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INTERACTIONS OF DELTA SHOCK WAVES FOR THE EQUATIONS OF CONSTANT PRESSURE FLUID DYNAMICS

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Abstract. The interactions among delta shock waves, vacuum states and contact discontinuities for the equations of constant pressure fluid dynamics are analyzed. By solving the Riemann problem with initial data of three piecewise constant states case by case, the global structures of solutions with four different configurations are constructed. Furthermore, the numerical simulations completely coinciding with theoretical analysis are presented.

1. Introduction

Consider the one-dimensional equations of constant pressure fluid dynamics

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \\ (\rho v)_t + (\rho u v)_x = 0, \end{cases} \tag{1.1}$$

where $\rho \geqslant 0$ and (u, v) represent the density and velocity of the fluid, respectively.

The system (1.1) is actually the special form for one-dimensional equations of fluid dynamics. We recall that the equations of fluid dynamics in Eulerian coordinates read [4]

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \\ (\rho v)_t + (\rho u v)_x = 0, \end{cases} \tag{1.2}$$

while $p = p(\rho)$ denotes the pressure. Noting that a flow is formed by two kinds of effects, namely, the effect of inertia and the effect of pressure difference, so neglecting the effect of pressure difference in (1.2), the zero-pressure system (1.1) is obtained.

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A distinct feature for (1.1) is that it is a weakly coupled system in the sense that the first two equations, which are the zero-pressure flow model consisting of conservation of mass and momentum, are independent of the third one. However, as pointed out in [7], it is necessary to consider the three equations combined in (1.1) in constructing the Riemann solution of the two-dimensional system of fluid dynamics with constant pressure. Nowadays, the zero-pressure type system (1.1) has been used to describe some important physical phenomena, such as the motion of free particles sticking together under collision [1] and the formation of large scale structures in the universe [12, 17].

Specifically, the Riemann problem for (1.1) with piecewise constant initial data

$$(\rho,u,v)(0,x)=\begin{cases}(\rho_-,u_-,v_-),\ x<0,\\(\rho_+,u_+,v_+),\ x>0\end{cases} \tag{1.3}$$

has been studied by Hu [8], where $\rho_\pm \geqslant 0$, $u_\pm$ and $v_\pm$ are constants. By employing the viscosity vanishing approach, the Riemann solution has been constructed. Interestingly, a special type of nonlinear singular shock wave called the delta shock wave appears in solution.

The delta shock wave is a generalization of the traditional shock wave. It is a new kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. Compared to the ordinary shock wave, the delta shock wave is more compressive because more characteristics will enter the discontinuity line. Physically, a delta shock wave is used to represent the process of concentration of mass. It is well known that, the study of the systems whose solutions admit a delta shock wave is very important in applications, because systems of this type often arise in modelling physical processes in gas dynamics, magnetohydrodynamics, filtration theory, and cosmology [5, 6, 17]. For more details about the delta shock waves, we refer to [9, 11, 14, 15, 16, 18, 19, 20], etc.

As for the study of delta shock wave, a hot topic is to investigate the wave interactions. In the past over two decades, the discussion for wave interactions has been increasingly active due to their significance in practical applications and basic role as building blocks for the general mathematical theory of quasi-linear hyperbolic equations. Moreover, the results on interactions are also touchstones for the numerical schemes. As far as the interaction of delta shock waves for zero-pressure type equations, we refer to Sheng and Zhang [14] and Cheng et al. [3] for the one-dimensional and two-dimensional zero-pressure gas dynamics, respectively. See also Shen and Sun [13] for the transport equations, Cai and Zhang et al. [2, 22] for the zero-pressure gas dynamics with energy conservation law, Zhang and Zhang [23] for the steady zero-pressure adiabatic flow, etc.

Motivated by the works mentioned above, we in this paper analyze the interactions among delta shock waves, vacuums, as well as contact discontinuities. For this purpose, we study the Riemann problem of (1.1) with the initial data of three piecewise constant states as follows

$$(\rho,u,v)(0,x)=\begin{cases}(\rho_-,u_-,v_-),\ x<-\varepsilon,\\(\rho_m,u_m,v_m),\ -\varepsilon<x<\varepsilon,\\(\rho_+,u_+,v_+),\ x>\varepsilon,\end{cases}\tag{1.4}$$

where $\rho_m \geqslant 0$, u_m and v_m are constants. The initial data (1.4) can be viewed as a perturbation of the corresponding Riemann initial data (1.3) with the small perturbed parameter $\varepsilon > 0$.

Clearly, discussing interaction of waves for (1.1) is equivalent to solving Riemann problem of (1.1) with (1.4). For this purpose, one has to justify if the two adjacent waves intersect and interact with each other when constructing the global solution. Nevertheless, it is not so easy to cope with the overtaking of two delta shock waves. Luckily, by skillfully discussing a special case when the initial data contain Dirac measure, we overcome this difficulty successfully, see Section 3 bellow. Rely on this result, four kinds of different explicit structures of Riemann solutions are established uniquely. It is found that, the solutions of the perturbed Riemann problem (1.1), (1.4) converge to nothing but the corresponding ones of (1.1), (1.3) as $\varepsilon \to 0$, from which the stability of the Riemann solutions with respect to this local small perturbation of the Riemann initial data are obtained.

This paper is organized as follows. In Section 2, we review the Riemann solution for (1.1) and (1.3). Section 3 discusses the interactions of delta shock waves and vacuum states, and constructs the global solutions of (1.1) and (1.4). Finally, some numerical simulations coinciding with the theoretical analysis are presented in Section 4.

2. Delta shock wave and vacuum state for (1.1) and (1.3)

This section reviews the delta-shock and vacuum solution of (1.1), (1.3). For more details, please see [8, 21], etc.

The system (1.1) has a triple eigenvalue $\lambda = u$ and two right eigenvectors $\overrightarrow{r_1} = (1,0,0)^T$ and $\overrightarrow{r_2} = (0,0,1)^T$. Since one can check that $\nabla\lambda \cdot \overrightarrow{r_i} \equiv 0$ for $i = 1,2$, so λ is linearly degenerate and the elementary waves involve only contact discontinuities.

Noting that the equations and the Riemann data are invariant under uniform stretching of coordinates: $(t,x) \to (\beta t, \beta x)$ (β is constant), it follows that if the solution is unique, then the solution must depend on x/t alone. Therefore, we can look for the self-similar solutions of (1.1) and (1.3) as

$$(\rho,u,v)(t,x) = (\rho,u,v)(\xi), \quad \xi = x/t,$$

for which the Riemann problem (1.1) and (1.3) is reduced to the boundary value problem

$$-\xi\rho_\xi + (\rho u)_\xi = 0, \quad -\xi(\rho u)_\xi + (\rho u^2)_\xi = 0, \quad -\xi(\rho v)_\xi + (\rho uv)_\xi = 0$$

with the boundary data $(\rho,u,v)(\pm\infty) = (\rho_\pm,u_\pm,v_\pm)$.

As in [8, 21], we can construct the solution of (1.1), (1.3) by two cases.

For the case $u_- < u_+$, the solution contains two contact discontinuities and a vacuum state besides two constants, which is

$$(\rho,u,v)(\xi) = \begin{cases} (\rho_-,u_-,v_-), & -\infty < \xi < u_-, \\ (0,u(\xi),v(\xi)), & u_- \leqslant \xi \leqslant u_+, \\ (\rho_+,u_+,v_+), & u_+ < \xi < +\infty, \end{cases} \tag{2.1}$$

where $u(\xi), v(\xi)$ are two smooth functions.

For the case $u_- > u_+$, the singularity of solutions must develop because of the overlap of characteristic lines. Thus, the delta-shock solution should be introduced. To do so, we define a weighted delta function supported on a curve.

DEFINITION 1. A two-dimensional weighted delta function $w(s)\,\delta_S$ supported on a smooth curve S parameterized as $t=t(s)$, $x=x(s)$ $(a \leqslant s \leqslant b)$ is defined by

$$\big\langle w(t(s))\delta_S, \varphi(t(s),x(s))\big\rangle = \int_a^b w(t(s))\varphi(t(s),x(s))\sqrt{x'(s)^2+t'(s)^2}ds \tag{2.2}$$

for all test functions $\varphi(t,x) \in C_0^\infty(R^+ \times R^1)$.

The definition of solutions to the system (1.1) in the sense of distributions is as follows.

DEFINITION 2. A triple (ρ,u,v) is called a delta-shock solution of (1.1) and (1.3) in the sense of distributions if there exist a smooth curve L and a function $w(t)$ such that ρ, u and v are represented in the following form

$$\rho = \bar\rho(t,x) + w(t)\delta_L, \quad u = \bar u(t,x), \quad v = \bar v(t,x),$$

$\bar\rho, \bar u, \bar v \in L^\infty([0,+\infty)\times R;R)$, $w(t) \in C^1(L)$, $u|_L = u_\delta(t)$, $v|_L = v_\delta(t)$, and they satisfy

$$\langle \rho,\phi_t\rangle + \langle \rho u,\phi_x\rangle = 0, \quad \langle \rho u,\phi_t\rangle + \langle \rho u^2,\phi_x\rangle = 0, \quad \langle \rho v,\phi_t\rangle + \langle \rho uv,\phi_x\rangle = 0, \tag{2.3}$$

for all test functions $\phi \in C_0^\infty(R^+\times R^1)$, where

$$\langle \rho,\phi\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \bar\rho\phi \,\mathrm{d}x\mathrm{d}t + \langle w(t)\delta_L,\phi\rangle,$$
$$\langle \rho u,\phi\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \bar\rho\bar u\phi \,\mathrm{d}x\mathrm{d}t + \langle w(t)u_\delta(t)\delta_L,\phi\rangle,$$

and v has the similar integral identities as above.

Based on this definition, we can introduce a piecewise smooth solution to the Riemann problem (1.1) and (1.3) in the form

$$(\rho,u,v)(t,x)=\begin{cases}(\rho_-,u_-,v_-), & x<x(t),\\ \big(w(t)\delta(x-x(t)),u_\delta(t),v_\delta(t)\big), & x=x(t),\\ (\rho_+,u_+,v_+), & x>x(t),\end{cases} \tag{2.4}$$

where $w(t)$ is the strength of the delta shock wave, and $u_\delta(t), v_\delta(t)$ are the corresponding values of u,v on the discontinuous curve $x=x(t)$.

The solution $(\rho,u,v)(t,x)$ defined above should satisfy the following generalized Rankine-Hugoniot relations

$$\begin{cases}\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}=u_\delta(t),\\[2mm] \dfrac{\mathrm{d}w(t)}{\mathrm{d}t}=-[\rho]u_\delta(t)+[\rho u],\\[2mm] \dfrac{\mathrm{d}\big(w(t)u_\delta(t)\big)}{\mathrm{d}t}=-[\rho u]u_\delta(t)+[\rho u^2],\\[2mm] \dfrac{\mathrm{d}\big(w(t)v_\delta(t)\big)}{\mathrm{d}t}=-[\rho v]u_\delta(t)+[\rho uv].\end{cases} \tag{2.5}$$

The derivation of (2.5) is similar to that in [14], so we omit it here.

What is more, in order to guarantee the uniqueness, the delta shock wave should satisfy the Lax entropy condition

$$u_+<u_\delta<u_-, \tag{2.6}$$

which means that the characteristics on both sides of the discontinuity are in-coming.

Under the entropy condition (2.6), by solving the ordinary differential equations (2.5) with the initial data $t=0: x(0)=0, w(0)=0$, one has

$$\begin{cases}x(t)=\dfrac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}t,\\[2mm] w(t)=\sqrt{\rho_-\rho_+}(u_--u_+)t,\\[2mm] u_\delta=\dfrac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}},\\[2mm] v_\delta=\dfrac{\sqrt{\rho_-}v_-+\sqrt{\rho_+}v_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}.\end{cases} \tag{2.7}$$

3. Interactions of waves

This section studies the wave interactions. Just for convince, we shall consider a special case when the initial data contain Dirac measure.

3.1. Solutions when the initial data contain Dirac measure

We set forth to solve the ordinary differential equations (2.5) with the initial values

$$t=0:\ x(0)=0,\quad w(0)=m_0,\quad u_\delta(0)=u_0,\quad v_\delta(0)=v_0, \tag{3.1}$$

satisfying $u_+ < u_0 < u_-$.

We obtain from the second and third ones of (2.5) that

$$\begin{cases} w(t)-m_0=-[\rho]x(t)+[\rho u]t, \\ w(t)u_\delta(t)-m_0u_0=-[\rho u]x(t)+[\rho u^2]t, \end{cases} \tag{3.2}$$

from which we have

$$[\rho]u_\delta(t)x(t)-\Big([\rho u]x(t)+[\rho u]u_\delta(t)t\Big)+[\rho u^2]t+m_0u_0-m_0u_\delta(t)=0,$$

namely,

$$\frac{d}{dt}\left(\frac{[\rho]}{2}x^2(t)-[\rho u]x(t)t+\frac{[\rho u^2]}{2}t^2+m_0u_0t-m_0x(t)\right)=0.$$

Then, we obtain the equation on $x(t)$ as follows

$$[\rho]x^2(t)-2\Big([\rho u]t+m_0\Big)x(t)+[\rho u^2]t^2+2m_0u_0t=0. \tag{3.3}$$

So, one can solve that

$$x(t)=\begin{cases} \dfrac{2m_0u_0t+\rho_+(u_-^2-u_+^2)t^2}{2(\rho_+(u_--u_+)t+m_0)}, & [\rho]=0, \\ \dfrac{[\rho u]t+m_0-w(t)}{[\rho]}, & [\rho]\neq 0, \end{cases} \tag{3.4}$$

and

$$\begin{cases} w(t)=\sqrt{m_0^2+\rho_-\rho_+(u_--u_+)^2t^2+2m_0(\rho_-(u_--u_0)+\rho_+(u_0-u_+))t}, \\ u_\delta(t)=\dfrac{m_0u_0+[\rho u^2]t-[\rho u]x(t)}{w(t)}, \\ v_\delta(t)=\dfrac{m_0v_0+[\rho uv]t-[\rho v]x(t)}{w(t)}. \end{cases} \tag{3.5}$$

Moreover, by a direct calculation, one can summarize that

LEMMA 1. *There exists a unique solution* $(x(t), w(t), u_\delta(t), v_\delta(t))$ *to the initial value problem* (2.5) *and* (3.1) *as* $u_+ < u_0 < u_-$, *which enjoys the following properties:*
(i) $x'(t)$ *is a momotone function of* t;
(ii) *If* $\rho_- + \rho_+ > 0$, *then*

$$\lim_{t\to+\infty} u_\delta(t) = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \quad \lim_{t\to+\infty} v_\delta(t) = \frac{\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+}{\sqrt{\rho_-} + \sqrt{\rho_+}};$$

While if $\rho_- = \rho_+ = 0$, *then* $u_\delta(t) = u_0, v_\delta(t) = v_0$;
(iii) $u_+ < u_\delta(t) < u_-$.

REMARK 1. When $(u_0, m_0) = (0,0)$, (3.4)-(3.5) is nothing but the solution expressed in (2.7), which shows that the delta-shock solution is stable under perturbations of initial data.

3.2. Interactions of delta shock waves

Now, let us analyze the interaction of delta shock waves. To ensure that all the cases are covered completely, according to the relation among u_-, u_m, u_+, our discussion is divided into four cases:

(1). $u_- > u_m > u_+$; (2). $u_- > u_+ > u_m$;
(3). $u_+ > u_- > u_m$; (4). $u_+ > u_m > u_-$.

Case 1. $u_- > u_m > u_+$.

In this case, two delta shock waves δ_1 and δ_2 will emit from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$, respectively, as shown in Fig. 3.1. These two delta shock waves are uniquely determined by

$$\delta_1 : \begin{cases} x_1(t) = \dfrac{\sqrt{\rho_-}u_- + \sqrt{\rho_m}u_m}{\sqrt{\rho_-} + \sqrt{\rho_m}}t - \varepsilon, \\ u_{\delta_1} = \dfrac{\sqrt{\rho_-}u_- + \sqrt{\rho_m}u_m}{\sqrt{\rho_-} + \sqrt{\rho_m}}, \\ v_{\delta_1} = \dfrac{\sqrt{\rho_-}v_- + \sqrt{\rho_m}v_m}{\sqrt{\rho_-} + \sqrt{\rho_m}}, \\ w_{\delta_1}(t) = \sqrt{\rho_-\rho_m}(u_- - u_m)t, \end{cases} \tag{3.6}$$

and

$$\delta_2 : \begin{cases} x_2(t) = \dfrac{\sqrt{\rho_m}u_m + \sqrt{\rho_+}u_+}{\sqrt{\rho_+} + \sqrt{\rho_m}}t + \varepsilon, \\ u_{\delta_2} = \dfrac{\sqrt{\rho_m}u_m + \sqrt{\rho_+}u_+}{\sqrt{\rho_m} + \sqrt{\rho_+}}, \\ v_{\delta_2} = \dfrac{\sqrt{\rho_m}v_m + \sqrt{\rho_+}v_+}{\sqrt{\rho_m} + \sqrt{\rho_+}}, \\ w_{\delta_2}(t) = \sqrt{\rho_m\rho_+}(u_m - u_+)t. \end{cases} \tag{3.7}$$

According to the entropy condition (2.6), one has $u_+ < u_{\delta_2} < u_m < u_{\delta_1} < u_-$, which means that δ_1 must overtake δ_2 at a finite time, and the intersection point (x^*,t^*) is determined by

$$\begin{cases} x^* + \varepsilon = u_{\delta_1} t^*, \\ x^* - \varepsilon = u_{\delta_2} t^*, \end{cases} \tag{3.8}$$

which yields that $(x^*,t^*) = \left(\frac{u_{\delta_1}+u_{\delta_2}}{u_{\delta_1}-u_{\delta_2}}\varepsilon,\ \frac{2\varepsilon}{u_{\delta_1}-u_{\delta_2}}\right)$.

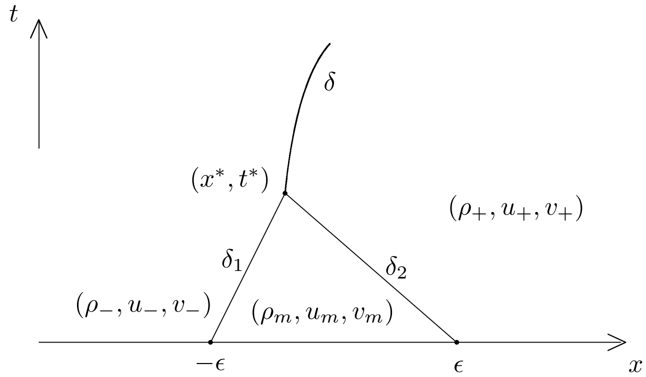


Fig. 3.1. $u_- > u_m > u_+$.

At the intersection (x^*,t^*), the new initial data are formed:

$$t = t^*: \begin{cases} x(t^*) = x_1(t^*) = x_2(t^*), \\ w(t^*) = w_{\delta_1}(t^*) + w_{\delta_2}(t^*) := w^*, \\ u_\delta(t^*) = \dfrac{w_{\delta_1}(t^*)u_{\delta_1} + w_{\delta_2}(t^*)u_{\delta_2}}{w_{\delta_1}(t^*) + w_{\delta_2}(t^*)} := u_\delta^*, \\ v_\delta(t^*) = \dfrac{w_{\delta_1}(t^*)v_{\delta_1} + w_{\delta_2}(t^*)v_{\delta_2}}{w_{\delta_1}(t^*) + w_{\delta_2}(t^*)} := v_\delta^*, \end{cases} \tag{3.9}$$

satisfying $u_{\delta_1} > u_\delta^* > u_{\delta_2}$. Since it holds that $u_- > u_+$, so a new delta shock wave will generate after interaction and we denote it with $\delta : x = x(t)$. By Lemma 3.1, the trajectory, velocity and weighs $(x(t), u_\delta(t), v_\delta(t), w(t))$ of δ can be uniquely obtained by solving the ordinary differential equations (2.5) with the initial date (3.9). The detail is omitted.

Thus, the result of interaction of two delta shock waves is still a single delta shock wave, which can be formulated as

$$\delta + \delta \to \delta.$$

Case 2. $u_- > u_+ > u_m$. (When $u_m > u_+ > u_-$, the structure of solution is similar.)

In this situation, as shown in Fig. 3.2, when t is small enough, a delta shock wave δ_1 determined by (3.6) emits from $(-\varepsilon, 0)$, and two contact discontinuities $J_1 : u = u_m$ and $J_2 : u = u_+$ with a vacuum in between from $(\varepsilon, 0)$.

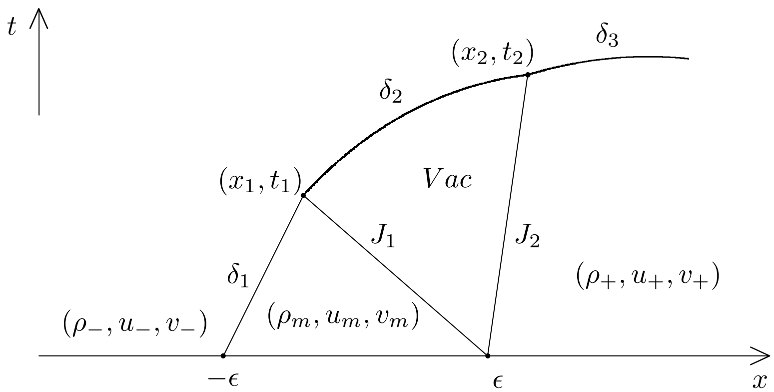


Fig. 3.2. $u_- > u_+ > u_m$.

Noting that the propagating speed of δ_1 satisfies $u_m < u_{\delta_1} < u_-$, so δ_1 must meet the contact discontinuity $J_1 : u = u_m$ at the point $(x_1, t_1) = \left(\frac{u_{\delta_1}+u_m}{u_{\delta_1}-u_m}\varepsilon, \frac{2\varepsilon}{u_{\delta_1}-u_m}\right)$, and a new delta shock wave $\delta_2 : x = x_2(t)$ forms. Obviously, the δ_2 connects the left state (ρ_-, u_-, v_-) and a vacuum state, so it obeys the generalized Rankine-Hugoniot relations

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = u_\delta(t), \\ \dfrac{\mathrm{d}w(t)}{\mathrm{d}t} = \rho_- u_- - \rho_- u_\delta(t), \\ \dfrac{\mathrm{d}\big(w(t)u_\delta(t)\big)}{\mathrm{d}t} = \rho_- u_-^2 - \rho_- u_- u_\delta(t), \\ \dfrac{\mathrm{d}\big(w(t)v_\delta(t)\big)}{\mathrm{d}t} = \rho_- u_- v_- - \rho_- v_- u_\delta(t), \end{cases} \tag{3.10}$$

with the initial data

$$t = t_1 : \begin{cases} x(t_1) = x_1(t_1) := x_1, \\ w(t_1) = w_{\delta_1}(t_1) := w_1, \\ u_\delta(t_1) = u_{\delta_1}(t_1) := u_{\delta_1} \\ v_\delta(t_1) = v_{\delta_1}(t_1) := v_{\delta_1}. \end{cases} \tag{3.11}$$

Thus, solving (3.10) and (3.11) yields that

$$\begin{cases} x_2(t) = \dfrac{\rho_- u_-(t-t_1)+w_1-\sqrt{w_1^2+2w_1\rho_-(u-u_{\delta_1})(t-t_1)}}{\rho_-}+x_1, \\ w_2(t) = \sqrt{w_1^2+2w_1\rho_-(u-u_{\delta_1})(t-t_1)}, \\ u_{\delta_2}(t) = u_- + \dfrac{w_1(u_{\delta_1}-u_-)}{\sqrt{w_1^2+2w_1\rho_-(u-u_{\delta_1})(t-t_1)}}, \\ v_{\delta_2}(t) = v_- + \dfrac{w_1(v_{\delta_1}-v_-)}{\sqrt{w_1^2+2w_1\rho_-(u-u_{\delta_1})(t-t_1)}}. \end{cases}$$

It is clear that δ_2 will cross the vacuum region with a varying propagation speed. By Lemma 3.1, one has $\lim\limits_{t\to+\infty} u_{\delta_2}(t) = u_- > u_+$, so δ_2 will penetrate over the whole vacuum region and then meet $J_2 : u = u_+$ at a finite time. The intersection (x_2,t_2) is determined by

$$\begin{cases} x_2 = \dfrac{\rho_- u_-(t_2-t_1)+w_1-\sqrt{w_1^2+2w_1\rho_-(u-u_{\delta_1})(t_2-t_1)}}{\rho_-}+x_1, \\ x_2 = u_+t_2+\varepsilon. \end{cases}$$

At this moment, a new initial value problem is formed and can be solved similarly to Case 1. Thus, after the interaction of δ_2 and J_2, a new delta shock wave connecting two constant states (ρ_-,u_-,v_-) and (ρ_+,u_+,v_+) is obtained, denoted by δ_3.

The conclusion of this case is that the delta shock wave will penetrate over the whole vacuum region between two contact discontinuities, which is expressed as

$$\delta + J + Vac + J \to \delta.$$

Case 3. $u_+ > u_- > u_m$. (When $u_m > u_- > u_+$, the structure of solution is similar.)

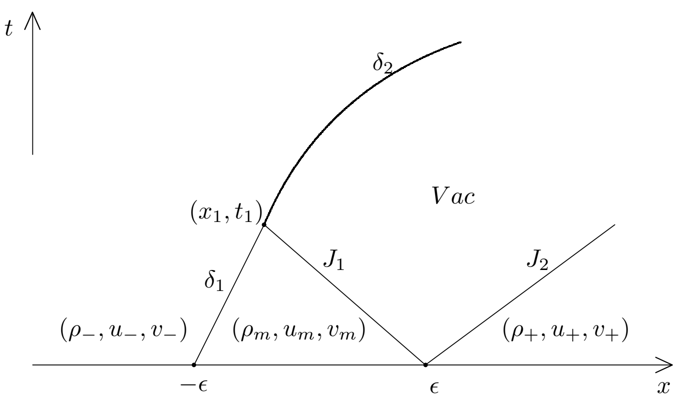


Fig. 3.3. $u_+ > u_- > u_m$.

Similar to Case 2, when t is small enough, there are a delta shock wave, two contact discontinuities and a vacuum on the (x,t)-plane, as shown in Fig. 3.3.

The delta shock wave δ_1 collides with J_1 at first and a new delta shock wave δ_2 generates. However, since $\lim\limits_{t\to+\infty} u_{\delta_2}(t) = u_- < u_+$, so δ_2 can not penetrate over the vacuum region and finally has $x_2(t) = u_-t + \varepsilon$ as its asymptote. This fact is different from the former one and can be symbolized as

$$\delta + J + Vac + J \to \delta + Vac + J.$$

Case 4. $u_+ > u_m > u_-$.

In this situation, both the contact discontinuities with a vacuum state in between emit from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$, respectively. Since the contact discontinuities J_2 and J_3 own the same propagating speed, thus there is no collision of waves, as shown in Fig. 3.4.

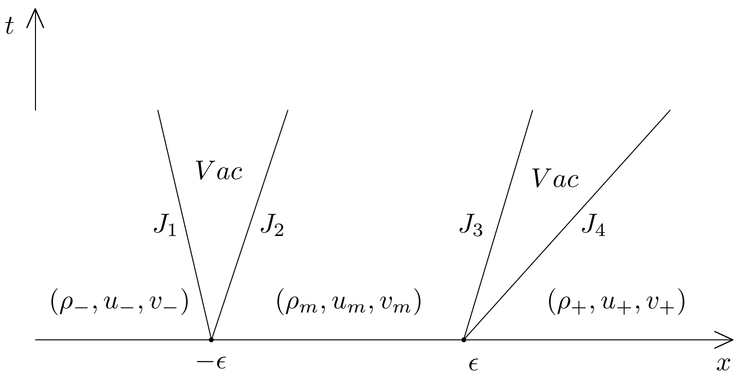


Fig. 3.4. $u_+ > u_m > u_-$.

The solution for this case is called a collisionless solution, which can be expressed as

$$(\rho,u,v)(t,x) = \begin{cases} (\rho_-,u_-,v_-), & x < u_-t - \varepsilon, \\ Vac, & u_-t - \varepsilon < x < u_mt - \varepsilon, \\ (\rho_m,u_m,v_m), & u_mt - \varepsilon < x < u_mt + \varepsilon, \\ Vac, & u_mt + \varepsilon < x < u_+t + \varepsilon, \\ (\rho_+,u_+,v_+), & x > u_+t + \varepsilon. \end{cases} \tag{3.12}$$

4. Numerical simulations

This section presents some representative numerical simulations to verify the validity of the interactions of waves mentioned in Section 3. Many more numerical tests have been performed to make sure that what are presented are not numerical artifacts.

To discretize the system, we employ the second-order non-oscillatory central schemes [10] with 300×300 cells and CFL=0.475. Then, by taking $\varepsilon=0.2$, we simulate the interaction of waves by four cases. For convenience, each situation will be simulated at two different time.

Case 1. $u_- > u_m > u_+$. For this case, the initial data are selected as

$$\text{Data 1:}\ \ (\rho,u,v)(0,x)=\begin{cases}(0.15,0.50,-0.60),\ x<-0.2,\\(0.10,0.01,0.02),\quad -0.2<x<0.2,\\(0.20,-0.35,0.35),\ x>0.2.\end{cases}$$

The numerical results are presented by Figs.4.1-4.3.

From Figs.4.1-4.3, it is easy to observe that, when $t=0.1$, two delta shock waves occur at $(-0.2,0)$ and $(0.2,0)$, respectively. As the time t increases, these two delta shock waves will overtake each other and finally unify into a new delta shock wave at $t=4.6$.

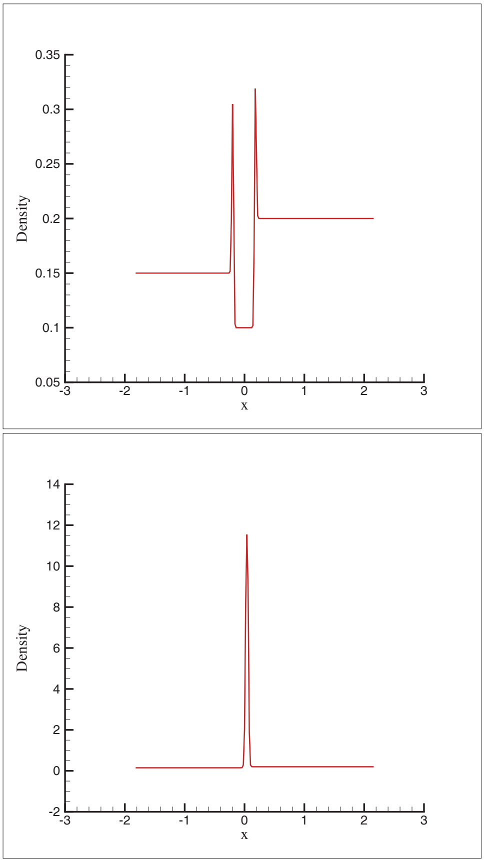


Fig. 4.1. Numerical results of ρ at $t=0.1$ (up) and $t=4.6$ (down) for Data 1.

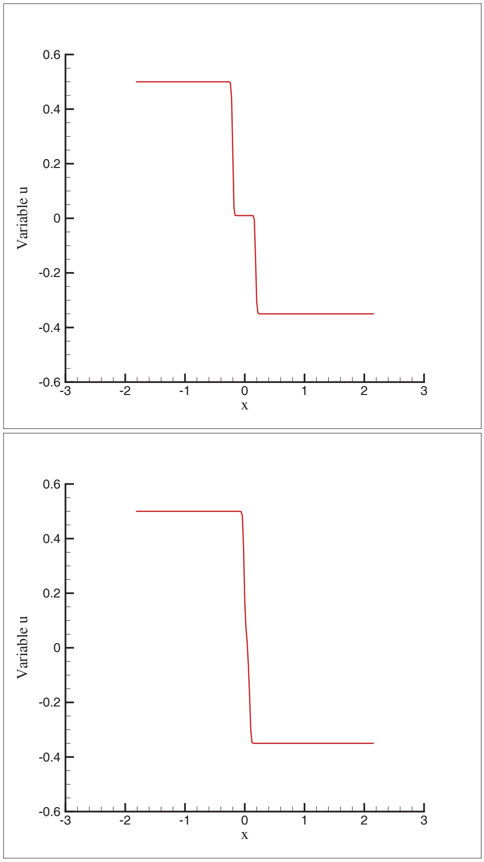
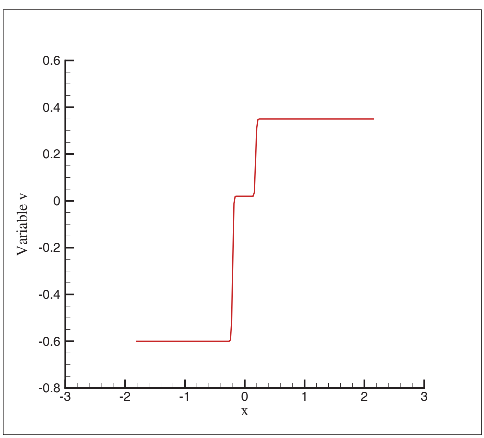


Fig. 4.2. Numerical results of u at $t = 0.1$ (up) and $t = 4.6$ (down) for Data 1.



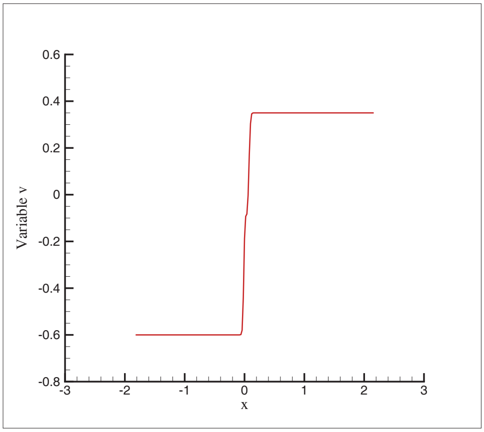
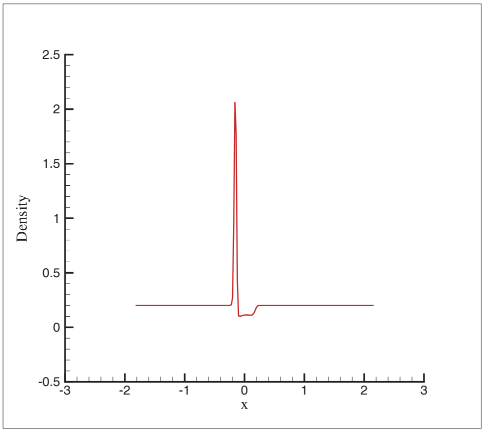


Fig. 4.3. Numerical results of v at $t = 0.1$ (up) and $t = 4.6$ (down) for Data 1.

Case 2. $u_- > u_+ > u_m$. For this case, the initial data are chosen as

$$\text{Data 2:}\ \ (\rho, u, v)(0, x) = \begin{cases} (0.20, 0.50, 0.20), & x < -0.2, \\ (0.10, -0.25, -0.1), & -0.2 < x < 0.2, \\ (0.20, -0.15, -0.06), & x > 0.2. \end{cases}$$

The numerical results are presented by Figs.4.4-4.6.



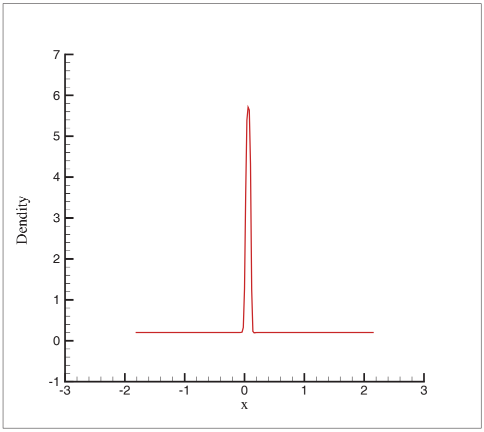
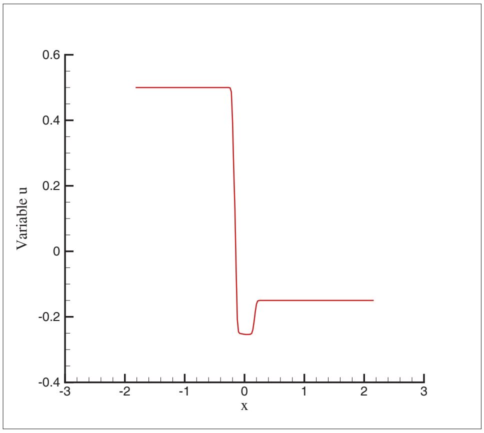


Fig. 4.4. Numerical results of ρ at $t = 0.8$ (up) and $t = 4.3$ (down) for Data 2.



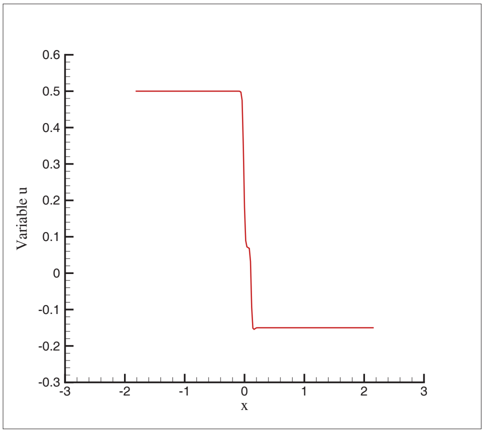


Fig. 4.5. Numerical results of u at $t = 0.8$ (up) and $t = 4.3$ (down) for Data 2.

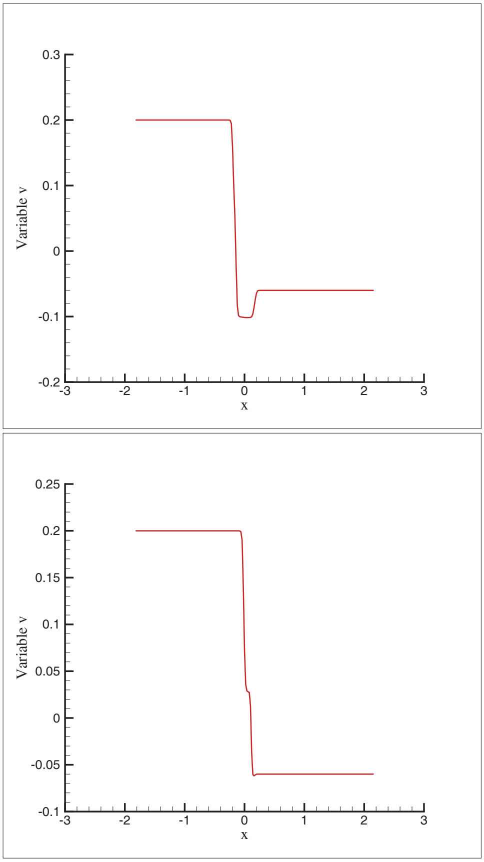


Fig. 4.6. Numerical results of v at $t=0.8$ (up) and $t=4.3$ (down) for Data 2.

From Figs.4.4-4.6, we can clearly see that, at $t=0.8$, a delta shock wave and two contact discontinuities with a vacuum state in between emit from $(-0.2,0)$ and $(0.2,0)$, respectively. However, the delta shock wave penetrates over the whole vacuum region, and a new delta shock wave generates at $t=4.3$.

Case 3. $u_+ > u_- > u_m$. For this case, we set the initial data as

$$\text{Data 3:}\ \ (\rho,u,v)(0,x)=\begin{cases}(0.40,0.18,0.21), & x<-0.2,\\(0.35,-0.06,-0.07), & -0.2<x<0.2,\\(1.20,0.41,0.38), & x>0.2.\end{cases}$$

The numerical results are presented by Figs.4.7-4.9.

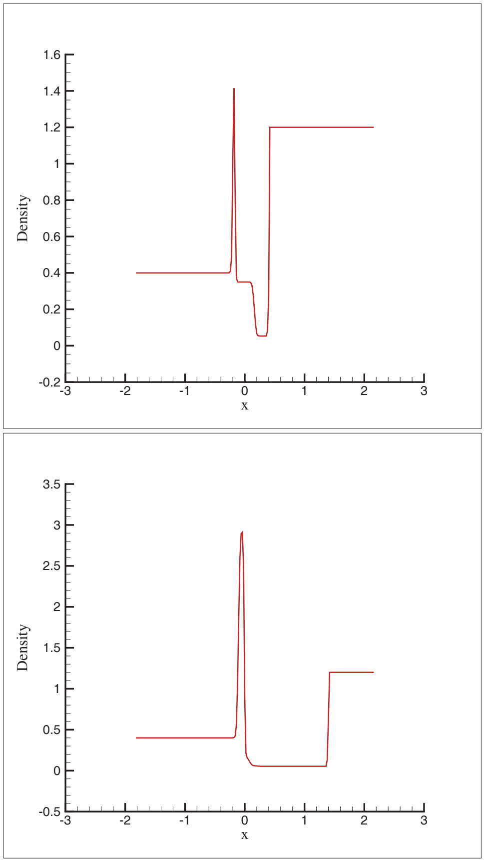
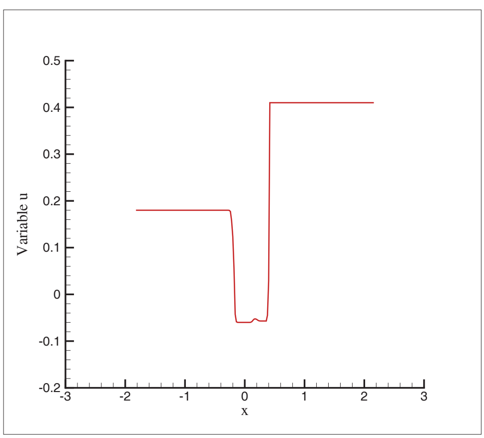


Fig. 4.7. Numerical results of ρ at $t = 0.5$ (up) and $t = 2.8$ (down) for Data 3.



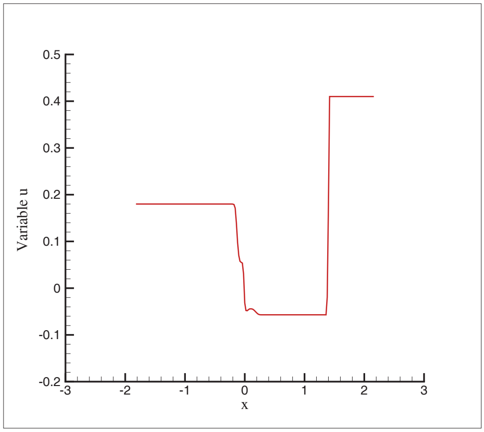


Fig. 4.8. Numerical results of u at $t = 0.5$ (up) and $t = 2.8$ (down) for Data 3.

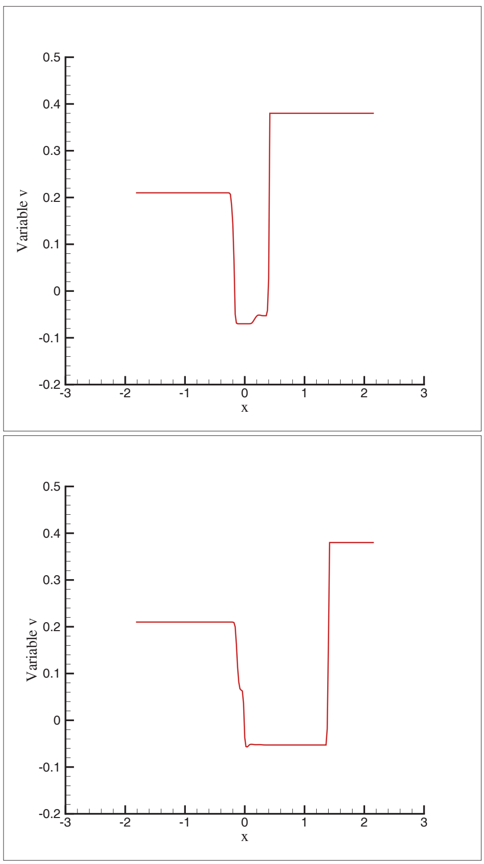


Fig. 4.9. Numerical results of v at $t = 0.5$ (up) and $t = 2.8$ (down) for Data 3.

Figs. 4.7–4.9 imply that, a delta shock wave emits from $(-0.2,0)$, and two contact discontinuities with a vacuum in between from $(0.2,0)$ at $t=0.5$. However, the delta shock wave will never penetrate over the whole vacuum region even though time is on the increase. That is to say, in this process, the region of vacuum state keeps expanding.

Case 4. $u_+ > u_m > u_-$. For this case, the initial data are selected to be

$$\text{Data 4:}\ (\rho,u,v)(0,x)=\begin{cases}(1.05,-0.37,-0.08), x<-0.2,\\(0.65,-0.01,0.00), \quad -0.2<x<0.2,\\(1.06,0.37,0.09), \quad x>0.2.\end{cases}$$

The numerical results are presented by Figs.4.10-4.12.

From Figs. 4.10–4.12, we find that, both the contact discontinuities with a vacuum state in between emit from $(-0.2,0)$ and $(0.2,0)$ at $t=0.8$, respectively. As time goes on, the vacuum state keeps continuously expanding and never disappears.

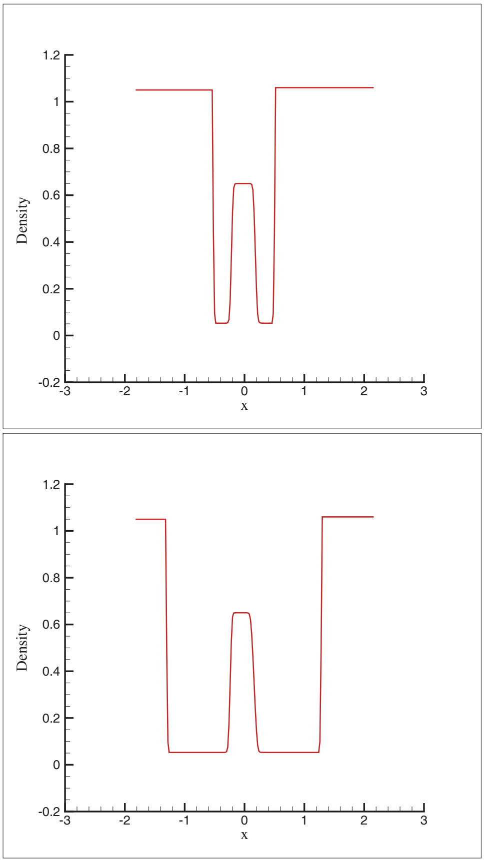


Fig. 4.10. Numerical results of ρ at $t=0.8$ (up) and $t=2.8$ (down) for Data 4.

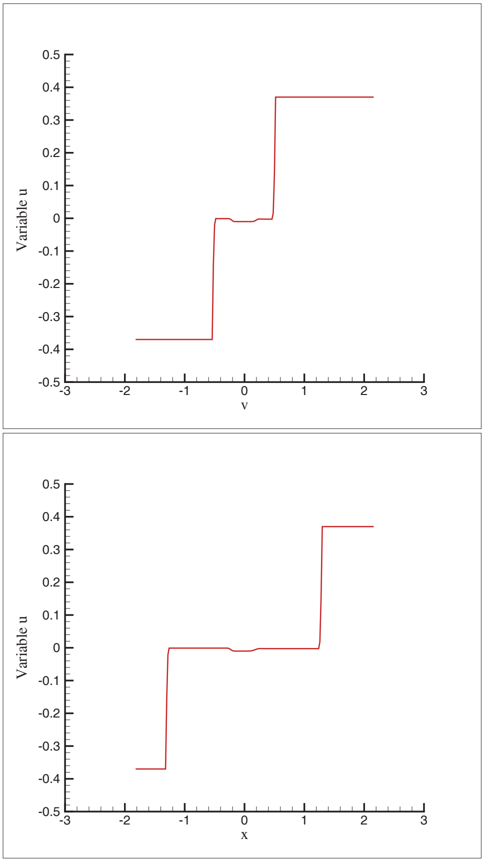
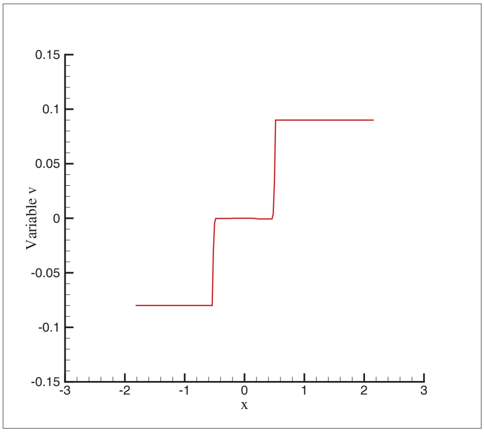


Fig. 4.11. Numerical results of u at $t = 0.8$ (up) and $t = 2.8$ (down) for Data 4.



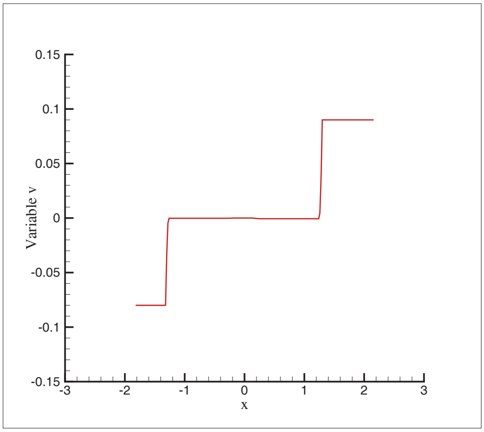


Fig. 4.12. Numerical results of v at $t = 0.8$ (up) and $t = 2.8$ (down) for Data 4.

To sum up, all of the above numerical results clearly reveal the interactions of delta shock waves and vacuum states discussed in Section 3. However, we also indicate that, because of the occurrence of singularity as the weighted Dirac delta functions, some slight oscillations appear in the numerical experiments, which may be a challenge for numerical schemes when delta shock waves develop in solutions.

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