Differential
Equations
& Applications
Volume 13, Number 2 (2021), 101–114
doi:10.7153/dea-2021-13-07

CAPUTO TYPE MODIFICATION OF THE ERDÉLYI–KOBER COUPLED IMPLICIT FRACTIONAL DIFFERENTIAL SYSTEMS WITH RETARDATION AND ANTICIPATION

MOKHTAR BOUMAAZA, MOUFFAK BENCHOHRA* AND JUAN J. NIETO

(*Communicated by J. R. Wang*)

Abstract. In this paper, we deal with the existence and uniqueness of solutions of a coupled system of nonlinear implicit fractional differential equations of Caputo-type modification of the Erdélyi-Kober involving both retarded and advanced arguments. The arguments are based upon the Banach contraction principle and Schauder's fixed point theorem. An example is included to show the applicability of our outcomes.

1. Introduction

Differential equations of fractional order are valuable in modeling phenomena in various fields of science and engineering. They can be found in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. For examples and details, we refer the reader to the monographs [1, 2, 3, 19, 24, 25, 28], and the references therein. On the other hand, coupled systems of fractional differential equations arise in various problems of applied nature. In recent years, some authors have investigated the existence and uniqueness of solutions for coupled systems of nonlinear fractional differential equations; see [8] and the references therein.

In [23] the authors provide some properties of Caputo-type modification of the Erdélyi-Kober fractional derivative. More details on the Erdélyi-Kober fractional integral and fractional derivative are given in [10, 16, 20, 21, 22]. Implicit differential equations have been considered by many authors [6, 11, 12, 14, 27]. In [4, 9, 13, 15, 18, 26], the authors studied the existence and uniqueness of solutions for boundary value problems of integer and fractional order functional differential equations and inclusions involving both retarded and advanced arguments. In [5], Abbas *et al.* studied a coupled Caputo-Hadamard fractional differential system with multipoint boundary conditions given by

$$\begin{cases} ({}^{Hc}D_1^{\alpha_1}u)(t) = f_1(t,u(t),v(t)) \\ ({}^{Hc}D_1^{\alpha_2}v)(t) = f_2(t,u(t),v(t)) \end{cases} ; \quad t \in I := [1,T],$$

Mathematics subject classification (2010): 34A08, 34K05.

Keywords and phrases: Coupled fractional differential system implicit of Caputo-type modification of the Erdélyi-Kober, existence, retarded arguments, advanced arguments, fixed point.

* Corresponding author.

Paper DEA-13-07

with the multipoint boundary conditions

$$\begin{cases} a_1u(1)-b_1u'(1)=d_1u(\xi_1) \\ a_2u(T)+b_2u'(T)=d_2u(\xi_2) \\ a_3v(1)-b_3v'(1)=d_3v(\xi_3) \\ a_4v(T)+b_4v'(T)=d_4v(\xi_4), \end{cases}$$

where $T>1,\ a_i,b_i,d_i\in\mathbb{R},\ \xi_i\in(1,T);\ i=1,2,3,4,\ \alpha_j\in(1,2],\ f_j:I\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m;\ j=1,2$ are given continuous functions, $\mathbb{R}^m;\ m\in\mathbb{N}^*$ is the Euclidian Banach space with a suitable norm $\|\cdot\|$, ${}^{Hc}D_1^{\alpha_j}$ is the Caputo–Hadamard fractional derivative of order $\alpha_j;\ j=1,2$. In [7], Abbas *et al.* studied implicit coupled Hilfer-Hadamard fractional differential systems under weak topologies given by

$$\begin{cases} {}^HD_1^{\alpha,\beta}u(t)=f_1(t,u(t),v(t),{}^HD_1^{\alpha,\beta}u(t),{}^HD_1^{\alpha,\beta}v(t)) \\ {}^HD_1^{\alpha,\beta}v(t)=f_2(t,u(t),v(t),{}^HD_1^{\alpha,\beta}u(t),{}^HD_1^{\alpha,\beta}v(t)) \end{cases} \quad t\in I:=[1,T],$$

with the initial conditions

$$\begin{cases} ({}^HI^{1-\gamma}u_1)(t)\ |_{t=1}=\phi_1 \\ ({}^HI^{1-\gamma}u_2)(t)\ |_{t=1}=\phi_2 \end{cases}$$

where $T>1,\ t\in I=[1,T],\ \alpha\in(0,1),\ \beta\in[0,1],\ f_i:I\times E^4\to E;\ i=1,2$ are given continuous functions, E is a real (or complex) Banach space with norm $\|\cdot\|_E$, ${}^HD_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β.

In this paper, we study the existence and uniqueness of solutions to the following Coupled system nonlinear implicit of Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments:

$$\begin{cases} {}_c^\rho D_{a^+}^\alpha u(t)=f_1(t,u^t,v^t,{}_c^\rho D_{a^+}^\alpha u(t),{}_c^\rho D_{a^+}^\alpha v(t)) \\ {}_c^\rho D_{a^+}^\alpha v(t)=f_2(t,u^t,v^t,{}_c^\rho D_{a^+}^\alpha u(t),{}_c^\rho D_{a^+}^\alpha v(t)) \end{cases} \quad t\in I:=[a,T], \tag{1.1}$$

$$\begin{cases} (u(t),v(t))=(\phi_1(t),\phi_2(t)),\ t\in[a-r,a], \quad r>0 \\ (u(t),v(t))=(\psi_1(t),\psi_2(t)),\ t\in[T,T+\beta], \quad \beta>0, \end{cases} \tag{1.2}$$

where ${}_c^\rho D_{a^+}^\alpha$ is the Caputo type modification of the Erdélyi-Kober fractional derivative and $f_i:I\times C([-r,\beta],\mathbb{R})^2\times\mathbb{R}^2\to\mathbb{R}$ is a given function, $\phi_i\in C([a-r,a],\mathbb{R})$ with $\phi_i(a)=0$ and $\psi_i\in C([T,T+\beta],\mathbb{R})$ with $\psi_i(T)=0,\ i=1,2$.

For $t\in I$, we denote by u^t the element of $C([-r,\beta])$ defined by:

$$u_t(s)=u(t+s):s\in[-r,\beta].$$

2. Preliminaries

In this part, we present notations and definitions, we will use throughout this paper. By $C([-r,\beta],\mathbb{R})$ we denote the Banach space of all continuous functions from $[-r,\beta]$ into $\mathbb{R}$ equipped with the norm

$$\|u\|_{[-r,\beta]} = \sup\{|u(t)| : -r \leqslant t \leqslant \beta\}$$

and $C([a,T],\mathbb{R})$ is the Banach space endowed with the norm.

$$\|u\|_{[a,T]} = \sup\{|u(t)| : a \leqslant t \leqslant T\}.$$

Also, let $E_1 = C([a-r,a],\mathbb{R})$, $E_2 = C([T,T+\beta],\mathbb{R})$, and

$$AC^1(I) := \{w : I \longrightarrow \mathbb{R} : w' \in AC(I)\},$$

where

$$w'(t) = t\frac{d}{dt}w(t), \;\; t \in I.$$

$AC(I,\mathbb{R})$ is the space of absolutely continuous functions on I.

$$\Omega = \{u : [a-r,T+\beta] \longmapsto \mathbb{R} : u\,|_{[a-r,a]} \in C([a-r,a]), u\,|_{[a,T]} \in AC^1([a,T])$$

and

$$u\,|_{[T,T+\beta]} \in C([T,T+\beta])\}$$

be the spaces endowed, respectively, with the norms

$$\|u\|_{[a-r,a]} = \sup\{|u(t)| : a-r \leqslant t \leqslant a\},$$

$$\|u\|_{[T,T+\beta]} = \sup\{|u(t)| : T \leqslant t \leqslant T+\beta\},$$

$$\|u\|_{\Omega} = \sup\{|u(t)| : a-r \leqslant t \leqslant T+\beta\}.$$

Define the weighted product space $\overline{\Omega} := \Omega \times \Omega$ with the norm

$$\|(u,v)\|_{\overline{\Omega}} := \|u\|_{\Omega} + \|v\|_{\Omega}.$$

Consider the space $X_c^p(a,b)$, $(c \in \mathbb{R},\ 1 \leqslant p \leqslant \infty)$ of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by:

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \;\; (1 \leqslant p < \infty,\ c \in \mathbb{R}).$$

In particular, where $c = \frac{1}{p}$ the space $X_c^p(a,b)$ coincides with $L^p(a,b)$, i.e.

$$X_{\frac{1}{p}}^p(a,b) = L^p(a,b).$$

And for $p = \infty$, we have

$$L^{\infty}(I) = \left\{ f : I \longrightarrow \mathbb{R} | \;\; \begin{array}{c} f \text{ is measurable and there is a constant } C \\ \text{such that } |f(x)| \leqslant C \text{ a.e. on } \;\; I \end{array} \right\},$$

with the norm

$$\|f\|_{L^{\infty}} = \inf\{C > 0;\;\; |f(x)| \leqslant C \text{ a.e. on } \;\; I\}.$$

DEFINITION 1. ([23]): (Erdélyi-Kober fractional integral) Let $\alpha \in \mathbb{R}$, $c \in \mathbb{R}$ and $g \in X_c^p(a,b)$ the Erdélyi-Kober fractional integral of order α is defined by:

$$({}^\rho I_{a^+}^\alpha g)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} g(s)ds, \qquad t > a,\ \rho > 0 \tag{2.1}$$

where Γ is the Euler gamma function defined by: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0.$

DEFINITION 2. ([23]) The generalized fractional derivative, corresponding to the generalized fractional integrals (2.1), is defined, for $0 \leqslant a < t$, by:

$$\begin{aligned}({}^\rho D_{a^+}^\alpha g)(t) &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(s^{1-\rho} \frac{d}{ds} \right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-n+\alpha}} g(s)ds \\ &= \delta_\rho^n ({}^\rho I_{a^+}^{n-\alpha} g)(t),\end{aligned} \tag{2.2}$$

where $\delta_\rho^n = \left(s^{1-\rho}\frac{d}{ds}\right)^n$.

DEFINITION 3. ([23]) The Caputo-type generalized fractional derivative ${}_c^\rho D_{a^+}^\alpha$ is defined via the above generalized fractional derivative (2.2) as follows

$${}_c^\rho D_{a^+}^\alpha g(t) = \left({}^\rho D_{a^+}^\alpha \left[g(s) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (s-a)^k \right] \right)(t). \tag{2.3}$$

LEMMA 1. ([23]) *Let* $\alpha, \rho \in \mathbb{R}^+$, *Then*

$$({}^\rho I_{a^+}^\alpha {}_c^\rho D_{a^+}^\alpha g)(t) = g(t) - \sum_{k=0}^{n-1} c_k \left(\frac{t^\rho - a^\rho}{\rho} \right)^k, \tag{2.4}$$

for some $c_k \in \mathbb{R}$, $n = [\alpha] + 1$.

THEOREM 1. ([17]) (Schauder's fixed point theorem) *Let* X *be a Banach space,* $D \subset X$ *a nonempty convex bounded closed set and let* $N : D \longmapsto D$ *be a completely continuous operator. Then,* N *has at least one fixed point.*

3. Existence of solutions

LEMMA 2. *Let* $1 < \alpha \leqslant 2$, $\phi \in C([a-r,a],\mathbb{R})$ *with* $\phi(a) = 0$, $\psi \in C([T, T+\beta],\mathbb{R})$ *with* $\psi(T) = 0$ *and* $h : I \to \mathbb{R}$ *be a continuous function. Then the linear problem*

$${}_c^\rho D_{a^+}^\alpha u(t) = h(t), \text{ for a.e } t \in I := [a,T],\ 1 < \alpha \leqslant 2, \tag{3.1}$$

$$u(t) = \phi(t),\ t \in [a-r,a],\ r > 0 \tag{3.2}$$

$$u(t) = \psi(t),\ t \in [T, T+\beta],\ \beta > 0, \tag{3.3}$$

has a unique solution, which is given by

$$u(t)=\begin{cases}\phi(t), \ \textit{if} \ \ t\in[a-r,a],\\ -\int_a^T G(t,s)h(s)ds, \ \textit{if} \ \ t\in I\\ \psi(t), \ \textit{if} \ \ t\in[T,T+\beta],\end{cases} \tag{3.4}$$

where

$$G(t,s)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\begin{cases}\dfrac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho-a^\rho)}-s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}, & a\leqslant s\leqslant t\leqslant T.\\ \dfrac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho-a^\rho)}, & a\leqslant t\leqslant s\leqslant T.\end{cases} \tag{3.5}$$

Here $G(t,s)$ is called the Green function of the boundary value problem (3.1)–(3.3).

Proof. From (2.4), we have

$$u(t)=c_0+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)+{}^\rho I_{a^+}^{\alpha}h(s), c_0,c_1\in\mathbb{R}, \tag{3.6}$$

therefore

$$u(a)=c_0=0,$$
$$u(T)=c_1\left(\frac{T^\rho-a^\rho}{\rho}\right)+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds,$$

and

$$c_1=-\frac{\rho^{2-\alpha}}{(T^\rho-a^\rho)\Gamma(\alpha)}\int_a^T(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds.$$

Substitute the value of c_0 and c_1 into equation (3.6), we get equation(3.4).

$$u(t)=\begin{cases}\phi(t), \ \text{if} \ \ t\in[a-r,a],\\ -\int_a^T G(t,s)h(s)ds, \ \text{if} \ \ t\in I\\ \psi(t), \ \text{if} \ \ t\in[T,T+\beta],\end{cases}$$

where G is defined by equation (3.5), the proof is complete. □

LEMMA 3. *Let $f_i : I\times C[-r,\beta]^2\times\mathbb{R}^2\longrightarrow\mathbb{R}$, $i=1,2$, be continuous functions. A function $(u,v)\in\Omega^2$ is solution of system* (1.1)–(1.2) *if and only if (u,v) satisfies the following coupled system of integral equations*

$$u(t)=\begin{cases}\phi_1(t), \ \textit{if} \ \ t\in[a-r,a],\\ -\int_a^T G(t,s)h_1(s)ds, \ \textit{if} \ \ t\in I\\ \psi_1(t), \ \textit{if} \ \ t\in[T,T+\beta],\end{cases}$$

$$v(t) = \begin{cases} \phi_2(t), \ if \ \ t \in [a-r,a], \\ -\int_a^T G(t,s)h_2(s)ds, \ if \ \ t \in I \\ \psi_2(t), \ if \ \ t \in [T,T+\beta], \end{cases}$$

where $h_i \in C(I)$ *satisfies the system of functional equations*

$$\begin{cases} h_1(t) = f_1(t,u^t,v^t,h_1(t),h_2(t)), \\ h_2(t) = f_2(t,u^t,v^t,h_1(t),h_2(t)). \end{cases}$$

The following hypotheses will be used in the sequel:

(H_1 **)** The functions $f_i : I \times C[-r,\beta]^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous.

(H_2 **)** There exist $K_i, \overline{K_i}, C_i, \overline{C_i} > 0$, $0 < \overline{C_2} < 1$, $0 < C_1 < 1$ such that

$$\begin{aligned} &|f_i(t,u,v,w,z) - f_i(t,\bar{u},\bar{v},\bar{w},\bar{z})| \\ &\qquad \leqslant K_i\|u-\bar{u}\|_{[-r,\beta]} + \overline{K_i}\|v-\bar{v}\|_{[-r,\beta]} + C_i|w-\bar{w}| + \overline{C_i}|z-\bar{z}| \end{aligned}$$

for any $u,\bar{u} \in C([-r,\beta])$ and $v,\bar{v} \in \mathbb{R}$, $i = 1,2$.

(H_3 **)** There exist $p_i, q_i \in L^\infty([a,T],\mathbb{R}_+)$ such that

$$|f_i(t,u,v,\bar{u},\bar{v})| \leqslant \frac{p_i(t)\|u\|_{[-r,\beta]} + q_i(t)\|v\|_{[-r,\beta]}}{1 + \|u\|_{[-r,\beta]} + \|v\|_{[-r,\beta]} + |\bar{u}| + |\bar{v}|}$$

for a.e. $t \in I$, and each $u,v \in C([-r,\beta])$ and $\bar{u},\bar{v} \in \mathbb{R}$.

Set

$$p_i^* = ess\sup_{t\in I} p_i(t), \quad q_i^* = ess\sup_{t\in I} q_i(t), \ \ i = 1,2$$

$$\widetilde{G} = \sup\left\{\int_a^T |G(t,s)|ds : t \in I\right\}.$$

Now, we state and prove our existence result for (1.1)–(1.2) based on the Banach fixed point theorem.

THEOREM 2. *Assume* (H_1) *and* (H_2) *hold. If*

$$\frac{C_2\overline{C_1}}{(1-C_1)(1-\overline{C_2})} < 1, \tag{3.7}$$

and

$$G_1^* + G_2^* < 1, \tag{3.8}$$

then the problem (1.1)–(1.2) *has a unique solution.*

Proof. Let the operator $N:\Omega\times\Omega\longmapsto\Omega\times\Omega$ defined by

$$N(u,v)(t) = (N_1(u,v)(t),N_2(u,v)(t)) = \begin{cases}(\phi_1(t),\phi_2(t)), \text{ if } & t\in[a-r,a],\\ -\left(\int_a^T G(t,s)h_1(s)ds, \int_a^T G(t,s)h_2(s)ds\right), \text{ if } & t\in I\\ (\psi_1(t),\psi_2(t)), \text{ if } & t\in[T,T+\beta].\end{cases} \tag{3.9}$$

By Lemma 3 it is clear that the fixed points of N are solutions (1.1)–(1.2).

Let $(u_2,v_2),(u_1,v_1)\in\Omega^2$. If $t\in[a-r,a]$ or $t\in[T,T+\beta]$ then

$$|N(u_2,v_2)(t)-N(u_1,v_1)(t)|=0.$$

For $t\in I$, we have

$$|N_1(u_2,v_2)(t)-N_1(u_1,v_1)(t)|\leqslant\int_a^T|G(t,s)||{}^\rho_c D^\alpha u_2(t)-{}^\rho_c D^\alpha u_1(t)|ds, \tag{3.10}$$

and by (H_2) we have

$$\begin{aligned}|{}^\rho_c D^\alpha_{a^+}u_2(t)-{}^\rho_c D^\alpha_{a^+}u_1(t)| &= |f_1(t,u_2^t,v_2^t,{}^\rho_c D^\alpha_{a^+}u_2(t),{}^\rho_c D^\alpha_{a^+}v_2(t))\\ &\quad -f_1(t,u_1^t,v_1^t,{}^\rho_c D^\alpha_{a^+}u_1(t),{}^\rho_c D^\alpha_{a^+}v_1(t))|\\ &\leqslant K_1\|u_2-u_1\|_{[-r,\beta]}+\overline{K_1}\|v_2-v_1\|_{[-r,\beta]}\\ &\quad +C_1|{}^\rho_c D^\alpha_{a^+}u_2(t)-{}^\rho_c D^\alpha_{a^+}u_1(t)|+\overline{C_1}|{}^\rho_c D^\alpha_{a^+}v_2(t)-{}^\rho_c D^\alpha_{a^+}v_1(t)|.\end{aligned}$$

Then

$$\begin{aligned}|{}^\rho_c D^\alpha_{a^+}u_2(t)-{}^\rho_c D^\alpha_{a^+}u_1(t)| &\leqslant \frac{K_1}{(1-C_1)}\|u_2-u_1\|_{[-r,\beta]}+\frac{\overline{K_1}}{(1-C_1)}\|v_2-v_1\|_{[-r,\beta]}\\ &\quad +\frac{\overline{C_1}}{(1-C_1)}|{}^\rho_c D^\alpha_{a^+}v_2(t)-{}^\rho_c D^\alpha_{a^+}v_1(t)|.\end{aligned}$$

Similarly, one can find that

$$\begin{aligned}|{}^\rho_c D^\alpha_{a^+}v_2(t)-{}^\rho_c D^\alpha_{a^+}v_1(t)| &\leqslant \frac{K_2}{(1-\overline{C_2})}\|u_2-u_1\|_{[-r,\beta]}+\frac{\overline{K_2}}{(1-\overline{C_2})}\|v_2-v_1\|_{[-r,\beta]}\\ &\quad +\frac{C_2}{(1-\overline{C_2})}|{}^\rho_c D^\alpha_{a^+}u_2(t)-{}^\rho_c D^\alpha_{a^+}u_1(t)|.\end{aligned}$$

Therefore

$$\begin{aligned}|{}^\rho_c D^\alpha_{a^+}u_2(t)-{}^\rho_c D^\alpha_{a^+}u_1(t)| &\leqslant \frac{K_1}{(1-C_1)}\|u_2-u_1\|_{[-r,\beta]}+\frac{\overline{K_1}}{(1-C_1)}\|v_2-v_1\|_{[-r,\beta]}\\ &\quad +\frac{\overline{C_1}}{(1-C_1)}\left[\frac{K_2}{(1-\overline{C_2})}\|u_2-u_1\|_{[-r,\beta]}+\frac{\overline{K_2}}{(1-\overline{C_2})}\|v_2-v_1\|_{[-r,\beta]}\right]\\ &\quad +\frac{C_2\overline{C_1}}{(1-C_1)(1-\overline{C_2})}|{}^\rho_c D^\alpha_{a^+}u_2(t)-{}^\rho_c D^\alpha_{a^+}u_1(t)|,\end{aligned}$$

then

$$|{}^{\rho}_{c}D^{\alpha}_{a^+}u_2(t) - {}^{\rho}_{c}D^{\alpha}_{a^+}u_1(t)| \leqslant \frac{K_1(1-\overline{C_2})+\overline{C_1}K_2}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{[-r,\beta]}$$
$$+\frac{\overline{K_1}(1-\overline{C_2})+\overline{C_1}\overline{K_2}}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{[-r,\beta]},$$

and

$$|{}^{\rho}_{c}D^{\alpha}_{a^+}v_2(t) - {}^{\rho}_{c}D^{\alpha}_{a^+}v_1(t)| \leqslant \frac{K_2(1-C_1)+C_2K_1}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{[-r,\beta]}$$
$$+\frac{\overline{K_2}(1-C_1)+\overline{K_1}C_2}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{[-r,\beta]}.$$

From it we get

$$|N_1(u_2,v_2)(t)-N_1(u_1,v_1)(t)| \leqslant \int_a^T |G(t,s)|\Big(\frac{K_1(1-\overline{C_2})+\overline{C_1}K_2}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{[-r,\beta]}$$
$$+\frac{\overline{K_1}(1-\overline{C_2})+\overline{C_1}\overline{K_2}}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{[-r,\beta]}ds\Big)$$
$$\leqslant \frac{\widetilde{G}\left(K_1(1-\overline{C_2})+\overline{C_1}K_2\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{[-r,\beta]}$$
$$+\frac{\widetilde{G}\left(\overline{K_1}(1-\overline{C_2})+\overline{C_1}\overline{K_2}\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{[-r,\beta]}.$$

Therefore, for each $t \in I$, we have

$$|N_1(u_2,v_2)(t)-N_1(u_1,v_1)(t)| \leqslant \frac{\widetilde{G}\left(K_1(1-\overline{C_2})+\overline{C_1}K_2\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{\Omega}$$
$$+\frac{\widetilde{G}\left(\overline{K_1}(1-\overline{C_2})+\overline{C_1}\overline{K_2}\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{\Omega}.$$

Thus

$$\|N_1(u_2,v_2)-N_1(u_1,v_1)\|_{\Omega} \leqslant G_1^*\left[\|u_2-u_1\|_{\Omega}+\|v_2-v_1\|_{\Omega}\right], \tag{3.11}$$

with

$$G_1^* = \frac{\widetilde{G}\left((\overline{K_1}+K_1)(1-\overline{C_2})+\overline{C_1}(K_2+\overline{K_2})\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}.$$

likewise, we get

$$\|N_2(u_2,v_2)-N_2(u_1,v_1)\|_{\Omega} \leqslant G_2^*\left[\|u_2-u_1\|_{\Omega}+\|v_2-v_1\|_{\Omega}\right], \tag{3.12}$$

with

$$G_2^* = \frac{\widetilde{G}\left((\overline{K_2}+K_2)(1-C_1)+C_2(K_1+\overline{K_1})\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}.$$

Thus it follows from (3.11) and (3.12), that

$$\|N(u_2,v_2)-N(u_1,v_1)\|_{\overline{\Omega}} \leqslant (G_1^*+G_2^*)\left[\|u_2-u_1\|_{\overline{\Omega}}+\|v_2-v_1\|_{\overline{\Omega}}\right],$$

with

$$G_1^*+G_2^*=\widetilde{G}\left(\frac{(\overline{K_1}+K_1)(1-\overline{C_2}+C_2)+(1-C_1+\overline{C_1})(K_2+\overline{K_2})}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\right).$$

So by (3.8) the operator N is a contraction. By the Banach contraction principle, N has a fixed point, which is solution to problem (1.1)–(1.2). □

We now prove an existence result for (1.1)–(1.2) by using the Schauder's fixed point theorem.

THEOREM 3. *Suppose that (H_1) and (H_3) hold. Then problem* (1.1)–(1.2) *has at least one solution.*

Step 1. N is continuous. Let $\{(u_n,v_n)\}$ be a sequence such that $(u_n,v_n)\longrightarrow(u,v)$ in $\Omega\times\Omega$. If $t\in[a-r,a]$ or $t\in[T,T+\beta]$ then

$$|N(u_n,v_n)(t)-N(u,v)(t)|=0.$$

For $t\in I$, we have

$$|N_i(u_n,v_n)(t)-N_i(u,v)(t)| \leqslant \int_a^T |G(t,s)||h_{i,n}(s)-h_i(s)|ds, \quad i=1,2, \tag{3.13}$$

where

$$h_{i,n}(t)=f_i(t,u_n^t,v_n^t,h_{1,n}(t),h_{2,n}(t)),$$

and

$$h_i(t)=f_i(t,u^t,v^t,h_1(t),h_2(t)).$$

Since $(u_n,v_n)\longrightarrow(u,v)$, and by (H_1) we get $h_{i,n}(t)\longrightarrow h(t)$, $i=1,2$ as $n\longrightarrow\infty$ for each $t\in I$. By (H_3) we have for each $t\in I$, $i=1,2$,

$$|h_{i,n}(t)| \leqslant p_i^*+q_i^*. \tag{3.14}$$

Then,

$$\begin{aligned}|G(t,s)||h_{i,n}(t)-h_i(t)| &\leqslant |G(t,s)|\left[|h_{i,n}(t)|+|h_i(t)|\right]\\ &\leqslant 2(p_i^*+q_i^*)|G(t,s)|.\end{aligned}$$

For each $t\in I$ the functions $s\longmapsto 2(p_i^*+q_i^*)|G(t,s)|$ are integrable on $[a,t]$, then by Lebesgue dominated convergence theorem, equation (3.13) implies

$$|N_i(u_n,v_n)(t)-N_i(u,v)(t)|\longrightarrow 0 \text{ as } n\longrightarrow\infty,$$

and hence

$$\|N(u_n,v_n)-N(u,v)\|_{\overline{\Omega}} \longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Consequently, N is continuous.

Let the constant R be such that:

$$R\geqslant \max\left\{L_1+L_2, \|\phi_1\|_{[a-r,a]}+\|\phi_2\|_{[a-r,a]}, \|\psi_1\|_{[T,T+\beta]}+\|\psi_2\|_{[T,T+\beta]}\right\}, \tag{3.15}$$

and define

$$D_R=\{(u,v)\in\Omega\times\Omega : \|(u,v)\|_{\overline{\Omega}}\leqslant R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $\overline{\Omega}$.

Step 2. $N(D_R)\subset D_R$.
Let $(u,v)\in D_R$ we show that $N(u,v)=(N_1(u,v),N_2(u,v))\in D_R$.
If $t\in[a-r,a]$, then

$$|N(u,v)(t)|\leqslant \|\phi_1\|_{[a-r,a]}+\|\phi_2\|_{[a-r,a]}\leqslant R,$$

and if $t\in[T,T+\beta]$, then

$$|N(u,v)(t)|\leqslant \|\psi_1\|_{[T,T+\beta]}+\|\psi_2\|_{[T,T+\beta]}\leqslant R.$$

For each $t\in I$, we have

$$|N_i(u,v)(t)|\leqslant \int_a^T |G(t,s)||h_i(s)|ds, \ \ i=1,2.$$

By (H_3), we have

$$\begin{aligned}|N_i(u,v)(t)| &\leqslant (p_i^*+q_i^*)\int_a^T |G(t,s)|ds\\ &\leqslant (p_i^*+q_i^*)\widetilde{G}=L_i\end{aligned}$$

from which it follows that for each $t\in[a-r,T+\beta]$, we have

$$|N_i(u,v)(t)|\leqslant L_i,$$

which implies that $\|N_i(u,v)\|_{\Omega}\leqslant L_i$, hence we get

$$\begin{aligned}\|N(u,v)\|_{\overline{\Omega}} &\leqslant L_1+L_2\\ &\leqslant R.\end{aligned}$$

Consequently,

$$N(D_R)\subset D_R.$$

Step 3. $N(D_R)$ is bounded and equicontinuous.
By Step 2 we have $N(D_R)$ is bounded.
Let $t_1, t_2 \in I = [a,T], t_1 < t_2,$ *and* $(u,v) \in D_R$ then

$$\begin{aligned}|N_i(u,v)(t_2) - N_i(u,v)(t_1)| &\leqslant \int_a^T |G(t_2,s) - G(t_1,s)||h_i(s)|ds \\ &\leqslant (p_i^* + q_i^*) \int_a^T |G(t_2,s) - G(t_1,s)|ds.\end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero.Therefore, the operator $N(u,v)$ is equicontinuous. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous and satisfies the assumptions of Schauder's fixed point theorem. Then N has a fixed point, which is a solution of the problem (1.1)–(1.2).

4. An example

Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation:

$$\begin{cases} (u(t),v(t)) = (e^{t-2} - 1, 2t - 4), \ \ t \in [1,2], \\ {}_c^{\frac{1}{2}}D_{2+}^{\frac{3}{2}} u(t) = \dfrac{\ln(t)}{200e^{t+2}\left(1 + |u^t| + |v^t| + \left|{}_c^{\frac{1}{2}}D_{2+}^{\frac{3}{2}} u(t)\right| + \left|{}_c^{\frac{1}{2}}D_{2+}^{\frac{3}{2}} v(t)\right|\right)}, \ \ t \in I = [2,e] \\ {}_c^{\frac{1}{2}}D_{2+}^{\frac{3}{2}} v(t) = \dfrac{\arctan(t)}{100e^{t+2}\left(1 + |u^t| + |v^t| + \left|{}_c^{\frac{1}{2}}D_{2+}^{\frac{3}{2}} u(t)\right| + \left|{}_c^{\frac{1}{2}}D_{2+}^{\frac{3}{2}} v(t)\right|\right)}, \ \ t \in I = [2,e] \\ (u(t),v(t)) = (\ln(t) - 1, t - e), \ \ \ t \in [e,6]. \end{cases} \tag{4.1}$$

Set

$$f_1(t,u,v,\bar{u},\bar{v}) = \frac{\ln(t)}{200e^{t+2}(1 + |u^t| + |v^t| + |\bar{u}| + |\bar{v}|)},$$

$$t \in [2,4], \ u,v \in C([-r,\beta]), \ \bar{u},\bar{v} \in \mathbb{R},$$

$$f_2(t,u,v,\bar{u},\bar{v}) = \frac{\arctan(t)}{100e^{t+2}(1 + |u^t| + |v^t| + |\bar{u}| + |\bar{v}|)},$$

$$t \in [2,4], \ u,v \in C([-r,\beta]), \ \bar{u},\bar{v} \in \mathbb{R},$$

$$v \in \mathbb{R}, \ \alpha = \frac{3}{2}, \ \rho = \frac{5}{2}, \ r = 1, \ \beta = 6 - e.$$

Condition (H_2) is satisfied, indeed, for each $u,v\in C([-r,\beta])$, $\bar{u},\bar{v}\in\mathbb{R}$ and $t\in[2,e]$, we have

$$|f_1(t,u_2,v_2,\bar{u}_2,\bar{v}_2)-f_1(t,u_1,v_1,\bar{u}_1,\bar{v}_1)|\leqslant \frac{1}{200e^{t+2}}(\|u_2-u_1\|_{[-r,\beta]}+\|u_2-v_1\|_{[-r,\beta]} +|\bar{u}_2-\bar{u}_1|+|\bar{v}_2-\bar{v}_1|),$$

and

$$|f_2(t,u_2,v_2,\bar{u}_2,\bar{v}_2)-f_2(t,u_1,v_1,\bar{u}_1,\bar{v}_1)|\leqslant \frac{\pi}{200e^{t+2}}(\|u_2-u_1\|_{[-r,\beta]}+\|u_2-v_1\|_{[-r,\beta]} +|\bar{u}_2-\bar{u}_1|+|\bar{v}_2-\bar{v}_1|).$$

Therefore, (H_2) is verified with

$$K_i=\overline{K}_i=C_i=\overline{C}_i=\begin{cases}\dfrac{\pi}{200e^4} & \text{for } i=2,\\[2mm] \dfrac{1}{200e^4} & \text{for } i=1.\end{cases}$$

For each $t\in I$ we have

$$\begin{aligned}\int_a^T|G(t,s)|ds &\leqslant \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{T^\rho-a^\rho}\right)\int_a^T\left|\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\right|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_a^t\left|\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\right|ds\\ &\leqslant \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}.\end{aligned}$$

Therefore

$$\widetilde{G}\leqslant \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}.$$

We have

$$\begin{aligned}G_1^*+G_2^* &\leqslant \frac{\frac{1}{100e^4}+\frac{\pi}{100e^4}}{((1-\frac{1}{200e^4})(1-\frac{\pi}{200e^4})-\frac{\pi}{(200e^4)^2})}\frac{2}{\Gamma(\frac{5}{2})}\left(\frac{e^{\frac{5}{2}}-2^{\frac{5}{2}}}{\frac{5}{2}}\right)^{\frac{3}{2}}\\ &\approx 6.689246337.10^{-7}\\ &<1.\end{aligned}$$

Hence (3.8) is satisfied with $T=e$, $a=2$ and $\alpha=\frac{3}{2}$. Hence all conditions of Theorem 2 are satisfied, it follows that the problem (4.1) admit a unique solution defined on I.

Acknowledgements. The work of Juan J. Nieto has been partially supported by the Agencia Estatal de Investigacin (AEI) of Spain, cofinanced by the European Fund for Regional Development (FEDER) corresponding to the 2014-2020 multiyear financial framework, project MTM2016-75140-P; and by Xunta de Galicia under grant ED431C 2019/02.

REFERENCES

[1] S. ABBAS, M. BENCHOHRA, J. R. GRAEF, J. HENDERSON, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, de Gruyter, Berlin 2018.

[2] S. ABBAS, M. BENCHOHRA, G. M. N'GUÉRÉKATA, *Topics in Fractional Differential Equations*, Springer, New York 2012.

[3] S. ABBAS, M. BENCHOHRA AND G. M. N'GUÉRÉKATA, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[4] S. ABBAS, M. BENCHOHRA, J. LAZREG AND G. N'GUÉRÉKATA, *Weak solutions for implicit Pettis Hadamard fractional differential equations with retarded and advanced arguments*, Nonlinear Stud. **24** (2) (2017), 355–365.

[5] S. ABBAS, S. AIBOUT, M. BENCHOHRA, M. BOHNER, *A coupled Caputo-Hadamard fractional differential system with multipoint boundary conditions*, Dyn. Cont., Disc. Impul. Syst. (to appear).

[6] S. ABBAS, M. BENCHOHRA, AND A. N. VITYUK, *On fractional order derivatives and Darboux problem for implicit differential equations*, Fract. Calc. Appl. Anal. **15** (2), (2012), 168–182.

[7] S. ABBAS, M. BENCHOHRA, N. HAMIDI AND Y. ZHOU, *Implicit coupled Hilfer-Hadamard fractional differential systems under weak topologies*, Adv. Difference Equ. **328** (2018), 17 pp.

[8] S. ALJOUDI, B. AHMAD, J. J. NIETO, A. A ALSAEDI, *Coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions*, Chaos Solitons Fractals **91** (2016), 39–46.

[9] R. P. AGARWAL, S. K. NTOUYAS, B. AHMAD, AND A. K. ALZAHRANI, *Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments*, Adv. Difference Equ. **2016**, 2016:92, 15 pp.

[10] B. AL-SAQABI, V. S. KIRYAKOVA, *Explicit solutions of fractional integral and differential equations involving Erdélyi-Kober operators*, Appl. Math. Comput. **95** (1998), no. 1, 1–13.

[11] T. D. BENAVIDES, *An existence theorem for implicit differential equations in a Banach space*, Ann. Mat. Pura Appl. (4) **118** (1978), 119–130.

[12] M. BENCHOHRA, S. BOURIAH, J. E. LAZREG AND J. J. NIETO, *Nonlinear implicit Hadamard fractional differential equations with delay in Banach space*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **55** (2016), 15–26.

[13] M. BENCHOHRA, S. BOURIAH AND J. HENDERSON, *Nonlinear implicit Hadamard's fractional differential equations with retarded and advanced arguments*, Azerbaijan J. Math. **8** (2) (2018), 72–85.

[14] M. BENCHOHRA AND J. E. LAZREG, *Nonlinear fractional implicit differential equations*, Commun. Appl. Anal. **17** (2013), 471–482.

[15] M. BENCHOHRA, J. E. LAZREG AND G. N'GUÉRÉKATA, *Nonlinear implicit Hadamard's fractional differential equations on Banach space with retarded and advanced arguments*, Intern. J. Evol. Equ. **10** (3 & 4) (2017), 283–295.

[16] F. Z. BERRABAH, B. HEDIA, J. HENDERSON, *A fully Hadamard and Erdélyi-Kober-type integral boundary value problem of a coupled system of implicit differential equations*, Turkish J. Math. **43** (2019), no. 3, 1308–1329.

[17] A. GRANAS AND J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[18] T. GNANA BHASKAR, V. LAKSHMIKANTHAM AND J. VASUNDHARA DEVI, *Monotone iterative technique for functional differential equations with retardation and anticipation*, Nonlinear Anal. **66** (2007), no. 10, 2237–2242.

[19] R. HILFER, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore 2000.

[20] A. A. KILBAS, HARI M. SRIVASTAVA, AND JUAN J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, 2006.

[21] V. KIRYAKOVA, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Math. 301, Longman, Harlow J. Wiley, New York, 1994.

[22] V. KIRYAKOVA, Y. LUCHKO, *Multiple Erdélyi-Kober integrals and derivatives as operators of generalized fractional calculus. Handbook of fractional calculus with applications*, Vol. 1, 127–158, De Gruyter, Berlin, 2019.

[23] Y. LUCHKO AND J. J. TRUJILLO, *Caputo-type modification of the Erdélyi-Kober fractional derivative*, Fract. Calc. Appl. Anal. **10** (3) (2007), 249–267.

[24] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[25] V. E. TARASOV, *Fractional Dynamics Application of Fractional Calculus to Dynamics of Particles*, Fields and Media, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[26] A. S. VATSALA, *Generalized monotone iterative technique for functional differential equations with retardation and anticipation*, Nonlinear Dyn. Syst. Theory **7** (2007), no. 4, 431–438.

[27] A. N. VITYUK AND A. V. MIKHALENKO, *The Darboux problem for an implicit differential equation of fractional order*, Ukr. Mat. Visn. **7**(4) (2010), 439–452.

[28] Y. ZHOU, J.-R. WANG, L. ZHANG, *Basic Theory of Fractional Differential Equations*, second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

(Received February 17, 2020)

Mokhtar Boumaaza
Laboratory of Mathematics, Department of Mathematics
Djillali Liabes University of Sidi Bel-Abbès
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
e-mail: boumaaza.mokhtar@yahoo.com

Mouffak Benchohra
Laboratory of Mathematics, Department of Mathematics
Djillali Liabes University of Sidi Bel-Abbès
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
e-mail: benchohra@yahoo.com

Juan J. Nieto
Departamento de Estatistica, Análise Matemática e Optimización
Instituto de Matemáticas, Universidade de Santiago de Compostela
Santiago de Compostela, Spain
e-mail: juanjose.nieto.roig@usc.es