Fixed Point Theorem [6].

Associated with each of the mentioned theorems we also have the following facets:

a) The Green's function that is used to define the fixed point operator;

b) Cones and convex sets where the fixed point operator compresses or expands.

The presence of high order terms in the f argument, implies the immediate creation of new Green's functions, cones and also, of course, the handling of new hypotheses. This work establishes the use of Avery's theorem to equation (1)–(2), which is a more general equation than those considered in previous works. To achieve this goal, formulations of convenient cones, convex sets and Green's function are established.

Some of the works presented in the literature explore a numerical analysis related to third order problems, the vast majority of which are dedicated only to determining the conditions for the component functions of the problem of establishing conditions for the existence of solutions. In [22], the authors use the Banach Fixed Point Theorem to define an iterative method. Methods that use this model of strategies are not indicated for determining multiple solutions. Common sense suggests that optimization methods combined with heuristic processes can provide good results to find multiple numerical solutions, as we can see in the works [25], [23], [24]. In this way, a method based on Sequential Quadratic Programming (SQP) is shown in Section 3. Still in Section 3, a simple but effective heuristic process is introduced in order to determine various numerical solutions. Final considerations are presented in Section 4.

The *contribution* of this work is threefold: **a**) an analysis regarding the existence of multiple solutions have been provided; **b**) a proposition and an analysis of an efficient and promising optimization method based on SQP are made; **c**) and non-trivial examples are presented to validate the proposed method.

2. Positive solutions

As presented in [22], we can represent the problem (1)–(2) as an integral equation. For this, given $x \in C^1[0,1]$ then we have a unique solution. In this context $C^1[0,1]$ denotes the Banach space of continuously differentiable functions in $[0,1]$. Moreover, this solution is expressed by

$$u(t) = \int_0^1 G(t,s)f(s,x(s),x'(s))ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)f(s,x(s),x'(s))ds + \frac{\lambda t^2}{2(1-\alpha\eta)}, \tag{3}$$

where G is the Green's function:

$$G(t,s) = \frac{1}{2}\begin{cases} (2t-t^2-s)s, & s \leqslant t \\ (1-s)t^2, & t \leqslant s \end{cases} \tag{4}$$

and

$$G_1(t,s) = \frac{\partial G(t,s)}{\partial t} = \begin{cases} (1-t)s, & s \leqslant t \\ (1-s)t, & t \leqslant s \end{cases}. \tag{5}$$

Defining $x(t) = u(t)$ in the expression (3), it is easy to see that the solution of (1)–(2) can be expressed as a fixed point of the operator $T : C^1[0,1] \to C^1[0,1]$ defined by:

$$Tu(t) = \int_0^1 G(t,s)f(s,u,u')ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)f(s,u,u')ds + \frac{\lambda t^2}{2(1-\alpha\eta)}. \tag{6}$$

We claim that T is continuous and completely continuous by Arzela-Ascoli's theorem.

REMARK 1. We list below some properties related to G and G_1, which will be useful to us.

- For all $(t,s) \in [0,1]\times[0,1]$:

$$0 \leqslant G_1(t,s) \leqslant (1-s)s$$

- For all $(t,s) \in [0,1]\times[0,1]$:

$$G(t,s) \leqslant G_1(1,s) = \frac{1}{2}(1-s)s$$

Let $\overline{m}$ be a constant in $[0,1/2]$. Thus, we can obtain the inequalities

$$G_1(t,s) \geqslant \overline{m}G_1(s,s), \forall x \in [\overline{m}, 1-\overline{m}] \tag{7}$$

and

$$G_1(t,s) \leqslant G_1(s,s), \forall t \in [0,1]. \tag{8}$$

In order to determine multiple solutions, consider the cone

$$P = \{u \in C^1[0,1] : u(0) = 0\}, \tag{9}$$

equipped with the norm

$$\|u\|_P = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

REMARK 2. If $u \in P$ then Tu satisfies $Tu(0) = 0$. Besides $\|(Tu)'\|_\infty \geqslant \|Tu\|_P$ and $\|u\|_\infty \leqslant \|u\|_P = \|u'\|_\infty$.

The proof of the existence of a solution demands some basic hypotheses:

$(H1)$ For problem (1)–(2) we assume that f is a continuous function and that there are positive constants A, B and d such that

- For all $(s, v_1, v_2) \in [0,1] \times [-d,d] \times [-d,d]$ then $0 \leqslant f(s, v_1, v_2) \leqslant \frac{d(1-\alpha\eta)6B}{1+\alpha(1-\eta)}$;

- $\lambda \leqslant Aa(1-\alpha\eta)$, where $0 < a < d$;

- $A + B \leqslant 1$.

In order to demonstrate the main result of this work, we need to present the main tool to be used. In this direction, consider P as a cone in the real Banach space X and define the following convex subsets:

$$P(\gamma,d) = \{x \in P | \gamma(x) < d\}$$

$$P(\gamma,\alpha,b,d) = \{x \in P | b \leqslant \alpha(x) \text{ and } \gamma(x) < d\}$$

$$P(\gamma,\theta,\alpha,b,c,d) = \{x \in P | b \leqslant \alpha(x), \theta(x) \leqslant c \text{ and } \gamma(x) < d\}$$

and the closed set:

$$R(\gamma,\psi,a,d) = \{x \in P | a \leqslant \psi(x) \text{ and } \gamma(x) < d\}.$$

THEOREM 1. (Avery-Peterson) *Let γ and θ nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leqslant \lambda\psi(x)$ for $0 \leqslant \lambda \leqslant 1$, such that for some positive numbers μ and d,*

$$\alpha(x) \leqslant \psi(x) \text{ and } \|x\| \leqslant \mu\gamma(x),$$

for all $x \in \overline{P(\gamma,d)}$. Suppose,

$$T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$$

is completely continuous and there exist positive numbers a, b, c with $a < b$, such that

$$\{u \in P(\gamma,\theta,\alpha,b,c,d) | \alpha(u) > b\} \neq \emptyset \text{ and}$$

$$u \in P(\gamma, \theta, \alpha, b, c, d) \Rightarrow \alpha(Tu) > b \tag{10}$$

$$\alpha(Tu) > b \textit{ for } u \in P(\gamma, \alpha, b, d) \textit{ with } \theta(Tu) > c, \tag{11}$$

$$0 \notin R(\gamma, \psi, a, d) \textit{ and } \psi(Tu) < a \textit{ for} \tag{12}$$

$$u \in R(\gamma, \psi, a, d) \textit{ with } \psi(u) = a.$$

Then T has at least three distinct fixed points in $\overline{P(\gamma,d)}$.

From this point on we will consider P as the cone defined in (9). The lemma presented below will be fundamental for demonstrating our main result.

LEMMA 1. *Suppose that $(H1)$ holds, then T defined in (6) fulfills $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$, where $\gamma(.) = \|.\|_P$.*

Proof. Let us consider $u \in P$ with $\|u\|_P \leqslant d$. As from H1 we can get:

$$\begin{aligned}
\|Tu\|_P &< \|(Tu)'\|_\infty \\
&= \max_{t\in[0,1]} |(Tu)'(t)|, \\
&\leqslant \max_{t\in[0,1]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} \int_0^1 |(1-s)s||f(s,u,u')|ds + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leqslant \max_{(s,v_1,v_2)\in[0,1]\times[-d,d]\times[-d,d]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} |f(s,v_1,v_2)| \int_0^1 (1-s)s\,ds + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leqslant \max_{(s,v_1,v_2)\in[0,1]\times[-d,d]\times[-d,d]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} \frac{|f(s,v_1,v_2)|}{6} + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leqslant \frac{1}{1-\alpha\eta}\left[\frac{1+\alpha(1-\eta)}{6}\max|f(s,v_1,v_2)| + \lambda\right] \\
&\leqslant \frac{1}{1-\alpha\eta}\left[\frac{1+\alpha(1-\eta)}{6}\frac{d(1-\alpha\eta)6B}{1+\alpha(1-\eta)} + \lambda\right] \\
&\leqslant \frac{1}{1-\alpha\eta}\left[d(1-\alpha\eta)B + Ad(1-\alpha\eta)\right] \\
&\leqslant dA + dB \leqslant d.
\end{aligned}$$

Therefore $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$. $\square$

In our main result (given by Theorem 2) we will show that the Problem (1)–(2) has at least three positive solutions.

THEOREM 2. *Suppose that the hypothesis $(H1)$ is satisfied. Suppose, in addition, that there exists a, $0 < a < d$ such that f satisfies the following conditions:*

$(H2)$ $f(s,u,v) < \dfrac{a(1-\alpha\eta)6B}{1+\alpha(1-\eta)}$, $\forall (s,u,v) \in [0,1]\times[0,a]\times[-a,a]$,

$(H3)$ $f(s,u,v) > \dfrac{2a}{r_2}$, $\forall(s,u,v) \in [0,1]\times[2a,4a]\times[-d,d]$, *where*

$$r_2 = \frac{1}{4}\left(\int_0^1 G_1(s,s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)ds\right).$$

Then, Problem (1)–(2) *has at least three positive solutions.*

Proof. We will apply Avery-Peterson theorem, then let us consider T and P as defined before. Furthermore, we need to define the following functionals:

$$\begin{aligned}
\gamma(u) &= \|u\|_P, \\
\psi(u) &= \max_{t\in[0,1]} |u'(t)|, \\
\theta(u) &= \max_{t\in[\frac{1}{4},\frac{3}{4}]} |u'(t)|, \\
\alpha(u) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} |u'(t)|.
\end{aligned}$$

Therefore, from Lemma1 we get

$$T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$$

and T is completely continuous and there exist positive numbers b and c with $a < b$, such that

$$\{u \in P(\gamma,\theta,\alpha,b,c,d) | \alpha(u) > b\} \neq \emptyset \text{ and}$$

$$u \in P(\gamma,\theta,\alpha,b,c,d) \Rightarrow \alpha(Tu) > b \tag{13}$$

$$\alpha(Tu) > b \text{ for } u \in P(\gamma,\alpha,b,d) \text{ with } \theta(Tu) > c, \tag{14}$$

$$0 \notin R(\gamma,\psi,a,d) \text{ and } \psi(Tu) < a \text{ for} \tag{15}$$

$$u \in R(\gamma,\psi,a,d) \text{ with } \psi(u) = a.$$

Now, we consider

$$b = 2a$$

and

$$c = 8a.$$

Clearly, we have $\{u \in P(\gamma,\theta,\alpha,b,c,d) | \alpha(u) > b\} \neq \emptyset$. Let us demonstrate (13). Using

(A3) we obtain:

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]}(Tu)'(t)\\
&= \min_{t\in[\frac{1}{4},\frac{3}{4}]}\Big(\int_0^1 G_1(t,s)f(s,u(s),u'(s))ds\\
&\quad +\frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)f(s,u(s),u'(s))ds+\frac{\lambda t}{1-\alpha\eta}\Big)\\
&\geqslant \Big(\int_0^1 \frac{1}{4}G_1(s,s)f(s,u(s),u'(s))ds\\
&\quad +\frac{\alpha\frac{1}{4}}{1-\alpha\eta}\int_0^1 G_1(\eta,s)f(s,u(s),u'(s))ds+\frac{\lambda\frac{1}{4}}{1-\alpha\eta}\Big)\\
&\geqslant \frac{1}{4}\Big(\int_0^1 G_1(s,s)f(s,u(s),u'(s))ds\\
&\quad +\frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)f(s,u(s),u'(s))ds+\frac{\lambda}{1-\alpha\eta}\Big)\\
&> \frac{2a}{r_2}\frac{1}{4}\Big(\int_0^1 G_1(s,s)ds+\frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)ds\Big)\\
&\geqslant 2a=b.
\end{aligned}$$

Let us demonstrate (14). Let $u\in P(\gamma,\alpha,b,d)$ with $\theta(Tu)>c$. Then

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]}(Tu)'(t)\\
&= \min_{t\in[\frac{1}{4},\frac{3}{4}]}\Big(\int_0^1 G_1(t,s)f(s,u(s),u'(s))ds\\
&\quad +\frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)f(s,u(s),u'(s))ds+\frac{\lambda t}{1-\alpha\eta}\Big)\\
&\geqslant \frac{1}{4}\Big(\int_0^1 G_1(s,s)f(s,u(s),u'(s))ds\\
&\quad +\frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)f(s,u(s),u'(s))ds+\frac{\lambda}{1-\alpha\eta}\Big)\\
&\geqslant \frac{1}{4}\max_{t\in[\frac{1}{4},\frac{3}{4}]}\Big(\int_0^1 G_1(t,s)f(s,u(s),u'(s))ds\\
&\quad +\frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)f(s,u(s),u'(s))ds+\frac{\lambda t}{1-\alpha\eta}\Big)\\
&\geqslant \frac{1}{4}\theta(Tu)\\
&> \frac{1}{4}c=b.
\end{aligned}$$

Now, let us show (15). Thus, let $u\in R(\gamma,\psi,a,d)$ with $\psi(u)=a$. From $(H1)$–

$(H2)$ we have,

$$\begin{aligned}
\psi(Tu) &= \max_{t\in[0,1]} |(Tu)'| \\
&\leqslant \max_{t\in[0,1]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} \int_0^1 |(1-s)s||f(s,u,u')|ds + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leqslant \max_{(s,v_1,v_2)\in[0,1]\times[-a,a]\times[-a,a]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta}|f(s,v_1,v_2)| \int_0^1 (1-s)s\,ds + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leqslant \max_{(s,v_1,v_2)\in[0,1]\times[-a,a]\times[-a,a]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} \frac{|f(s,v_1,v_2)|}{6} + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leqslant \frac{1}{1-\alpha\eta}\left[\frac{1+\alpha(1-\eta)}{6}\max|f(s,v_1,v_2)|+\lambda\right] \\
&\leqslant \frac{1}{1-\alpha\eta}\left[\frac{1+\alpha(1-\eta)}{6}\frac{a(1-\alpha\eta)6B}{1+\alpha(1-\eta)}+\lambda\right] \\
&\leqslant \frac{1}{1-\alpha\eta}[a(1-\alpha\eta)B + Aa(1-\alpha\eta)] \\
&\leqslant aA + aB < a.
\end{aligned}$$

Applying Avery-Peterson theorem we obtain the result. □

The example presented below illustrates the hypotheses assumed in Theorem 2.

EXAMPLE 1. Let us consider (1)–(2) with

$$f(t,u,v) = \begin{cases} \frac{t}{10} + \frac{1}{5}(10u)^7 + \frac{v^2}{10} & 0 \leqslant u \leqslant 0.2 \\ \frac{t}{10} + 25.4 + u + \frac{v^2}{10} & 0.2 \leqslant u \leqslant 20 \end{cases}$$

$$\eta = \frac{1}{10}, \quad \alpha = \frac{1}{3}, \quad \lambda = 0.009$$

Choosing the constants

$$d = 20, \;\; A = 0.1, \;\; B = 0.89, \;\; a = 0.1,$$

we can easily verify that in these conditions the hypothesis $(H1)$ and the hypotheses of Theorem 2 are satisfied.

3. Numerical solutions

Most studies dedicated to problem (1)–(2) do not explore the determination of numerical solutions. In [22], the numerical solutions are obtained by fixed point methods. However, the convergence of these methods depends on the contraction of the operator in the neighborhood of the solution and, consequently, depends on the quality of the starting points. In order not to depend on these characteristics and to determine various solutions, our proposal is based on the Sequential Quadratic Programming [26] method. An algorithm of this method is proposed to solve the Problem (1)–(2).

3.1. Sequential Quadratic Programming (SQP)

The sequential quadratic programming method basically consists of the sequential approximation of the nonlinear programming problem as a quadratic programming problem. As highlighted in [7], SQP is backed by robust implementations and a well established theory. A didact explanation of the main aspects of SQP theory can be found in [26] and a good review on the most prominent developments in this subject is given in [13]. In this section, the progress is quite limited and we just briefly describe the idea of how to handle SQP.

In this sense, let us consider the following optimization problem

$$\begin{aligned} \min_u \quad & g(u) \\ \text{s.t. } & c_i(u) = 0, \;\; i = 1,\dots,m, \\ & lb \leqslant u \leqslant ub \end{aligned} \tag{16}$$

where $g : \mathbb{R} \to \mathbb{R}$ is the objective function, $c_i : \mathbb{R}^n \to \mathbb{R}$ are equality constraint, lb, $ub \in \mathbb{R}\;\; n$ are lower and upper bound, respectively. As usual in optimization $lb \leqslant u \leqslant ub$ means $lb_j \leqslant u_j \leqslant ub_j$, $\forall j = 1,\dots,n$.

The kernel idea in SQP optimization approach is the formulation of a quadratic programming subproblem based on a quadratic approximation of the Lagrangian function:

$$\mathscr{L}(u,\mu) = g(u) + \sum_{i=1}^{m} \mu_i c_i(u). \tag{17}$$

Thus, at each iteration, the SQP method solves the following quadratic programming problem:

$$\begin{aligned} \min_u \quad & g(u^k) + \nabla g(u^k)^T d + \frac{1}{2} d \nabla^2_{u^k} \mathscr{L} d \\ \text{s.t.} \quad & \nabla c_i(u^k) + c_i(u^k)^T d = 0, \quad i = 1 \dots m, \\ & lb \leqslant u \leqslant ub \end{aligned} \tag{18}$$

where $\nabla^2_{u^k}\mathscr{L} = \nabla^2 \mathscr{L}(u^k) = \nabla^2 g(u^k) - \sum_{i=1}^{K} \mu_i \nabla^2 c_i(u^k)$, with the aiming of determining the best search direction, denoted by d^k (from the point u_k) as the solution to (18) and associated with the Lagrange multipliers μ_k at k-th iteration; and then proceed to update to the next point u^{k+1} is obtained by projecting $u_k + d^k$ into the box $lb \leqslant u \leqslant ub$.

Hence, using SQP method, a nonlinear problem can be solved in fewer iterations, with similar complexity as a quadratic problem. Algorithm 1 summarizes the SQP method.

In Algorithm 1 the orthogonal projection operator $\mathscr{P}$ is defined by:

$$\mathscr{P}(u) = \overline{u}, \qquad \text{where} \quad \overline{u}_i = \begin{cases} lb, & u_i \leqslant lb \\ ub, & u_i \geqslant ub \\ u_i, & \text{otherwise} \end{cases} \tag{19}$$

Algorithm 1 : SQP procedure

Data: Choose an initial u_0 and λ_0;
Result: u^* approximated solution of (3.1);
Set $k \leftarrow 0$;
while $\|u^{k+1} - u^k\| > \varepsilon$ or $k = 0$ **do**
  Evaluate $g(u^k), \nabla g(u^k), \nabla^2_{u^k}\mathscr{L}, c_i(u^k)$ and $\nabla c_i(u^k)$;
  Solve (18) to obtain d^k and μ_{k+1};
  $u^{k+1} \leftarrow \mathscr{P}(u^k + d^k)$; where $\mathscr{P}$ is the projection operator in the box $lb \leqslant u \leqslant ub$;
  $k = k+1$.
end while

3.2. Proposed Numerical Method

In this subsection we present our main algorithm, which is based on the SQP method, in which we will consider a restricted non-linear programming problem defined from the problem (1)–(2). The construction of this optimization problem is done in such a way that solutions to the problem (1)–(2) are global solutions to this problem.

The most important observation to understand how our numerical approach works is to understand how to model the discrete problem in terms of optimization. In this direction let us consider $\{t_j, j = 0, 1, \ldots, n\}$ a discretization of $[0, 1]$ by an equal spaced mesh where $h = t_{j+1} - t_j$, $j = 0, 1, \ldots, n-1$ and $u_j \approx u(t_j)$, $j = 0, 1, \ldots, n$. Following classical finite difference schemes given in [12], we have:

$$u_j''' = \frac{u_{j+2} - u_{j-2} - 2(u_{j+1} - u_{j-1})}{2h^3}, \quad j = 2, \ldots, n-2, \tag{20}$$

$$u_j' = \frac{u_{j+1} - u_{j-1}}{2h}, \quad j = 1, \ldots, n-1, \tag{21}$$

Replacing (20) and (21) in (1) we obtain the nonlinear system:

$$u_{j+2} - u_{j-2} - 2(u_{j+1} - u_{j-1}) + 2h^3 f\left(t_j, u_j, \frac{u_{j+1} - u_{j-1}}{2h}\right) = 0, \quad j = 2, \ldots, n-2. \tag{22}$$

The nonlinear system (22) provides $n-3$ equations. Assuming $u_0 = 0$ we need to determine $u_1, u_2, \ldots, u_n$ and the approximation u'_η for $u'(\eta)$, that is, we need to find $n+1$ variables. Once the values of u'_j, $j = 1, \ldots, n-1$ obtained as in (21), so you can approximate the value of $u'(\eta)$ by $u'_\eta = Sp(u'_{s-2}, u'_{s-1}, u'_s, u'_{s+1})$, the function Sp is defined by Cubic Splines [2] from the points $(u'_{s-2}, u'_{s-1}, u'_s, u'_{s+1})$, these are chosen so that $\eta \in [t_{s-1}, t_s]$.

Considering the first condition in (2), it is tempting to want to impose the condition:

$$u_0 = 0, \quad u'_0 = 0.$$

On the other hand, it is perceptive that u'_0 will be close to zero but not necessarily zero, since imposing $u_1 = 0$ in the finite difference scheme could be too restrictive. Then we can define a deviation $r_1(u) = (u'_0)^2$, where $u = (u_0, u_1, \ldots, u_n, u_\eta)$. Now considering the second condition in (2), the ideal would be to obtain equality with the approximations of the vector u:

$$u'_{n+1} - \alpha u'_{\eta} - \lambda = 0.$$

However, we expect the above equation to be close to zero if we use spline interpolation instead of u'_{η}. Then we can define a deviation

$$r_2(u) = (u'_{n+1} - \alpha Sp(u'_{s-2}, u'_{s-1}, u'_{s}, u'_{s+1}) - \lambda)^2,$$

where $u = (u_0, u_1, \ldots, u_n, u_{\eta})$. An approximate solution u^* to the problem defined in (1)–(2) should imply that the deviation $r_1(u^*) + r_2(u^*)$ is close to zero, so we can determine approximate solutions by minimizing this deviation while maintaining the constraints (22) and $u_0 = 0$.

We present in the sequence an algorithm of this method proposed to solve the Problem (1)–(2).

Algorithm 2 : Main Method

Data: Given in problem (1)–(2), an uniformly spaced mesh $\{t_j\}$, $j = 0, 1, \ldots, n$, in $[0, 1]$; choose $lb < ub$.

Result: u^* solution approach to problem (1)–(2);

1. Choose initial approximation $u_j^0 = u^0(t_j)$.
2. Apply Algorithm 1 to the following optimization problem:

$$\begin{aligned}
\min_{(u_0, u_1, \ldots, u_n)} \quad & r_1(u) + r_2(u) \\
\text{s.t.} \quad & u_{j+2} - u_{j-2} - 2(u_{j+1} - u_{j-1}) + 2h^3 f\left(t_j, u_j, \frac{u_{j+1} - u_{j-1}}{2h}\right) = 0, \\
& j = 2, \ldots, n-2; \\
& u_0 = 0; \\
& lb \leqslant u_j \leqslant ub, \quad j = 0, 1, \ldots, n.
\end{aligned} \tag{23}$$

23 with Algorithm 1 and update u^{k+1}.

The main motivation for Algorithm 2 is that fixed point methods are biased towards finding solutions in which the T operator is a contraction. If we apply Algorithm several times, we can find the multiple solutions mentioned by Theorem 2. Therefore, the development of a heuristic to find a better initial approximation is relevant.

3.3. An heuristic procedure for initial guesses

We know that the solutions we are looking for must be continuous and must satisfy the condition $u(0) = 0$ and $u'(0) = 0$, so it is expected that the solutions are convex. Thus, exponential approaches are reasonable ways to approach the solution. In this sense, our heuristic procedure is to generate parables about starting points as follows:

$$u^0(x) = \zeta(e^{x\xi} - 1 - x\xi)(x),$$

where the constants ζ, ξ are random numbers in $[-d,d]\times(0,1]$. For practical purposes, the proposed procedure is defined by Algorithm 3.

Algorithm 3 : Multi-start Procedure

Data: Given the problem (1)–(2);
Result: $u^{*,1}$, $u^{*,2},\ldots,u^{*,N}$ approach solution to the problem (1)–(2);
Choose a vector $(\zeta,\xi)\in[-d,d]^N\times(0,1]^N$.
for $k=1,\ldots,N$ **do**
  Compute $u^0_{k,i}=u^0_k(x_i)=\zeta_k(e^{x_i\xi_k}-1-x_i\xi_k)$, $i=1,\ldots,n$;
  Run the Algorithm 2 with initial guess u^0_k.
end for

It is expected that this procedure returns several solutions; therefore, it is necessary to establish a way to compare these solutions. Note that the magnitude of the solutions may be different. In this sense, we say that the numeric solutions u^* and u^{**} are *equivalent* if

$$\|u^*-u^{**}\|\leqslant\max\{10^{-4},10^{-2}\min\{\|u^*\|,\|u^{**}\|\}\}. \tag{24}$$

is satisfied.

3.4. Numerical examples

The examples that follow show how the Algorithm 2 can be promise in order to find multiple solutions. We run the Algorithm 2 with $N=50$ and $n=20$. For Algorithm 2 we consider as the criterion of stop $\|u^{k+1}-u^k\|<10^{-4}$ it is considered that convergence was obtained if u^* satisfy:

$$\max\{r_1(u^*),r_2(u^*)\}<10^{-3}$$

and

$$\max_{j\in\{2,\ldots,n-2\}}\left|u^*_{j+2}-u^*_{j-2}-2(u^*_{j+1}-u^*_{j-1})+2h^3f\left(t_j,u^*_j,\frac{u^*_{j+1}-u^*_{j-1}}{2h}\right)\right|<10^{-3}.$$

In this way, possible non-global solutions of the problem (23) will not be accepted, since a solution of (1)–(2) is necessarily a global solution of the problem of non-linear programming (23).

EXAMPLE 2. Consider the example where (1)–(2) defined by

$$f(t,u,u')=\begin{cases}-u', & 0\leqslant u\\ u', & 0>u\end{cases}$$

$$\eta=0.15,\ \lambda=1,\ \alpha=6,\ \text{and } d=10.$$

The solutions are $u^{*,1}(x)=\frac{\cos(t)-1}{0.005515781}$ and $u^{*,2}(x)=\frac{\cosh(t)-1}{0.2718224}$. Applying Algorithm 3, of the 50 times that Algorithm 2 was called it obtained convergence 29 times. Where

4 converged to the solution $u^{*,1}$ and 25 converged to the solution $u^{*,2}$. In the 21 times that the method diverged, the Algorithm 2 returned local solutions, these solutions were basically variations close to zero that approximately satisfied the discretized equation, thus satisfying the feasibility but not satisfying $r_1(u^*)+r_2(u^*)\approx 0$.

EXAMPLE 3. We introduce an additional test. Let's run the Algorithm 3 using the functions defined in Example 1. In this example we are defining $n=20$ and $d=10$. Using the criterion established in (24) we obtain three solutions that are different. These results illustrate the result of existence given in Theorem 2. In Figure 1 we have a graphical representation of these solutions.

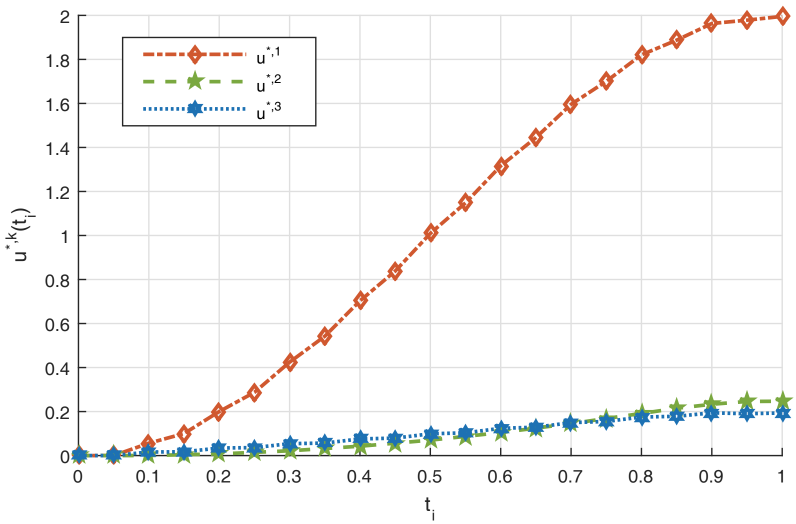

Figure 1: *Solutions found by Algorithm 3 in Example 1.*

4. Final remarks and future work

We prove, using the Avery-Peterson theorem, that the problem (1), (2) can have several solutions if the f function meets certain conditions. The inclusion of problem with one derivative present is due to the fact that few studies are dedicated to this variation for equations with multiple border points and third order due to the greater difficulty in obtaining conditions for the existence of solutions. The complexity of Green's function in this case is the biggest flaw, in turn in problems with multiple border points and second order the dependence of f on u' is more common, due to its Green function simpler. Another point that we highlight is the construction of cones and convex sets that allow the application of the present technique. Considering the proposed problem, we do not currently have a cone and convex sets that fits the objective and contemplates the requirement of the second order derivative in the considered equation. Consequently, we expect to provide results in this direction in the future.

Regarding the numerical aspects of this work, we present a new algorithm and a heuristic that allow us to obtain multiple solutions to the problem addressed. The proposed method proved to be robust in solving the problem. However, the cost of this robustness is a slightly higher cost of computational processing when compared to the classic method based on the principle of contraction (using the operator defined in (6)), however this strategy has more restrictions since it depends on the integral operator being a contraction in a solution neighborhood.

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