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A CLASS OF NONLINEAR THIRD–ORDER BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITION AT RESONANCE

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Abstract. We are interested in the existence result for a class of nonlinear third-order three-point boundary value problem with integral condition at resonance. By constructing suitable operators, we establish an existence theorem upon the coincidence degree theory of Mawhin. The result are illustrated with an example.

1. Introduction

The focus of this paper is to provide sufficient conditions that ensure the existence of solutions for the following nonlinear third-order boundary value problem

$$u'''(t) = f\left(t, u(t), u'(t)\right), \quad 0 < t < T, \tag{1.1}$$

$$u(0) = u''(0) = 0,\ u(T) = \frac{2T}{\eta^2}\int_0^\eta u(t)\,dt \tag{1.2}$$

where $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $0 < \eta < T$. We say that the boundary value problem (1.1)–(1.2) is a resonance problem if the linear equation $Lu = u'''$, with the boundary value conditions (1.2) has non-trivial solution that is $\dim KerL \geqslant 1$.

Boundary value problems involving ordinary differential equations with integral boundary conditions arise in various fields of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can produce boundary-value problems with integral boundary conditions [8, 12, 14]. Many authors have studied third-order boundary value problems with different boundary conditions using different approaches. In the non-resonance case, we can mention the works of ([1, 2, 4, 5, 11, 17]). For the resonance case, we refer the reader to see ([3, 6, 7, 9, 10, 13, 16]).

Inspired and motivated by the works mentioned above, in the present article, we use the coincidence degree theory of Mawhin [15] to discuss the existence of solution for third-order nonlocal boundary value problem (1.1)–(1.2) at resonance case, and establish an existence theorem. The layout of this paper is as follows. In section 2, we give the background information from coincidence degree theory. We also define appropriate mappings and projections that will be used in the sequel. We state and prove our main result in section 3, and we illustrate the obtained results by an example.

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2. Preliminaries

In this section, we introduce some notations and an abstract existence result of the coincidence degree theory (Mawhin 1979).

DEFINITION 1. Let Y, Z be two real Banach spaces. A linear operator $L : domL \subset Y \rightarrow Z$ is said to be a Fredholm map of index zero provided that $\ker L$, the kernel of L, is of the same finite dimension as the Y/ImL, where ImL is the image of L.

Let L be a Fredholm map of index zero, and $P : Y \longrightarrow Y$, $Q : Z \longrightarrow Z$ be continuous projectors, such that $ImP = \ker L$, $KerQ = ImL$. Then $Y = \ker L \oplus \ker P$, $Z = ImL \oplus ImQ$, thus $L\mid_{domL \cap KerP} \rightarrow ImL$ is invertible, denote its inverse by K_P.

DEFINITION 2. Let L be a Fredholm map of index zero and Ω be an open bounded subset of Y, such that $domL \cap \Omega \neq \emptyset$, the map $N : Y \rightarrow Z$ is said to be $L-compact$ on $\overline{\Omega}$, if $QN\left(\overline{\Omega}\right)$ bounded and $K_P\left(I-Q\right)N : \overline{\Omega} \rightarrow Y$ is compact.

We will formulate the boundary value problem (1.1)–(1.2) as $Lu = Nu$ where L and N are appropriate operators. To obtain our existence results we use the following fixed point theorem of Mawhin.

THEOREM 1. ([15]) *Let L be a Fredholm operator of index zero and let N be $L-compact$ on $\overline{\Omega}$. Assume that the following conditions are satisfied:*
(i) $Lu \neq \lambda Nu$ for every $(u,\lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0,1)$.
(ii) $Nu \notin ImL$ for every $u \in KerL \cap \partial\Omega$.
(iii) $deg\left(QN\mid_{KerL}, KerL \cap \Omega, 0\right) \neq 0$,
where $Q : Z \rightarrow Z$ is a projection as above with $ImL = KerQ$.
Then the abstract equation $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$.

For $u \in C^2[0,T]$, we use the norm $\|u\|_\infty = \max\limits_{t\in[0,T]} |u(t)|$ and $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, and denote the norm in $L^1[0,T]$ by $\|\cdot\|_1$. We will use the Sobolev space

$$W^{3,1}(0,T) = \left\{u : [0,T] \rightarrow \mathbb{R} : u, u', u'' \text{ are absolutely continuous on } [0,T] \text{ with } u''' \in L^1[0,T]\right\}$$

Let $Y = C^2[0,T]$, $Z = L^1[0,T]$, define the linear operator $L : domL \subset Y \rightarrow Z$ by

$$Lu = u''', \quad u \in domL$$

where

$$domL = \left\{u \in W^{3,1}(0,T) : u(0) = u''(0) = 0,\, u(T) = \alpha \int_0^\eta u(t)\,dt\right\}$$

and define $N : Y \rightarrow Z$ by

$$Nu(t) = f\left(t, u(t), u'(t)\right),\ t \in (0,T)$$

Then the boundary value problem (1.1)–(1.2) can be written as $Lu = Nu$.

3. Existence results

We will assume that the following conditions hold.

(H_1) There exist functions $\alpha, \beta, \gamma \in L^1[0,T]$, such that for $(u,v) \in \mathbb{R}^2, t \in [0,T]$, satisfying

$$|f(t,u,v)| \leqslant \alpha(t)|u| + \beta(t)|v| + \gamma(t) \tag{3.1}$$

(H_2) There exists a constant $M > 0$, such that for $u \in domL$, if $|u'(t)| > M$ for all $t \in [0,T]$, it holds

$$\int_0^T (T-s)^2 f(s,u(s),u'(s))\,ds - \frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,u(s),u'(s))\,ds \neq 0 \tag{3.2}$$

(H_3) There exists a constant $M^* > 0$, such that for any $u(t) = bt \in KerL$ with $|b| > M^*$, either

$$b\left[\int_0^T (T-s)^2 f(s,u(s),u'(s))\,ds - \frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,u(s),u'(s))\,ds\right] < 0 \tag{3.3}$$

or else

$$b\left[\int_0^T (T-s)^2 f(s,u(s),u'(s))\,ds - \frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,u(s),u'(s))\,ds\right] > 0 \tag{3.4}$$

THEOREM 2. *Assume that conditions $(H_1)-(H_3)$ hold and that*

$$\|\alpha\|_1 + \|\beta\|_1 < \frac{1}{T+1}, \tag{3.5}$$

then boundary value problem (1.1)–(1.2) has at least one solution in $C^2[0,T]$.

For the Proof of Theorem 2 we shall apply Theorem 1 and the following Lemmas.

LEMMA 1. *The operator $L: domL \subset X \to Z$ is a Fredholm operator of index zero. Furthermore, define the linear continuous projector operator $Q: Z \to Z$ by*

$$Qy(t) = \frac{1}{C}\left[\int_0^T (T-s)^2 y(s)\,ds - \frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 y(s)\,ds\right] t$$

where $\frac{1}{C} = \frac{60}{5T^4 - 2T\eta^3}$ and the linear operator $K_P: ImL \to domL \cap KerP$ by

$$K_P y(t) = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds, \quad \forall y \in ImL$$

Furthermore

$$\|K_P y\| \leqslant \|y\|_1, \quad \forall y \in ImL$$

Proof. It is clear that

$$\begin{aligned} KerL &= \{u \in domL : \ Lu = 0\} \\ &= \left\{u \in domL : \ u''' = 0\right\} \\ &= \{u \in domL : \ u(t) = bt, \ b \in \mathbb{R}\} \simeq \mathbb{R}. \end{aligned}$$

Now, we show that

$$ImL = \left\{ y \in Z : \int_0^T (T-s)^2 y(s)\,ds - \frac{2T}{3\eta^2} \int_0^\eta (\eta - s)^3 y(s)\,ds = 0 \right\}. \tag{3.6}$$

Since the problem

$$u''' = y \tag{3.7}$$

has a solution $u(t)$ that satisfies the conditions $u(0) = u''(0) = 0$, $u(T) = \frac{2T}{\eta^2} \int_0^\eta u(t)\,dt$ if and only if

$$\int_0^T (T-s)^2 y(s)\,ds - \frac{2T}{3\eta^2} \int_0^\eta (\eta - s)^3 y(s)\,ds = 0. \tag{3.8}$$

From (3.7), we have

$$u(t) = u(0) + u'(0)t + u''(0)\frac{t^2}{2} + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds.$$

Thus from the conditions $u(0) = u''(0) = 0$, we have

$$u(t) = u'(0)t + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds.$$

According to $u(T) = \frac{2T}{\eta^2} \int_0^\eta u(t)\,dt$, we obtain

$$\int_0^T (T-s)^2 y(s)\,ds - \frac{2T}{3\eta^2} \int_0^\eta (\eta - s)^3 y(s)\,ds = 0.$$

Hence

$$ImL = \left\{ y \in Z : \int_0^T (T-s)^2 y(s)\,ds - \frac{2T}{3\eta^2} \int_0^\eta (\eta - s)^3 y(s)\,ds = 0 \right\}.$$

On the other hand, if (3.8) hold, setting

$$u(t) = bt + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds,$$

where b is arbitrary constants, then $u(t)$ is solution of (3.7). Hence (3.6) holds.

Setting

$$Ry = \int_0^T (T-s)^2 y(s)\,ds - \frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 y(s)\,ds$$

Let $C = \int_0^T (T-t)^2 t dt - \frac{2T}{3\eta^2}\int_0^\eta (\eta-t)^3 t dt \neq 0,\ t\in(0,1]$. By simple calculation, we get $C = \frac{5T^4-2T\eta^3}{60}$.

Define $Qy(t) = \frac{1}{C}\cdot(Ry)\cdot t$, it is clair that $\dim ImQ = 1$. We have

$$\begin{aligned}
(Q^2y)(t) &= (Q(Qy))(t)\\
&= \frac{1}{C}\left(\frac{1}{C}Ry\right)\left(\int_0^T (T-t)^2 t^2 ds - \frac{2T}{3\eta^2}\int_0^\eta (\eta-t)^3 t^2 ds\right)t\\
&= \frac{1}{C}(Ry)t\\
&= (Qy)(t),
\end{aligned}$$

which implies that the operator Q is projector. Furthermore, $ImL = KerQ$. For $y\in Z$, let $y = (y-Qy)+Qy$, since $Q(y-Qy) = Qy - Q^2y = 0$, we know $(y-Qy)\in \ker Q = ImL$, and $Qy\in ImQ$. Thus $Z = ImL + ImQ$.

Let $y\in ImL\cap ImQ$. Since $y\in ImQ$, then there exists $\rho\in\mathbb{R}$ such that $y(t) = \rho t,\ t\in[0,T]$. Since $y\in ImL = KerQ$, then

$$0 = \rho(Ry)(t) = \rho\left(\int_0^T (T-t)^2 t ds - \frac{2T}{3\eta^2}\int_0^\eta (\eta-t)^3 t ds\right) = \rho C,$$

since $C\neq 0$, then $\rho = 0$, so we have $y(t) = 0,\ t\in[0,T]$, which implies $ImL\cap ImQ = \{0\}$.

Consequently, $Z = ImL\oplus ImQ$, and

$$dimKerL = codimImL = dimImQ = 1.$$

Thus L is Fredholm operator of index zero ($IndL = dimKerL - codimImL = 1-1 = 0$).

Define the other projector $P: X\to X$ by

$$(Pu)(t) = u'(0)t,\ t\in[0,T]. \tag{3.9}$$

Note that $KerP = \{u\in X:\ u'(0)t = 0\} = \{u\in X\ :\ u'(0) = 0\}$ and $ImP = KerL$. Since $(Pu)'(t) = u'(0)$, then $(P^2u)(t) = P(t),\ t\in[0,T]$ for all $u\in X$, we have

$$u = (u-Pu)+Pu$$

$$u(t) = \left(u(t) - u'(0)t\right) + u'(0)t.$$

For $u \in X$, let $u = (u - Pu) + Pu$. Since $P(u - Pu) = Pu - P^2u = Pu - Pu = 0$, we know, $(u - Pu) \in KerP$ and $Pu \in ImP = KerL$, thus $X = KerP + KerL$.

Let $u \in KerL \cap KerP$, since $u \in KerL = ImP$, there exists $\mu \in \mathbb{R}$ such that $u(t) = \mu t^2$ and since $u \in KerP$, then $\mu = u'(0) = 0$ and so $u(t) = 0, t \in [0, T]$. Consequently, $KerL \cap KerP = \{0\}$. Then $X = KerP \oplus KerL$.

Define the generalized inverse operator $K_P : ImL \to domL \cap KerP$ of L by

$$K_P y(t) = \frac{1}{2} \int_0^t (t - s)^2 y(s) \, ds.$$

It follows that

$$\|K_P y\|_\infty \leqslant \frac{1}{2} \int_0^T (T - s)^2 |y(s)| \, ds \leqslant \int_0^T |y(s)| \, ds = \|y\|_1$$

from $(K_P y)'(t) = \int_0^t (t - s) y(s) \, ds$, we obtain

$$\|(K_P y)'\|_\infty \leqslant \int_0^T (T - s) |y(s)| \, ds \leqslant \int_0^T |y(s)| \, ds = \|y\|_1 .$$

As such we have

$$\|K_P y\| = max \left\{ \|K_P y\|, \|(K_P y)'\| \right\} \leqslant \|y\|_1 \tag{3.10}$$

then, we have

$$\|K_p y\| \leqslant \|y\|_1 .$$

Also, if $y \in ImL$, then

$$(LK_P) y(t) = [(K_P y)(t)]''' = y(t)$$

and for $u \in domL \cap KerP$, we know

$$(K_P L) u(t) = (K_P) u'''(t) = \frac{1}{2} \int_0^t (t - s)^2 u'''(s) \, ds = u(t) - u(0) - u'(0) t - u''(0) \frac{t^2}{2}$$

in view of $u \in domL \cap KerP$, $u(0) = u''(0) = 0$ and $Pu = 0$, thus

$$(K_P L) u(t) = u(t)$$

This shows that $K_P = (L|_{domL \cap KerP})^{-1}$. □

LEMMA 2. *Let* $\Omega_1 = \{u \in domL \setminus KerL : Lu = \lambda Nu, \, for\, some\, \lambda \in [0,1]\}$. *Then* Ω_1 *is buonded.*

Proof. Suppose that $u \in \Omega_1$, and $Lu = \lambda Nu$. Thus $\lambda \neq 0$ and $QNu = 0$, so it yields

$$\int_0^T (T-s)^2 f\left(s, u(s), u'(s)\right) ds - \frac{2T}{3\eta^2} \int_0^\eta (\eta - s)^3 f\left(s, u(s), u'(s)\right) ds = 0.$$

Thus, by condition (H_2), there exists $t_1 \in [0,T]$, such that $|u'(t_1)| \leqslant M$. In view of

$$u'(0) = u(t_1) - \int_0^{t_1} u''(s)\, ds, \quad u''(0) = u''(t_1) + \int_0^t u'''(s)\, ds.$$

Hence

$$|u(0)| = \left| u(t_1) - \int_0^{t_1} u'(s)\, ds \right| \leqslant M + \left\|u'\right\|_1$$

and

$$\left|u''(0)\right| = \left| u''(t_1) + \int_0^{t_1} u'''(t)\, dt \right| \leqslant M + \left\|u'''\right\|_1$$

then, we have

$$\left|u'(0)\right| \leqslant M + \int_0^T \left(\int_0^T \left|u'''(s)\right| ds \right) dt = M + T\left\|u'''\right\|_1 = M + T\left\|Lu\right\|_1 \leqslant M + T\left\|Nu\right\|_1. \tag{3.11}$$

Again for $u \in \Omega_1$, then $(I-P)u \in domL \cap KerP = ImK_P$ and $LPu = 0$, $0 < \lambda < 1$ and $Nu = \frac{1}{\lambda} Lu \in ImL$, thus from Lemma 1, we know

$$\|(I-P)u\| = \|K_P L(I-P)u\| \leqslant \|L(I-P)u\|_1 = \|Lu\|_1 \leqslant \|Nu\|_1. \tag{3.12}$$

From (3.11), (3.12) and $\|Pu\| = |u'(0)|$, we have

$$\|u\| \leqslant \|Pu\| + \|(I-P)u\| = \left|u'(0)\right| + \|(I-P)u\| \leqslant M + (T+1)\|Nu\|_1. \tag{3.13}$$

From (3.1) and (3.13), we obtain

$$\|u\| \leqslant (T+1)\left[\|\alpha\|_1 \|u\|_\infty + \|\beta\|_1 \left\|u'\right\|_\infty + \|\gamma\|_1 + \frac{M}{T+1} \right]. \tag{3.14}$$

Thus, from $\|u\|_\infty \leqslant \|u\|$ and (3.13), we have

$$\|u\|_\infty \leqslant \frac{T+1}{1-(T+1)\|\alpha\|_1} \left[\|\beta\|_1 \left\|u'\right\|_\infty + \|\gamma\|_1 + \frac{M}{T+1} \right]. \tag{3.15}$$

From $\|u'\|_\infty \leqslant \|u\|$, (3.14) and (3.15), we have

$$\|u'\|_\infty \leqslant \|u\|$$
$$\|u'\|_\infty \leqslant (T+1)\left[1+\frac{(T+1)\|\alpha\|_1}{1-(T+1)\|\alpha\|_1}\right]\left[\|\beta\|_1\|u'\|_\infty+\|\gamma\|_1+\frac{M}{T+1}\right]$$
$$=\frac{T+1}{1-(T+1)\|\alpha\|_1}\left[\|\beta\|_1\|u'\|_\infty+\|\gamma\|_1+\frac{M}{T+1}\right]$$

i.e.,

$$\|u'\|_\infty\left[1-\frac{(T+1)\|\beta\|_1}{1-(T+1)\|\alpha\|_1}\right]\leqslant\frac{T+1}{1-(T+1)\|\alpha\|_1}\left[\|\gamma\|_1+\frac{M}{T+1}\right].$$

Therefore

$$\|u'\|_\infty\left[\frac{1-(T+1)\|\alpha\|_1-(T+1)\|\beta\|_1}{1-(T+1)\|\alpha\|_1}\right]\leqslant\frac{1}{1-(T+1)\|\alpha\|_1}\left[(T+1)\|\gamma\|_1+M\right]$$

i.e.,

$$\|u'\|_\infty\leqslant\frac{(T+1)\left[\|\gamma\|_1+\frac{M}{T+1}\right]}{1-(T+1)\|\alpha\|_1-(T+1)\|\beta\|_1}=M_1 \tag{3.16}$$

thus, from (3.16), there exists $M_1>0$ such that

$$\|u'\|_\infty\leqslant M_1 \tag{3.17}$$

therefore, from (3.17) and (3.16), there exists $M_2>0$, such that

$$\|u\|_\infty\leqslant M_2. \tag{3.18}$$

Concequently

$$\|u\|=\max\left\{\|u\|_\infty,\|u'\|_\infty\right\}\leqslant\max\left\{M_1,M_2\right\}.$$

Again, from (3.1), (3.17) and (3.18) , we have

$$\|u'''\|_1=\|Lu\|_1\leqslant\|Nu\|_1\leqslant\|\alpha\|_1 M_2+\|\beta\|_1 M_1+\|\gamma\|_1.$$

Which shows that Ω_1 is bounded. □

LEMMA 3. *The set* $\Omega_2=\{u\in KerL : Nu\in ImL\}$ *is bounded.*

Proof. Let $u\in\Omega_2$, then $u\in KerL=\{u\in domL: \ u=bt,\, b\in\mathbb{R},\, t\in[0,T]\}$. Also, since $KerQ=ImL$, then $QNu=0$, therefore

$$\int_0^T (T-s)^2 f(s,bs,b)\,ds-\frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,bs,b)\,ds=0$$

From condition (H_2), $\|u\|_\infty=|b|T\leqslant MT$, so $\|u\|\leqslant MT$, thus Ω_2 is bounded. □

LEMMA 4. *If the first part of condition* (H_3) *holds, then*

$$b\left(\frac{60}{5T^4-2T\eta^3}\right)\left[\int_0^T (T-s)^2 f(s,bs,b)\,ds-\frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,bs,b)\,ds\right]<0 \tag{3.19}$$

for all $|b|>M^*$. *Let* $\Omega_3=\{u\in KerL:\ -\lambda Ju+(1-\lambda)QNu=0,\ \lambda\in[0,1]\}$ *where* $J:\ker L\to ImQ$ *is the linear isomorphism given by* $J(bt)=bt,\ \forall b\in\mathbb{R},\ t\in[0,T]$. *Then* Ω_3 *is bounded.*

Proof. Suppose that $u=b_0t\in\Omega_3$, then we obtain

$$\lambda b_0=(1-\lambda)\left(\frac{60}{5T^4-2T\eta^3}\right)\left(\int_0^T (T-s)^2 f(s,b_0s,b_0)-\frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,b_0s,b_0)\right).$$

If $\lambda=1$, then $b_0=0$, in this case Ω_3 is bounded.

If $\lambda\neq 1$, there exists $M^*>0$ such that $|b_0|>M^*$, then in view of the first part of (H_3), we have

$$\lambda b_0^2=b_0(1-\lambda)\left(\frac{30}{1-\eta^3}\right)\left(\int_0^T (T-s)^2 f(s,b_0s,b_0)-\frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,b_0s,b_0)\right)<0$$

which contradicts the fact that $\lambda b_0^2\geqslant 0$. Then $|u|=|b_0t|\leqslant|b_0|T\leqslant M^*T$, we obtain $\|u\|\leqslant M^*T$, hence $\Omega_3\subset\{u\in KerL:\ \|u\|\leqslant M^*T\}$ is bounded.

If $\lambda=0$, it yields

$$\int_0^T (T-s)^2 f(s,b_0s,b_0)\,ds-\frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,b_0s,b_0)\,ds=0,$$

taking condition (H_2) into account, we obtain $\|u\|=|b|T\leqslant M^*T$. □

LEMMA 5. *If the second part of* (H_3) *holds, then*

$$b\left(\frac{60}{5T^4-2T\eta^3}\right)\left[\int_0^T (T-s)^2 f(s,bs,b)\,ds-\frac{2T}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s,bs,b)\,ds\right]>0 \tag{3.20}$$

for all $|b|>M^*$ *and* $\Omega_3=\{u\in KerL:\ -\lambda Ju+(1-\lambda)QNu=0,\ \lambda\in[0,1]\}$ *is bounded, here* J *is defined as in Lemma 4.*

Proof. A similar argument as above shows that Ω_3 is bounded. □

Proof of Theorem 2. Let Ω to be an open bounded subset of X such that $\cup_{i=1}^3\overline{\Omega}_i\subset\Omega$. By using the fact that u''' is bounded and Arzela-Ascoli Theorem, we can prove that

$K_P(I-QN): \overline{\Omega} \to X$ is compact, thus N is $L-compact$ on $\overline{\Omega}$. Then by Lemmas 2 and 3, we have

(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0,1)$.

(ii) $Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$.

(iii) $H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu$, $\lambda \in [0,1]$.

According to Lemmas 4 and 5, we know that $H(u,\lambda) \neq 0$ for every $u \in \ker L \cap \partial\Omega$. Thus, by the homotopy property of degree, we obtain

$$\begin{aligned}
\deg\left(QN\,|_{\ker L}, \Omega\cap\ker L, 0\right) &= \deg\left(H(\cdot,0), \Omega\cap\ker L, 0\right)\\
&= \deg\left(H(\cdot,1), \Omega\cap\ker L, 0\right)\\
&= \deg\left(\pm J, \Omega\cap\ker L, 0\right) \neq 0.
\end{aligned}$$

Then, by Theorem 1, $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$, so the boundary value problem (1.1)–(1.2) has at least one solution in $C^2[0,T]$. The proof is complete. □

We construct an example to illustrate the applicability of the results presented.

EXAMPLE 1. Consider the following boundary value problem

$$u'''(t) = f\left(t, u(t), u'(t)\right),\ t \in (0,T) \tag{3.21}$$

$$u(0) = u''(0) = 0,\ u(T) = \frac{2T}{\eta^2}\int_0^{\eta} u(t)\,dt,\ \eta \in (0,T) \tag{3.22}$$

where

$$f\left(t, u(t), u'(t)\right) = \frac{(1-t^2)}{6}u(t) + \frac{1}{7}u'(t) + t,\ t \in (0,T)$$

Here, we take $T = \frac{3}{4}$, $\eta = \frac{1}{4}$.

We have

$$\left|f\left(t, u(t), u'(t)\right)\right| \leqslant \alpha(t)|u| + \beta(t)\left|u'\right| + \gamma(t)$$

where $\alpha(t) = \frac{1-t^2}{6}$, $\beta(t) = \frac{1}{7}$ and $\gamma(t) = t$, then α, β and γ are nonnegative and belong to $L^1[0,T]$, so, hypothesis (H_1) is satisfied.

Set $I = \int_0^T (T-s)^2 f(s,u(s),u'(s))\,ds - \frac{2T}{3\eta^2}\int_0^{\eta}(\eta-s)^3 f(s,u(s),u'(s))\,ds$.
For $M = 1,13317$ and $u \in domL$, $u(t) = bt$, if $|u'(t)| > M$, for all $t \in [0,T]$, then,

$$I = \frac{11211b + 12704}{491520} \neq 0.$$

Then, the condition (H_2) is satisfied.

Now, for $M^* = 2 > 0$, and any $u(t) = bt \in \ker L$ with $|b| > M^*$, we have

$$I = \frac{11211b^2 + 12704b}{491520} > 0,$$

consequently, condition (H_3) is satisfied.

Finally, a simple calculus gives $\|\alpha\|_1 + \|\beta\|_1 = \frac{13}{128} + \frac{3}{28} = \frac{187}{896} \leqslant \frac{1}{T+1} = \frac{4}{7}$. We conclude from Theorem 2 that the problem (3.21)–(3.22) has at least one solution in $C^2[0,T]$

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