EXISTENCE OF SOLUTIONS FOR A COUPLED SYSTEM OF CAPUTO TYPE FRACTIONAL–ORDER DIFFERENTIAL INCLUSIONS WITH NON–SEPARATED BOUNDARY CONDITIONS ON MULTIVALUED MAPS

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Abstract. Sufficient conditions for the existence of solutions to a coupled system of fractional-order differential inclusions associated with fractional non-separated boundary conditions for multivalued maps are established, by employing the nonlinear alternative of Leray–Schauder type. We emphasize that the methods of fixed point theory used in our analysis are standard, although their application to a system of fractional-order differential inclusions is new.

1. Introduction

Fractional-order differential equations (FDEs) arise in the mathematical modeling of several real-world phenomena and have now attained immense value due to their broad applications in the fields of engineering and science. Recently, there has been shown a great deal of interest in the study of fractional-order boundary value problems (FBVPs). Fractional-order models are more realistic and beneficial than their corresponding integer-order counterparts. See, for example, [18, 26] for additional information and explanations.

The existence of solutions for FDEs has been studied and used by authors under different initial or boundary conditions. For the recent development on the topic, we refer the reader to the works [15, 27, 28, 21, 35, 22] and the references cited therein. In particular, coupled systems of FDEs have aroused significant attention in the mathematical modeling of physical phenomena such as disease models [12, 25], chaos synchronization [14, 34], ecological effects [17], anomalous diffusion [32], etc. For some prior theoretical findings on coupled systems of FDEs [31, 33].

Differential inclusions play an essential role as a tool in the analysis of various dynamical processes such as sweeping processes [2, 24, 29], nonlinear dynamics of wheeled vehicles [6], control problems [20], granular systems [30], etc. Information on the critical issues in stochastic processes, control, differential games, optimization and their application to the economy, manufacturing, queuing and climate control can be found in [19] and also we refer [11] for application of fractional-order differential

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inclusions in synchronization processes. In [5], the existence of solutions for the BVP of coupled fractional-order differential inclusions supplemented by coupled boundary conditions studied.

To the best of our knowledge, the fractional-order coupled system with nonlocal and non-separated boundary conditions has not yet been studied thoroughly. We emphasize here that our problem is new in the sense of implemented the non-separated coupled boundary conditions. The objective of this work is to develop criteria for the existence of solutions for a coupled system of fractional-order differential inclusions associated with coupled boundary conditions for multi-valued maps F and G that include convex and non-convex values by using the non-linear alternative of Leray–Schauder type. In this article, we consider a coupled system of fractional-order differential inclusions

$$\left.\begin{array}{ll} {}^cD_{0+}^{q_1}y_1(t) \in F\big(t,y_1(t),y_2(t)\big), & 0<t<1, \\ {}^cD_{0+}^{q_2}y_2(t) \in G\big(t,y_1(t),y_2(t)\big), & 0<t<1, \end{array}\right\} \tag{1}$$

subject to the fractional non-separated boundary conditions

$$\left.\begin{array}{ll} y_1(0)=\vartheta_{11}y_2(1), & {}^cD_{0+}^{q_3}y_1(1)=\vartheta_{12}\,{}^cD_{0+}^{q_3}y_2(\eta), \\ y_2(0)=\vartheta_{21}y_1(1), & {}^cD_{0+}^{q_3}y_2(1)=\vartheta_{22}\,{}^cD_{0+}^{q_3}y_1(\eta), \end{array}\right\} \tag{2}$$

where $q_1,q_2 \in (1,2], 0<\eta,q_3<1$, ${}^cD^{q_i}$, for $i=1,2,3$ are the Caputo type fractional-order derivatives and $\vartheta_{11},\vartheta_{12},\vartheta_{21}$ and ϑ_{22} are real constants.

Throughout the article, we assume the following conditions hold:

(A0) $\Delta = 1 - \vartheta_{12}\vartheta_{22}\eta^{2(1-q_3)} \neq 0$ and $\vartheta_{21}\vartheta_{11} \neq 1$,

(A1) $F,G : [0,1] \times \mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ are given multivalued maps, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$,

(A2) the maps $F,G : [0,1] \times \mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ are L^1-Carathéodory and have convex values,

(A3) there exist continuous functions $b_1,b_2 : [0,1] \to \mathbb{R}^+$ and continuous nondecreasing functions $\Psi_1,\Psi_2,\Phi_1,\Phi_2 : [0,\infty) \to \mathbb{R}^+$ such that

$$\|F(t,y_1,y_2)\|_{\mathscr{P}} = \sup\Big\{|f| : f \in F(t,y_1,y_2)\Big\} \leqslant b_1(t)\Big(\Psi_1(\|y_1\|)+\Phi_1(\|y_2\|)\Big),$$

$$\|G(t,y_1,y_2)\|_{\mathscr{P}} = \sup\Big\{|g| : g \in G(t,y_1,y_2)\Big\} \leqslant b_2(t)\Big(\Psi_2(\|y_1\|)+\Phi_2(\|y_2\|)\Big),$$

for each $(t,y_1,y_2) \in [0,1] \times \mathbb{R}^2$,

(A4) there exist positive constants m_1,m_2 and N such that

$$\frac{m_1\|b_1\|\big(\Psi_1(N)+\Phi_1(N)\big)+m_2\|b_2\|\big(\Psi_2(N)+\Phi_2(N)\big)}{N} < 1.$$

The rest of this article is organized as follows. Section 2 consists some preliminary results. The main results are presented in Section 3. In Section 4, as an application, we demonstrate our results with an example.

2. Preliminaries

We recall some basic definitions of fractional calculus.

DEFINITION 1. [26] The fractional integral of a function $f : [a, +\infty) \to \mathbb{R}$ is given by

$$_aD_t^{-p}f(t) =_a I_t^p f(t) = \frac{1}{\Gamma(p)} \int_a^t (t - \tau)^{p-1} f(\tau)d\tau,$$

provided the right-hand side is defined and $p > 0$.

DEFINITION 2. The Caputo derivative of a function $f : [0, \infty) \to \mathbb{R}$ can be expressed as

$$^cD_{0+}^m f(t) = D_{0+}^m \left[f(t) - \sum_{j=0}^{n-1} \frac{t^j}{j!} f^{(j)}(0) \right], \quad t > 0, \quad n - 1 < m < n.$$

For the sake of convenience, we will use $^cD^m$ instead of $^cD_{0+}^m$ in the rest of this paper.

REMARK 1. If $f(t) \in C^n[0, \infty)$, then

$$^cD^m f(t) = \frac{1}{\Gamma(n-m)} \int_0^t (t - \tau)^{n-m-1} f^{(n)}(\tau)d\tau, \quad t > 0, \, n - 1 < m < n.$$

Some basic concepts of multivalued analysis are listed here [1, 10, 16]. Let $(X, ||\cdot||)$ be a Banach space and K be a subset of X. Then

$$(i) \quad \mathscr{P}_{cl}(X) = \{K \subset \mathscr{P}(X) : K \text{ is closed}\},$$
$$(ii) \quad \mathscr{P}_{cv}(X) = \{K \subset \mathscr{P}(X) : K \text{ is convex}\},$$
$$(iii) \quad \mathscr{P}_b(X) = \{K \subset \mathscr{P}(X) : K \text{ is bounded}\},$$
$$(iv) \quad \mathscr{P}_{cp}(X) = \{K \subset \mathscr{P}(X) : K \text{ is compact}\},$$
$$(v) \quad \mathscr{P}_{cv,cp}(X) = \mathscr{P}_{cv}(X) \cap \mathscr{P}_{cp}(X).$$

A multivalued map $G : X \to \mathscr{P}(X)$ is

(a) convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$,

(b) upper semicontinuous (u.s.c.) on X if the set $G(x_0)$ is a nonempty, closed subset of X for each open set U of X containing $G(x_0)$ is a nonempty, $\exists$ an open neighborhood V of x_0 such that $G(V) \subset U$.

(c) lower semicontinuous (l.s.c.) if the set $\{x \in X : G(x) \cap V \neq \emptyset\}$ is open for any open set V in X.

(d) completely continuous if $G(\mathscr{B})$ is relatively compact for every $\mathscr{B} \in \mathscr{P}_b(X)$.

REMARK 2. If the multivalued map G is completely continuous with nonempty compact values then G is u.s.c. $\iff$ G has a closed graph, $i.e., x_n \to x_*, y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. The set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ defines the graph of G.

DEFINITION 3. A multivalued map $G : [c,d] \to \mathscr{P}_{cl}(\mathbb{R})$ is said to be measurable if the function $t \mapsto d(y_1, G(t)) = \inf\{|y_1 - y_2| : y_2 \in G(t)\}$ is measurable for every $y_1 \in \mathbb{R}$.

DEFINITION 4. A multivalued map $G : [c,d] \times \mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto G(t,x,y)$ is measurable $\forall x, y \in \mathbb{R}$,

(ii) $(x,y) \mapsto G(t,x,y)$ is u.s.c. for almost all $t \in [c,d]$.

DEFINITION 5. A Carathéodory function G is said to be L^1-Carathéodory if for each $\rho > 0$, $\exists \; \varphi_\rho \in L^1([c,d],\mathbb{R}^+)$ such that $\|G(t,x,y)\|_{\mathscr{P}} = \sup\{|g| : g \in G(t,x,y)\} \leqslant \varphi_\rho(t) \; \forall x, y \in \mathbb{R}$ with $\|x\|, \|y\| \leqslant \rho$ and for a.e. $t \in [c,d]$.

LEMMA 1. ([10, Proposition 1.2]) *If $G : X \to \mathscr{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$, i.e., for every sequence $\{x_n\}_{n \in N} \subset X$ and $\{y_n\}_{n \in N} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$ when $n \to \infty$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semicontinuous.*

LEMMA 2. ([23]) *Let X be a separable Banach space. Let $G : [0,1] \times \mathbb{R}^2 \to \mathscr{P}_{cp,cv}(\mathbb{R})$ be an L^1- Carathéodory multivalued map and let χ be a linear continuous mapping from $L^1([0,1],\mathbb{R})$ to $C([0,1],\mathbb{R})$. Then the operator*

$$\chi \circ S_{G,x} : C([0,1],\mathbb{R}) \to \mathscr{P}_{cp,cv}(C([0,1],\mathbb{R})), \quad x \mapsto (\chi \circ S_{G,x})(x) = \chi(S_{G,x})$$

is a closed graph operator in $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$.

LEMMA 3. (Nonlinear alternative for Kakutani maps [13]) *Let $\mathscr{E}_1$ be a closed convex subset of a Banach space $\mathscr{E}$, and $\mathscr{U}$ be an open subset of $\mathscr{E}_1$ with $0 \in \mathscr{U}$. Suppose that $F : \overline{\mathscr{U}} \to \mathscr{P}_{cp,cv}(\mathscr{E}_1)$ is an upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{\mathscr{U}}$, or*

(ii) *there is a $u \in \partial \mathscr{U}$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

We now present an auxiliary lemma, which will play a key role in the forthcoming analysis.

LEMMA 4. *Let $\phi, \psi \in C([0,1],\mathbb{R})$ and Suppose that the condition (A0) is fulfilled. Then the solution of the associated linear FDEs*

$$\left. \begin{array}{l} {}^cD_{0+}^{q_1} y_1(t) = \phi(t), \quad t \in [0,1], \\ {}^cD_{0+}^{q_2} y_2(t) = \psi(t), \quad t \in [0,1], \end{array} \right\} \tag{3}$$

supplemented with the boundary conditions (2) *is given by*

$$
\begin{aligned}
y_1(t) =& \frac{\Gamma(2-q_3)}{\Delta}\left[\frac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\frac{\vartheta_{12}t}{\eta^{q_3-1}}\right]\left(\vartheta_{22}A_3-B_3\right) \\
&+\frac{\Gamma(2-q_3)}{\Delta}\left[\frac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right]\left(\vartheta_{12}B_2-A_2\right) \\
&+\frac{\vartheta_{11}(\vartheta_{21}A_1+B_1)}{1-\vartheta_{21}\vartheta_{11}}+\int_0^t\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}\phi(s)ds
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
y_2(t) =& \frac{\Gamma(2-q_3)}{\Delta}\left[\frac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right]\left(\vartheta_{22}A_3-B_3\right) \\
&+\frac{\Gamma(2-q_3)}{\Delta}\left[\frac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\frac{\vartheta_{22}t}{\eta^{q_3-1}}\right]\left(\vartheta_{12}B_2-A_2\right) \\
&+\frac{\vartheta_{21}(A_1+\vartheta_{11}B_1)}{1-\vartheta_{21}\vartheta_{11}}+\int_0^t\frac{(t-s)^{q_2-1}}{\Gamma(q_2)}\psi(s)ds,
\end{aligned}
\tag{5}
$$

where

$$
A_1 = \int_0^1\frac{(1-s)^{q_1-1}}{\Gamma(q_1)}\phi(s)ds, \quad B_1 = \int_0^1\frac{(1-s)^{q_2-1}}{\Gamma(q_2)}\psi(s)ds,
\tag{6}
$$

$$
A_2 = \int_0^1\frac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}\phi(s)ds, \quad B_2 = \int_0^\eta\frac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}\psi(s)ds,
\tag{7}
$$

$$
A_3 = \int_0^\eta\frac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}\phi(s)ds, \quad B_3 = \int_0^1\frac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}\psi(s)ds.
\tag{8}
$$

DEFINITION 6. A function $(y_1,y_2)\in C^2([0,1],\mathbb{R})\times C^2([0,1],\mathbb{R})$ is a solution of the coupled system of FBVP (1)–(2) if it satisfies the coupled boundary conditions (2) and there exist functions $f,g\in L^1([0,1],\mathbb{R})$ such that $f(t)\in F(t,y_1(t),y_2(t))$, $g(t)\in G(t,y_1(t),y_2(t))$ a.e. on $[0,1]$ and

$$
\begin{aligned}
y_1(t) =& \frac{\Gamma(2-q_3)}{\Delta}\left[\frac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\frac{\vartheta_{12}t}{\eta^{q_3-1}}\right]\left(\vartheta_{22}A_{3f}-B_{3g}\right) \\
&+\frac{\Gamma(2-q_3)}{\Delta}\left[\frac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right]\left(\vartheta_{12}B_{2g}-A_{2f}\right) \\
&+\frac{\vartheta_{11}(\vartheta_{21}A_{1f}+B_{1g})}{1-\vartheta_{21}\vartheta_{11}}+\int_0^t\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
y_2(t) = & \frac{\Gamma(2-q_3)}{\Delta} \left[\frac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}} + t \right] \left(\vartheta_{22}A_{3f} - B_{3g} \right) \\
& + \frac{\Gamma(2-q_3)}{\Delta} \left[\frac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}} + \frac{\vartheta_{22}t}{\eta^{q_3-1}} \right] \left(\vartheta_{12}B_{2g} - A_{2f} \right) \\
& + \frac{\vartheta_{21}(\vartheta_{11}B_{1g}+A_{1f})}{1-\vartheta_{21}\vartheta_{11}} + \int_0^t \frac{(t-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds,
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
A_{1f} &= \int_0^1 \frac{(1-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds, \quad B_{1g} = \int_0^1 \frac{(1-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds, \\
A_{2f} &= \int_0^1 \frac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds, \quad B_{2g} = \int_0^\eta \frac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds, \\
A_{3f} &= \int_0^\eta \frac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds, \quad B_{3g} = \int_0^1 \frac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds.
\end{aligned}
$$

3. Main results

Consider the Banach space $\mathscr{B} = X \times X$, where $X = \{x : x \in C[0,1]\}$ equipped with the norm $\|(y_1,y_2)\| = \|y_1\| + \|y_2\|$, for $(y_1,y_2) \in \mathscr{B}$ and the norm is defined as

$$
\|y_1\| = \sup_{t\in[0,1]} |y_1(t)|.
$$

Define the selection sets of F,G by

$$
\begin{aligned}
S_{F,(y_1,y_2)} &= \left\{ f \in L^1\left([0,1],\mathbb{R}\right) : f(t) \in F\left(t,y_1(t),y_2(t)\right) \text{ for a.e. } t \in [0,1] \right\}, \\
S_{G,(y_1,y_2)} &= \left\{ g \in L^1\left([0,1],\mathbb{R}\right) : g(t) \in G\left(t,y_1(t),y_2(t)\right) \text{ for a.e. } t \in [0,1] \right\}.
\end{aligned}
$$

In view of Lemma 4, we define the operators $\mathscr{A}_1,\mathscr{A}_2 : \mathscr{B} \to \mathscr{P}(\mathscr{B})$ by

$$
\begin{aligned}
\mathscr{A}_1(y_1,y_2) = & \left\{ h_1 \in \mathscr{B} : \exists f \in S_{F,(y_1,y_2)}, g \in S_{G,(y_1,y_2)} \text{ such that} \right. \\
& \left. h_1(y_1,y_2)(t) = Q_1(t,y_1,y_2), \forall t \in [0,1] \right\} \\
\mathscr{A}_2(y_1,y_2) = & \left\{ h_2 \in \mathscr{B} : \exists f \in S_{F,(y_1,y_2)}, g \in S_{G,(y_1,y_2)} \text{ such that} \right. \\
& \left. h_2(y_1,y_2)(t) = Q_2(t,y_1,y_2), \forall t \in [0,1] \right\},
\end{aligned}
\tag{11}
$$

where

$$
Q_1(y_1,y_2)(t) = \begin{cases} \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\dfrac{\vartheta_{12}t}{\eta^{q_3-1}}\right]\left(\vartheta_{22}A_{3f}-B_{3g}\right) \\[3mm] +\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right]\left(\vartheta_{12}B_{2g}-A_{2f}\right) \\[3mm] +\dfrac{\vartheta_{11}(\vartheta_{21}A_{1f}+B_{1g})}{1-\vartheta_{21}\vartheta_{11}}+\displaystyle\int_0^t\dfrac{(t-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds \end{cases}
$$

and

$$
Q_2(y_1,y_2)(t) = \begin{cases} \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right]\left(\vartheta_{22}A_{3f}-B_{3g}\right) \\[3mm] +\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\dfrac{\vartheta_{22}t}{\eta^{q_3-1}}\right]\left(\vartheta_{12}B_{2g}-A_{2f}\right) \\[3mm] +\dfrac{\vartheta_{21}(\vartheta_{11}B_{1g}+A_{1f})}{1-\vartheta_{21}\vartheta_{11}}+\displaystyle\int_0^t\dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds. \end{cases}
$$

Then we define an operator $\mathscr{A}:\mathscr{B}\to\mathscr{P}(\mathscr{B})$ by

$$
\mathscr{A}(y_1,y_2)(t)=\begin{pmatrix}\mathscr{A}_1(y_1,y_2)(t)\\\mathscr{A}_2(y_1,y_2)(t)\end{pmatrix},
$$

where $\mathscr{A}_1$ and $\mathscr{A}_2$ are respectively defined by (11).

For the sake of computational convenience, we set

$$
\begin{aligned}
\mathscr{M}_1 = {}& \left[\frac{|\vartheta_{11}|(|\vartheta_{21}\vartheta_{12}\eta^{1-q_3}|+1)}{|1-\vartheta_{11}\vartheta_{21}|}+\left|\frac{\vartheta_{12}}{\eta^{q_3-1}}\right|\right]\frac{|\vartheta_{22}\eta^{q_1-q_3}|}{|\Gamma(q_1-q_3+1)|} \\
& \times\frac{|\Gamma(2-q_3)|}{|\Delta|}+\left[\frac{|\vartheta_{11}|(|\vartheta_{21}|+|\vartheta_{22}\eta^{1-q_3}|)}{|1-\vartheta_{11}\vartheta_{21}|}+1\right]\frac{|\Gamma(2-q_3)|}{|\Delta|} \\
& \times\frac{1}{|\Gamma(q_1-q_3+1)|}+\left[\frac{|\vartheta_{11}\vartheta_{21}|}{|1-\vartheta_{21}\vartheta_{11}|}+1\right]\frac{1}{|\Gamma(q_1+1)|},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\mathscr{M}_2 = {}& \left[\frac{|\vartheta_{11}|(|\vartheta_{21}\vartheta_{12}\eta^{1-q_3}|+1)}{|1-\vartheta_{11}\vartheta_{21}|}+\left|\frac{\vartheta_{12}}{\eta^{q_3-1}}\right|\right]\left|\frac{1}{\Gamma(q_2-q_3+1)}\right| \\
& \times\frac{|\Gamma(2-q_3)|}{|\Delta|}+\left[\frac{|\vartheta_{11}|(|\vartheta_{21}|+|\vartheta_{22}\eta^{1-q_3}|)}{|1-\vartheta_{11}\vartheta_{21}|}+1\right]\frac{|\Gamma(2-q_3)|}{|\Delta|} \\
& \times\frac{|\vartheta_{12}\eta^{q_2-q_3}|}{|\Gamma(q_2-q_3+1)|}+\frac{|\vartheta_{11}|}{|1-\vartheta_{21}\vartheta_{11}|}\frac{1}{|\Gamma(q_2+1)|},
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\mathscr{M}_3 = {}& \left[\frac{|\vartheta_{21}|(|\vartheta_{11}\vartheta_{22}\eta^{1-q_3}|+1)}{|1-\vartheta_{11}\vartheta_{21}|}+\left|\frac{\vartheta_{22}}{\eta^{q_3-1}}\right|\right]\frac{1}{|\Gamma(q_1-q_3+1)|} \\
& \times\frac{|\Gamma(2-q_3)|}{|\Delta|}+\left[\frac{|\vartheta_{21}|(|\vartheta_{11}|+|\vartheta_{12}\eta^{1-q_3}|)}{|1-\vartheta_{11}\vartheta_{21}|}+1\right]\frac{|\Gamma(2-q_3)|}{|\Delta|} \\
& \times\frac{|\vartheta_{22}\eta^{q_1-q_3}|}{|\Gamma(q_1-q_3+1)|}+\frac{|\vartheta_{21}|}{|1-\vartheta_{21}\vartheta_{11}|}\frac{1}{|\Gamma(q_1+1)|},
\end{aligned} \tag{14}
$$

$$\mathcal{M}_4 = \left[\frac{|\vartheta_{21}|(|\vartheta_{11}\vartheta_{22}\eta^{1-q_3}|+1)}{|1-\vartheta_{11}\vartheta_{21}|} + \left|\frac{\vartheta_{22}}{\eta^{q_3-1}}\right| \right] \frac{|\vartheta_{12}\eta^{q_2-q_3}|}{|\Gamma(q_2-q_3+1)|} \right.$$
$$\times \frac{|\Gamma(2-q_3)|}{|\Delta|} + \left[\frac{|\vartheta_{21}|(|\vartheta_{11}|+|\vartheta_{12}\eta^{1-q_3}|)}{|1-\vartheta_{11}\vartheta_{21}|} + 1 \right] \frac{|\Gamma(2-q_3)|}{|\Delta|} \qquad (15)$$
$$\left. \times \frac{1}{|\Gamma(q_2-q_3+1)|} + \left[\frac{|\vartheta_{11}\vartheta_{21}|}{|1-\vartheta_{21}\vartheta_{11}|} + 1 \right] \frac{1}{|\Gamma(q_2+1)|}. \right\}$$

THEOREM 1. *Suppose the conditions* (A0)–(A4) *are fulfilled. Then the coupled system of FBVP* (1)–(2) *has at least one solution on* $[0,1]$.

Proof. Consider the operators $\mathscr{A}_1, \mathscr{A}_2 : \mathscr{B} \to \mathscr{P}(\mathscr{B})$ defined by (11). From $(A2)$, it follows that the sets $S_{F,(y_1,y_2)}$ and $S_{G,(y_1,y_2)}$ are nonempty for each $(y_1,y_2) \in \mathscr{B}$. Then, for $f \in S_{F,(y_1,y_2)}$, $g \in S_{G,(y_1,y_2)}$ for $(y_1,y_2) \in \mathscr{B}$, we have

$$h_1(y_1,y_2)(t) = \begin{cases} \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}} + \dfrac{\vartheta_{12}t}{\eta^{q_3-1}}\right]\left(\vartheta_{22}A_{3f} - B_{3g}\right) \\[3mm] + \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}} + t\right]\left(\vartheta_{12}B_{2g} - A_{2f}\right) \\[3mm] + \dfrac{\vartheta_{11}(\vartheta_{21}A_{1f}+B_{1g})}{1-\vartheta_{21}\vartheta_{11}} + \displaystyle\int_0^t \dfrac{(t-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds, \end{cases}$$

$$h_2(y_1,y_2)(t) = \begin{cases} \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}} + t\right]\left(\vartheta_{22}A_{3f} - B_{3g}\right) \\[3mm] + \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}} + \dfrac{\vartheta_{22}t}{\eta^{q_3-1}}\right]\left(\vartheta_{12}B_{2g} - A_{2f}\right) \\[3mm] + \dfrac{\vartheta_{21}(\vartheta_{11}B_{1g}+A_{1f})}{1-\vartheta_{21}\vartheta_{11}} + \displaystyle\int_0^t \dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds, \end{cases}$$

where $h_1 \in \mathscr{A}_1(y_1,y_2), h_2 \in \mathscr{A}_2(y_1,y_2)$ and so $(h_1,h_2) \in \mathscr{A}(y_1,y_2)$.

Now we verify that the operator $\mathscr{A}$ satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. This can be done in a couple of steps. In the first step, we will show that $\mathscr{A}(y_1,y_2)$ is convex valued. Let $(h_i, \bar{h}_i) \in (\mathscr{A}_1, \mathscr{A}_2), i = 1,2$. Then there exist $f_i \in S_{F,(y_1,y_2)}$, $g_i \in S_{G,(y_1,y_2)}$, $i = 1,2$ such that, for each $t \in [0,1]$, we have

$$h_i(t) = \begin{cases} \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}} + \dfrac{\vartheta_{12}t}{\eta^{q_3-1}}\right]\left(\vartheta_{22}A_{3f_i} - B_{3g_i}\right) \\[3mm] + \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}} + t\right]\left(\vartheta_{12}B_{2g_i} - A_{2f_i}\right) \\[3mm] + \dfrac{\vartheta_{11}(\vartheta_{21}A_{1f_i}+B_{1g_i})}{1-\vartheta_{21}\vartheta_{11}} + \displaystyle\int_0^t \dfrac{(t-s)^{q_1-1}}{\Gamma(q_1)}f_i(s)ds, \end{cases}$$

$$\bar{h}_i(t) = \begin{cases} \dfrac{\Gamma(2-q_3)}{\Delta} \left[\dfrac{\vartheta_{21}(\vartheta_{11} + \vartheta_{12}\eta^{1-q_3})}{1 - \vartheta_{11}\vartheta_{21}} + t \right] \left(\vartheta_{22}A_{3f_i} - B_{3g_i} \right) \\[4mm] + \dfrac{\Gamma(2-q_3)}{\Delta} \left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3} + 1)}{1 - \vartheta_{11}\vartheta_{21}} + \dfrac{\vartheta_{22}t}{\eta^{q_3-1}} \right] \left(\vartheta_{12}B_{2g_i} - A_{2f_i} \right) \\[4mm] + \dfrac{\vartheta_{21}(\vartheta_{11}B_{1g_i} + A_{1f_i})}{1 - \vartheta_{21}\vartheta_{11}} + \displaystyle\int_0^t \dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)} g_i(s)ds, \end{cases}$$

where

$$A_{1f_i} = \int_0^1 \frac{(1-s)^{q_1-1}}{\Gamma(q_1)} f_i(s)ds,$$

$$B_{1g_i} = \int_0^1 \frac{(1-s)^{q_2-1}}{\Gamma(q_2)} g_i(s)ds,$$

$$A_{2f_i} = \int_0^1 \frac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)} f_i(s)ds,$$

$$B_{2g_i} = \int_0^\eta \frac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)} g_i(s)ds,$$

$$A_{3f_i} = \int_0^\eta \frac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)} f_i(s)ds,$$

$$B_{3g_i} = \int_0^1 \frac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)} g_i(s)ds.$$

Let $0 \leqslant \omega \leqslant 1$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned}
&\left[\omega h_1 + (1-\omega)h_2 \right](t) \\
&= \begin{cases} \dfrac{\Gamma(2-q_3)}{\Delta} \left[\dfrac{\vartheta_{21}(\vartheta_{11} + \vartheta_{12}\eta^{1-q_3})}{1 - \vartheta_{11}\vartheta_{21}} + t \right] \left(\vartheta_{22} \displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)} [\omega f_1(s) \right. \\[4mm] \left. + (1-\omega)f_2(s)]ds - \displaystyle\int_0^1 \dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)} [\omega g_1(s) + (1-\omega)g_2(s)]ds \right) \\[4mm] + \dfrac{\Gamma(2-q_3)}{\Delta} \left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3} + 1)}{1 - \vartheta_{11}\vartheta_{21}} + \dfrac{\vartheta_{22}t}{\eta^{q_3-1}} \right] \left(\vartheta_{12} \displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)} \right. \\[4mm] \times [\omega g_1(s) + (1-\omega)g_2(s)]ds - \displaystyle\int_0^1 \dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)} [\omega f_1(s) \\[4mm] \left. + (1-\omega)f_2(s)]ds \right) + \dfrac{\vartheta_{21}}{1 - \vartheta_{21}\vartheta_{11}} \left(\vartheta_{11} \displaystyle\int_0^1 \dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)} [\omega g_1(s) \right. \\[4mm] \left. + (1-\omega)g_2(s)]ds + \displaystyle\int_0^1 \dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)} [\omega f_1(s) + (1-\omega)f_2(s)]ds \right) \\[4mm] + \displaystyle\int_0^t \dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)} [\omega g_1(s) + (1-\omega)g_2(s)]ds, \end{cases}
\end{aligned}$$

$[\omega\bar{h}_1 + (1-\omega)\bar{h}_2](t)$

$$= \begin{cases} \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right]\left(\vartheta_{22}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}[\omega f_1(s)\right. \\[2mm] \left.+(1-\omega)f_2(s)]ds - \displaystyle\int_0^1 \dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}[\omega g_1(s)+(1-\omega)g_2(s)]ds\right) \\[3mm] +\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\dfrac{\vartheta_{22}t}{\eta^{q_3-1}}\right]\left(\vartheta_{12}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}\right. \\[2mm] \times[\omega g_1(s)+(1-\omega)g_2(s)]ds - \displaystyle\int_0^1 \dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}[\omega f_1(s) \\[2mm] \left.+(1-\omega)f_2(s)]ds\right) + \dfrac{\vartheta_{21}}{1-\vartheta_{21}\vartheta_{11}}\left(\vartheta_{11}\displaystyle\int_0^1 \dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}[\omega g_1(s)\right. \\[2mm] \left.+(1-\omega)g_2(s)]ds + \displaystyle\int_0^1 \dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}[\omega f_1(s)+(1-\omega)f_2(s)]ds\right) \\[3mm] +\displaystyle\int_0^t \dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)}[\omega g_1(s)+(1-\omega)g_2(s)]ds. \end{cases}$$

Since F,G are considered as convex valued, we conclude that $S_{F,(y_1,y_2)}$, $S_{G,(y_1,y_2)}$ are convex valued. Clearly $\omega h_1 + (1-\omega)h_2 \in \mathscr{A}_1$, $\omega\bar{h}_1 + (1-\omega)\bar{h}_2 \in \mathscr{A}_2$ and hence $\omega(h_1,\bar{h}_1)+(1-\omega)(h_2,\bar{h}_2) \in \mathscr{A}$. Now we show that $\mathscr{A}$ maps bounded sets into bounded sets in $\mathscr{B}$. For a positive number r, let $B_r = \{(y_1,y_2) \in \mathscr{B} : \|(y_1,y_2)\| \leqslant r\}$ be a bounded set in $\mathscr{B}$. Then, $\exists f \in S_{F,(y_1,y_2)}, g \in S_{G,(y_1,y_2)}$ such that

$\left|h_1(y_1,y_2)(t)\right|$

$$\leqslant \begin{cases} \dfrac{|\Gamma(2-q_3)|}{|\Delta|}\left(\dfrac{|\vartheta_{11}|(|\vartheta_{21}\vartheta_{12}\eta^{1-q_3}|+1)}{|1-\vartheta_{11}\vartheta_{21}|}+\dfrac{|\vartheta_{12}|}{|\eta^{q_3-1}|}\right)\left(\dfrac{|\vartheta_{22}\eta^{q_1-q_3}|}{|\Gamma(q_1-q_3+1)|}\right. \\[2mm] \left.\times\|b_1\|\big(\Psi_1(r)+\Phi_1(r)\big) + \dfrac{1}{|\Gamma(q_2-q_3+1)|}\|b_2\|\big(\Psi_2(r)+\Phi_2(r)\big)\right) \\[3mm] +\dfrac{|\Gamma(2-q_3)|}{|\Delta|}\left(\dfrac{|\vartheta_{11}|(|\vartheta_{21}|+|\vartheta_{22}\eta^{1-q_3}|)}{|1-\vartheta_{11}\vartheta_{21}|}+1\right)\left(\dfrac{|\vartheta_{12}\eta^{q_2-q_3}|}{|\Gamma(q_2-q_3+1)|}\right. \\[2mm] \left.\times\|b_2\|\big(\Psi_2(r)+\Phi_2(r)\big) + \dfrac{1}{|\Gamma(q_1-q_3+1)|}\|b_1\|\big(\Psi_1(r)+\Phi_1(r)\big)\right) \\[3mm] +\dfrac{|\vartheta_{11}|}{|1-\vartheta_{21}\vartheta_{11}|}\left(\dfrac{|\vartheta_{21}|}{|\Gamma(q_1+1)|}\|b_1\|\big(\Psi_1(r)+\Phi_1(r)\big) + \dfrac{1}{|\Gamma(q_2+1)|}\right. \\[2mm] \left.\times\|b_2\|\big(\Psi_2(r)+\Phi_2(r)\big)\right) + \dfrac{1}{|\Gamma(q_1+1)|}\|b_1\|\big(\Psi_1(r)+\Phi_1(r)\big) \\[3mm] \leqslant \mathscr{M}_1\|b_1\|\big(\Psi_1(r)+\Phi_1(r)\big) + \mathscr{M}_2\|b_2\|\big(\Psi_2(r)+\Phi_2(r)\big). \end{cases}$$

and

$$
\begin{aligned}
&\left|h_2(y_1,y_2)(t)\right| \\
&\leqslant
\begin{cases}
\dfrac{|\Gamma(2-q_3)|}{|\Delta|}\left(\dfrac{|\vartheta_{21}|(|\vartheta_{11}|+|\vartheta_{12}\eta^{1-q_3}|)}{|1-\vartheta_{11}\vartheta_{21}|}+1\right) \\[3mm]
\quad\times\left(\dfrac{|\vartheta_{22}\eta^{q_1-q_3}|}{|\Gamma(q_1-q_3+1)|}||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)\right. \\[3mm]
\quad\left.+\dfrac{1}{|\Gamma(q_2-q_3+1)|}||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)\right) \\[3mm]
\quad+\dfrac{|\Gamma(2-q_3)|}{|\Delta|}\left(\dfrac{|\vartheta_{21}|(|\vartheta_{11}\vartheta_{22}\eta^{1-q_3}|+1)}{|1-\vartheta_{11}\vartheta_{21}|}+\left|\dfrac{\vartheta_{22}}{\eta^{q_3-1}}\right|\right) \\[3mm]
\quad\times\left(\dfrac{1}{|\Gamma(q_1-q_3+1)|}||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)\right. \\[3mm]
\quad\left.+\dfrac{|\vartheta_{12}\eta^{q_2-q_3}|}{|\Gamma(q_2-q_3+1)|}||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)\right) \\[3mm]
\quad+\dfrac{|\vartheta_{21}|}{|1-\vartheta_{21}\vartheta_{11}|}\left(\dfrac{|\vartheta_{11}|}{|\Gamma(q_2+1)|}||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)\right. \\[3mm]
\quad\left.+\dfrac{1}{|\Gamma(q_2+1)|}||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)\right) \\[3mm]
\quad+\dfrac{1}{|\Gamma(q_2+1)|}||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)
\end{cases}
\end{aligned}
$$

$$
\leqslant \mathscr{M}_3||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)+\mathscr{M}_4||b_2||\big(\Psi_2(r)+\Phi_2(r)\big),
$$

where $\mathscr{M}_1,\mathscr{M}_2,\mathscr{M}_3$ and $\mathscr{M}_4$ are given in (12), (13), (14) and (15) respectively. Thus,

$$
||h_1(y_1,y_2)||\leqslant \mathscr{M}_1||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)+\mathscr{M}_2||b_2||\big(\Psi_2(r)+\Phi_2(r)\big),
$$

and

$$
||h_2(y_1,y_2)||\leqslant \mathscr{M}_3||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)+\mathscr{M}_4||b_2||\big(\Psi_2(r)+\Phi_2(r)\big).
$$

Hence we obtain

$$
\begin{aligned}
||(h_1,h_2)|| &= ||h_1(y_1,y_2)||+||h_2(y_1,y_2)|| \\
&\leqslant m_1||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)+m_2||b_2||\big(\Psi_2(r)+\Phi_2(r)\big) \\
&=\ell,
\end{aligned}
$$

where

$$
m_1=\mathscr{M}_1+\mathscr{M}_3 \text{ and } m_2=\mathscr{M}_2+\mathscr{M}_4 \tag{16}
$$

and

$$\ell = m_1||b_1||\big(\Psi_1(r)+\Phi_1(r)\big) + m_2||b_2||\big(\Psi_2(r)+\Phi_2(r)\big) \tag{17}$$

are constants. Next, we show that $\mathscr{A}$ is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, there exist $f \in S_{F,(y_1,y_2)}, g \in S_{G,(y_1,y_2)}$ such that

$$\big|h_1(y_1,y_2)(t_2) - h_1(y_1,y_2)(t_1)\big|$$

$$\leqslant \begin{cases} ||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)\dfrac{|\eta^{q_1-q_3}|}{|\Gamma(q_1-q_3+1)|}\dfrac{|\vartheta_{22}\vartheta_{12}\eta^{1-q_3}\Gamma(2-q_3)|}{|\Delta|}(|t_2-t_1|) \\[2ex] + ||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)\dfrac{1}{|\Gamma(q_2-q_3+1)|}\dfrac{|\vartheta_{22}\eta^{1-q_3}\Gamma(2-q_3)|}{|\Delta|}(|t_2-t_1|) \\[2ex] + ||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)\dfrac{1}{|\Gamma(q_1-q_3+1)|}\dfrac{|\Gamma(2-q_3)|}{|\Delta|}(|t_2-t_1|) \\[2ex] + ||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)\dfrac{|\eta^{q_2-q_3}|}{|\Gamma(q_2-q_3+1)|}\dfrac{|\vartheta_{22}\Gamma(2-q_3)|}{|\Delta|}(|t_2-t_1|) \\[2ex] + ||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)\dfrac{(|t_2^{q_1}-t_1^{q_1}|)}{|\Gamma(q_1+1)|}. \end{cases}$$

Analogously, we can obtain

$$\big|h_2(y_1,y_2)(t_2) - h_2(y_1,y_2)(t_1)\big|$$

$$\leqslant \begin{cases} ||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)\dfrac{|\eta^{q_1-q_3}|}{|\Gamma(q_1-q_3+1)|}\dfrac{|\vartheta_{22}\Gamma(2-q_3)|}{|\Delta|}(|t_2-t_1|) \\[2ex] + ||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)\dfrac{1}{|\Gamma(q_2-q_3+1)|}\dfrac{|\Gamma(2-q_3)|}{|\Delta|}(|t_2-t_1|) \\[2ex] + ||b_1||\big(\Psi_1(r)+\Phi_1(r)\big)\dfrac{1}{|\Gamma(q_1-q_3+1)|}\dfrac{|\Gamma(2-q_3)|}{|\Delta|}(|t_2-t_1|) \\[2ex] + ||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)\dfrac{|\eta^{q_2-q_3}|}{|\Gamma(q_2-q_3+1)|}\dfrac{|\vartheta_{12}\vartheta_{22}\eta^{1-q_3}\Gamma(2-q_3)|}{|\Delta|}(|t_2-t_1|) \\[2ex] + ||b_2||\big(\Psi_2(r)+\Phi_2(r)\big)\dfrac{(|t_2^{q_2}-t_1^{q_2}|)}{|\Gamma(q_2+1)|}. \end{cases}$$

It follows from the above arguments that the operator $\mathscr{A}(y_1,y_2)$ is equicontinuous, and so it is completely continuous by the Arzelá–Ascoli theorem.

For our next step, we show that the operator $\mathscr{A}(y_1,y_2)$ has a closed graph. Let $(y_{1n}, y_{2n}) \to (y_{1*}, y_{2*})$, $(h_n, \bar{h}_n) \in \mathscr{A}(y_{1n}, y_{2n})$ and $(h_n, \bar{h}_n) \to (h_*, \bar{h}_*)$, then we have to show $(h_*, \bar{h}_*) \in \mathscr{A}(y_{1*}, y_{2*})$. Observe that $(h_n, \bar{h}_n) \in \mathscr{A}(y_{1n}, y_{2n})$ implies that there exist $f_n \in S_{F,(y_{1n},y_{2n})}$ and $g_n \in S_{G,(y_{1n},y_{2n})}$ such that

$h_n(y_{1n}, y_{2n})(t)$

$$
= \begin{cases}
\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\dfrac{\vartheta_{12}t}{\eta^{q_3-1}}\right] \\
\times\left(\vartheta_{22}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f_n(s)ds-\int_0^1\dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g_n(s)ds\right) \\
+\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right] \\
\times\left(\vartheta_{12}\displaystyle\int_0^\eta\dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g_n(s)ds-\int_0^1\dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f_n(s)ds\right) \\
+\left[\dfrac{\vartheta_{11}}{1-\vartheta_{21}\vartheta_{11}}\right]\left(\vartheta_{21}\displaystyle\int_0^1\dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}f_n(s)ds+\int_0^1\dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}g_n(s)ds\right) \\
+\displaystyle\int_0^t\dfrac{(t-s)^{q_1-1}}{\Gamma(q_1)}f_n(s)ds
\end{cases}
$$

and

$\bar{h}_n(y_{1n}, y_{2n})(t)$

$$
= \begin{cases}
\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right] \\
\times\left(\vartheta_{22}\displaystyle\int_0^\eta\dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}\times f_n(s)ds-\int_0^1\dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g_n(s)ds\right) \\
+\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}+\frac{\vartheta_{22}t}{\eta^{q_3-1}}}\right] \\
\times\left(\vartheta_{12}\displaystyle\int_0^\eta\dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g_n(s)ds\right. \\
\left.-\displaystyle\int_0^1\dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f_n(s)ds\right)+\dfrac{\vartheta_{21}}{1-\vartheta_{21}\vartheta_{11}} \\
\times\left(\vartheta_{11}\displaystyle\int_0^1\dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}g_n(s)ds+\int_0^1\dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}f_n(s)ds\right) \\
+\displaystyle\int_0^t\dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)}g_n(s)ds.
\end{cases}
$$

Let us consider the continuous linear operators $\Theta_1, \Theta_2 : L^1([0,1], \mathscr{B}) \to C([0,1], \mathscr{B})$

given by

$$\Theta_1(y_1, y_2)(t)$$

$$
= \begin{cases}
\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}} + \dfrac{\vartheta_{12}t}{\eta^{q_3-1}}\right] \\[2mm]
\times\left(\vartheta_{22}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds - \int_0^1 \dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds\right) \\[3mm]
+ \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}} + t\right] \\[2mm]
\times\left(\vartheta_{12}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds - \int_0^1 \dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds\right) \\[3mm]
+ \left[\dfrac{\vartheta_{11}}{1-\vartheta_{21}\vartheta_{11}}\right]\left(\vartheta_{21}\displaystyle\int_0^1 \dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds + \int_0^1 \dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds\right) \\[3mm]
+ \displaystyle\int_0^t \dfrac{(t-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds
\end{cases}
$$

and

$$\Theta_2(y_1, y_2)(t)$$

$$
= \begin{cases}
\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}} + t\right] \\[2mm]
\times\left(\vartheta_{22}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds - \int_0^1 \dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds\right) \\[3mm]
+ \dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}} + \dfrac{\vartheta_{22}t}{\eta^{q_3-1}}\right] \\[2mm]
\times\left(\vartheta_{12}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds - \int_0^1 \dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds\right) \\[3mm]
+ \dfrac{\vartheta_{21}}{1-\vartheta_{21}\vartheta_{11}}\left(\vartheta_{11}\displaystyle\int_0^1 \dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds + \int_0^1 \dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds\right) \\[3mm]
+ \displaystyle\int_0^t \dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds.
\end{cases}
$$

From Lemma 2, we know that $(\Theta_1, \Theta_2)\circ(S_F, S_G)$ is a closed graph operator. Further, we have $(h_n, \bar{h}_n)\in(\Theta_1, \Theta_2)\circ(S_{F,(y_{1n},y_{2n})}, S_{G,(y_{1n},y_{2n})})$ for all n. Since $(y_{1n}, y_{2n})\to (y_{1*}, y_{2*})$, $(h_n, \bar{h}_n)\to(h_*, \bar{h}_*)$, it follows that $f_*\in S_{F,(y_1,y_2)}$ and $g_*\in S_{G,(y_1,y_2)}$ such that

$h_*(y_{1*}, y_{2*})(t)$

$$
= \begin{cases}
\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\dfrac{\vartheta_{12}t}{\eta^{q_3-1}}\right] \\[2mm]
\times\left(\vartheta_{22}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f_*(s)ds-\int_0^1\dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g_*(s)ds\right) \\[2mm]
+\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right] \\[2mm]
\times\left(\vartheta_{12}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g_*(s)ds-\int_0^1\dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f_*(s)ds\right) \\[2mm]
+\left[\dfrac{\vartheta_{11}}{1-\vartheta_{21}\vartheta_{11}}\right]\left(\vartheta_{21}\displaystyle\int_0^1\dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}f_*(s)ds+\int_0^1\dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}g_*(s)ds\right) \\[2mm]
+\displaystyle\int_0^t \dfrac{(t-s)^{q_1-1}}{\Gamma(q_1)}f_*(s)ds
\end{cases}
$$

and

$\bar{h}_*(y_{1*}, y_{2*})(t)$

$$
= \begin{cases}
\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right] \\[2mm]
\times\left(\vartheta_{22}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f_*(s)ds-\int_0^1\dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g_*(s)ds\right) \\[2mm]
+\dfrac{\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\dfrac{\vartheta_{22}t}{\eta^{q_3-1}}\right] \\[2mm]
\times\left(\vartheta_{12}\displaystyle\int_0^\eta \dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g_*(s)ds-\int_0^1\dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f_*(s)ds\right) \\[2mm]
+\dfrac{\vartheta_{21}}{1-\vartheta_{21}\vartheta_{11}}\left(\vartheta_{11}\displaystyle\int_0^1\dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}g_*(s)ds+\int_0^1\dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}f_*(s)ds\right) \\[2mm]
+\displaystyle\int_0^t \dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)}g_*(s)ds.
\end{cases}
$$

that is, $(h_n, \bar{h}_n) \in \mathscr{A}(y_{1*}, y_{2*})$.

Eventually, we discuss a priori bounds on solutions. Let $(y_1, y_2) \in \vartheta\mathscr{A}(y_1, y_2)$ for $\vartheta \in (0, 1)$. Then there exist $f \in S_{F,(y_1,y_2)}$ and $g \in S_{G,(y_1,y_2)}$ such that

$$y_1(t) = \begin{cases} \dfrac{\vartheta\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}\vartheta_{12}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\dfrac{\vartheta_{12}t}{\eta^{q_3-1}}\right] \\ \times\left(\vartheta_{22}\displaystyle\int_0^{\eta}\dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds-\int_0^1\dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds\right) \\ +\dfrac{\vartheta\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{11}(\vartheta_{21}+\vartheta_{22}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right] \\ \times\left(\vartheta_{12}\displaystyle\int_0^{\eta}\dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds-\int_0^1\dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds\right) \\ +\left[\dfrac{\vartheta\vartheta_{11}}{1-\vartheta_{21}\vartheta_{11}}\right]\left(\vartheta_{21}\displaystyle\int_0^1\dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds+\int_0^1\dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds\right) \\ +\vartheta\displaystyle\int_0^t\dfrac{(t-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds \end{cases}$$

and

$$y_2(t) = \begin{cases} \dfrac{\vartheta\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}+\vartheta_{12}\eta^{1-q_3})}{1-\vartheta_{11}\vartheta_{21}}+t\right] \\ \times\left(\vartheta_{22}\displaystyle\int_0^{\eta}\dfrac{(\eta-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}\times f(s)ds-\int_0^1\dfrac{(1-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds\right) \\ +\dfrac{\vartheta\Gamma(2-q_3)}{\Delta}\left[\dfrac{\vartheta_{21}(\vartheta_{11}\vartheta_{22}\eta^{1-q_3}+1)}{1-\vartheta_{11}\vartheta_{21}}+\dfrac{\vartheta_{22}t}{\eta^{q_3-1}}\right] \\ \times\left(\vartheta_{12}\displaystyle\int_0^{\eta}\dfrac{(\eta-s)^{q_2-q_3-1}}{\Gamma(q_2-q_3)}g(s)ds-\int_0^1\dfrac{(1-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)}f(s)ds\right) \\ +\dfrac{\vartheta\vartheta_{21}}{1-\vartheta_{21}\vartheta_{11}}\left(\vartheta_{11}\displaystyle\int_0^1\dfrac{(1-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds+\int_0^1\dfrac{(1-s)^{q_1-1}}{\Gamma(q_1)}f(s)ds\right) \\ +\vartheta\displaystyle\int_0^t\dfrac{(t-s)^{q_2-1}}{\Gamma(q_2)}g(s)ds. \end{cases}$$

Using the arguments employed in Step 2, for each $t \in [0,1]$, we obtain

$$\|y_1\| \leqslant \mathscr{M}_1\|b_1\|\Big(\Psi_1(\|y_1\|)+\Phi_1(\|y_2\|)\Big)+\mathscr{M}_2\|b_2\|\Big(\Psi_2(\|y_1\|)+\Phi_2(\|y_2\|)\Big),$$

and

$$\|y_2\| \leqslant \mathscr{M}_3\|b_1\|\Big(\Psi_1(\|y_1\|)+\Phi_1(\|y_2\|)\Big)+\mathscr{M}_4\|b_2\|\Big(\Psi_2(\|y_1\|)+\Phi_2(\|y_2\|)\Big).$$

In consequence, we have

$$\begin{aligned}\|(y_1,y_2)\| &= \|y_1\|+\|y_2\| \\ &\leqslant m_1\|b_1\|\Big(\Psi_1(\|y_1\|)+\Phi_1(\|y_2\|)\Big)+m_2\|b_2\|\Big(\Psi_2(\|y_1\|)+\Phi_2(\|y_2\|)\Big)\end{aligned}$$

which implies that

$$\frac{\|(y_1,y_2)\|}{m_1\|b_1\|\big(\Psi_1(\|y_1\|)+\Phi_1(\|y_2\|)\big)+m_2\|b_2\|\big(\Psi_2(\|y_1\|)+\Phi_2(\|y_2\|)\big)}\leqslant 1.$$

where m_1 and m_2 are given in (16). In view of (A4), $\exists\ N$ such that $\|(y_1,y_2)\|\neq N$. Let us set

$$U=\{(y_1,y_2)\in\mathscr{B}:\|(y_1,y_2)\|<N\}.$$

Note that the operator $\mathscr{A}:\overline{U}\to\mathscr{P}_{cp,cv}(X)\times\mathscr{P}_{cp,cv}(X)$ is upper semicontinuous and completely continuous. From the choice of U, there is no $(y_1,y_2)\in\partial U$ such that $(y_1,y_2)\in\vartheta\mathscr{A}(y_1,y_2)$ for some $\vartheta\in(0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type [13], we conclude that $\mathscr{A}$ has a fixed point $(y_1,y_2)\in\overline{U}$ which is a solution of the coupled system of FBVP (1)–(2). The proof is completed. $\square$

4. Example

In this section, as an application, we demonstrate our results with an example. Consider the FBVP,

$$\left.\begin{array}{ll}{}^{c}D_{0+}^{\frac{8}{5}}y_1(t)\in F\big(t,y_1,y_2\big), & 0<t<1,\\[2mm]{}^{c}D_{0+}^{\frac{8}{5}}y_2(t)\in G\big(t,y_1,y_2\big), & 0<t<1,\end{array}\right\}\tag{18}$$

$$\left.\begin{array}{ll}y_1(0)=\dfrac{6}{11}y_2(1), & {}^{c}D_{0+}^{\frac{3}{5}}y_1(1)=\dfrac{7}{23}{}^{c}D_{0+}^{\frac{3}{5}}y_2\left(\dfrac{13}{50}\right),\\[3mm]y_2(0)=\dfrac{27}{64}y_1(1), & {}^{c}D_{0+}^{\frac{3}{5}}y_2(1)=\dfrac{17}{28}{}^{c}D_{0+}^{\frac{3}{5}}y_1\left(\dfrac{13}{50}\right).\end{array}\right\}\tag{19}$$

Here $q_1=q_2=\dfrac{8}{5},\eta=\dfrac{13}{50},q_3=\dfrac{3}{5},\vartheta_{11}=\dfrac{6}{11},\vartheta_{12}=\dfrac{7}{23},\vartheta_{21}=\dfrac{27}{64}$ and $\vartheta_{22}=\dfrac{17}{28}$. And also we have $\Gamma(q_1-q_3+1)=\Gamma(q_2-q_3+1)=1,\ \Gamma(2-q_3)=0.8872,\ \Gamma(q_1+1)=\Gamma(q_2+1)=1.4296$. $F,G:[0,1]\times\mathbb{R}^2\to\mathscr{P}(\mathbb{R})$ are multivalued maps given by

$$F\big(t,y_1,y_2\big)=\left[\sin t+e^{-|y_1|}-\frac{|y_2|}{1+|y_2|},\quad\frac{|y_1|}{1+y_1^2}+\cos(y_2)+7t^2+9\right],$$

$$G\big(t,y_1,y_2\big)=\left[e^{-y_1^2}+\frac{|y_2|}{1+|y_2|}+t^2,\quad\frac{|y_1|}{1+|y_1|}+t+3\cos(y_2)+16\right].$$

Then we have $\|F(t,y_1,y_2)\|_{\mathscr{P}}=\sup\big\{|f|:f\in F(t,y_1,y_2)\big\}\leqslant 18=b_1(t)\big(\Psi_1(\|y_1\|)+\Phi_1(\|y_2\|)\big)$ with $b_1(t)=1,\Psi_1(\|y_1\|)=3,\Phi_1(\|y_2\|)=15$ and $\|G(t,y_1,y_2)\|_{\mathscr{P}}=\sup\big\{|g|:g\in G(t,y_1,y_2)\big\}\leqslant 21=b_2(t)\big(\Psi_2(\|y_1\|)+\Phi_2(\|y_2\|)\big)$ with $b_2(t)=1,\Psi_2(\|y_1\|)=3,\Phi_2(\|y_2\|)=18$. From all information and also by (12), (13), (14), (15), we can

find that $\Delta = 1 - \left(\dfrac{7}{23}\right)\left(\dfrac{17}{28}\right)\left(\dfrac{13}{50}\right)^{2\left(1-\frac{3}{5}\right)} = 0.8887$, $\mathscr{M}_1 = 3.4546$, $\mathscr{M}_2 = 1.5571$, $\mathscr{M}_3 = 2.2717$, $\mathscr{M}_4 = 2.1580$, $m_1 = \mathscr{M}_1 + \mathscr{M}_3 = 5.7263$ and $m_2 = \mathscr{M}_2 + \mathscr{M}_4 = 3.7151$. Using the condition $(A4)$ we find that $N > 181.0905$. Hence, by applying Theorem (1), we deduce that the FBVP (18)–(19) has at least one solution on $[0,1]$.

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