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EXISTENCE OF SOLUTIONS TO NONLINEAR STURM–LIOUVILLE PROBLEMS WITH LARGE NONLINEARITIES

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Abstract. In this paper, we present results which allow us to establish the existence of solutions to nonlinear Sturm-Liouville problems with unbounded nonlinearities. We consider both regular and singular problems. Our main results rely on a variant of the Lyapunov-Schmidt used in conjunction with topological degree theory.

1. Introduction

In this paper, we consider nonlinear Sturm-Liouville boundary value problems. The results presented here enable us to establish the existence of solutions to both regular and singular problems. The class of unbounded nonlinearities in the differential equation includes as a special case those which exhibit sublinear behavior. For regular problems and for a class of singular problems, we allow weakly nonlinear boundary conditions.

As in previous related work, boundary value problems are analyzed by formulating them as operator equations of the form $\mathscr{L}x = \mathscr{F}(x)$ where $\mathscr{L}$ is linear and $\mathscr{F}$ is nonlinear. We will be mainly interested in the case where $\mathscr{L}$ does not have an inverse. Using an approach similar to the Lyapunov-Schmidt procedure together with topological degree theory we provide criteria for the solvability of boundary value problems. For general theory regarding properties of the topological degree, the reader may consult [13] and [30].

Use of the Lyapanov-Schmidt procedure in the study of nonlinear boundary value problems appears in [3], [4], [8], [12], [13], [19], [28], and [29]. In [21], the reader will find conditions for the solvability of nonlinear Sturm-Liouville problems where the boundary conditions are global and the nonlinearities in the dynamics are sublinear. There is often an intimate relation between discrete-time systems and differential equations. Those interested in the connection between the present paper and previous results on discrete-time systems are encouraged to see [9], [18], [19], and [20]. In [18] and [20], problems with linear boundary conditions are analyzed and [19] is devoted to the study of periodic behavior in discrete dynamical systems.

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2. Regular Sturm-Liouville problems

Let $a,b \in \mathbb{R}$ with $b > a$. Let X denote the space $(\mathscr{C}[a,b], \|\cdot\|)$ where $\|\cdot\|$ denotes the supremum norm. In the following, let $p:[a,b] \to \mathbb{R}$, $q:[a,b] \to \mathbb{R}$, and $f:\mathbb{R} \to \mathbb{R}$ be continuous and $p(t) > 0$ on $[a,b]$. We use μ and ε to denote real-valued parameters, and assume that α, β, γ and δ are constants such that $\alpha^2+\beta^2 > 0$ and $\gamma^2+\delta^2 > 0$. Let $G_i : X \to \mathbb{R}$ be a nonlinear map for $i = 1,2$ that maps bounded sets in X into bounded subsets of $\mathbb{R}$. Let $\mathscr{D} \subset X$ be the collection of all twice-continuously differentiable functions in X. Consider the following nonlinear boundary value problems on (a,b):

$$(p(t)x'(t))' + q(t)x(t) + \mu x(t) = f(x(t)) \tag{1}$$

subject to

$$\alpha x(a) + \beta x'(a) = \varepsilon G_1(x), \tag{2a}$$
$$\delta x(b) + \gamma x'(b) = \varepsilon G_2(x). \tag{2b}$$

For general theory regarding Sturm-Liouville problems, the reader may consult [5], [14], and [10]. It is well-known from general Sturm-Liouville theory that the closely related linear problem

$$(p(t)x'(t))' + q(t)x(t) = \lambda x(t)$$

subject to

$$\alpha x(a) + \beta x'(a) = 0$$
$$\delta x(b) + \gamma x'(b) = 0.$$

has countably many simple eigenvalues $\{\lambda_k)\}_{k=0}^{\infty}$ with corresponding eigenfunctions $\{\psi_k\}_{k=0}^{\infty}$. Without loss of generality, we will assume that $\|\psi_k\| = 1$ for all $k = 0,1,2,\ldots$. Define the map $L : \mathscr{D} \to X$ by

$$[Lx](t) = (p(t)x'(t))' + q(t)x(t) + \mu x(t).$$

and $\mathscr{L} : \mathscr{D} \to X \times \mathbb{R}^2$ by

$$[\mathscr{L}x](t) = \begin{bmatrix} Lx \\ B_1(x) \\ B_2(x) \end{bmatrix}.$$

It is straightforward to show that if μ is not an eigenvalue of the linear problem above, $\mathscr{L}$ is a bijection from $\mathscr{D}$ onto $X \times \mathbb{R}^2$. If $\mu = \lambda_k$ for some nonnegative integer k then the kernel of L is nontrivial and spanned by $\psi \equiv \psi_k$. Let ϕ be a solution to $Lx = 0$ such that $\{\psi, \phi\}$ forms a basis for the solution space to $Lx = 0$ and $\|\phi\| = 1$. Let $v_1 = B_1(\phi)$ and $v_2 = B_2(\phi)$. Then It is straightforward to show that $[h,[w_1,w_2]] \in X \times \mathbb{R}^2$ is in the image of L if and only if

$$\int_a^b h(t)\psi(t)dt + \frac{w_1}{v_1} - \frac{w_2}{v_2} = 0.$$

We define $U : X \to X$ by

$$[Ux](t) = \psi(t) \int_a^b x(s)\psi(s)ds.$$

Define $E : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$ by

$$E(h, w_1, w_2) = m\left(\int_a^b h(t)\psi(t)dt + \frac{w_1}{v_1} - \frac{w_2}{v_2}\right)(h, w_1, w_2)$$

where $m = \frac{1}{1+\sqrt{(v_1^{-1})^2+(v_2^{-1})^2}}$. Also note that the map L restricted to $\mathscr{D} \cap \operatorname{Im}(\mathscr{L})$ is a bijection onto $\operatorname{Im}(L) = \operatorname{Im}(I-U)$. Therefore, it follows that there exists a linear map $M : \operatorname{Im}(I-U) \to \mathscr{D} \cap \operatorname{Im}(\mathscr{L})$ satisfying

$$LMh = h$$

for all $h \in \operatorname{Im}(I-U)$ and

$$MLx = (I-U)x$$

for all $x \in \mathscr{D}$. It is clear that M is a compact operator between $\operatorname{Im}(I-U)$ and $\mathscr{D}$. Define $F : X \to X$ by

$$F(x)(t) = f(x(t)).$$

and $\mathscr{F} : X \times \mathbb{R} \to X \times \mathbb{R}^2$ by

$$[\mathscr{F}x](t) = \begin{bmatrix} F(x) \\ \varepsilon G_1(x) \\ \varepsilon G_2(x) \end{bmatrix}.$$

We note that solving

$$\mathscr{L}x = \mathscr{F}(x)$$

is equivalent to solving the system

$$\begin{cases} x - ME\mathscr{F}(\alpha\psi + x) = 0 \\ \qquad \text{and} \\ \int_{-1}^1 \psi(s) f(\alpha\psi(s) + x(s))ds + \frac{\varepsilon G_1(\alpha\psi+x)}{v_1} - \frac{\varepsilon G_2(\alpha\psi+x)}{v_2} = 0. \end{cases}$$

For $\eta \in [0,1]$ we denote $A_\eta = \{t \in [-1,1] : |\psi_k(t)| \geqslant \eta\}$ and for $T \subset \mathbb{R}$ we denote $m(T)$ as the Lebesgue measure of the set T. For the sake of notation in the following lemma, define $C = 0$ if ψ has no zeroes on $[a,b]$. Otherwise, let $\{s_1, \ldots, s_l\}$ be the set of roots for ψ and let C be the constant defined by

$$C = \left(\sum_{i=1}^{l} \frac{1}{|\psi'(s_i)|}\right).$$

For $l > 0$ we will write

$$\|f\|_l = \sup_{s\in[-l,l]} |f(s)|.$$

REMARK 1. We are focusing primarily on the case where μ is an eigenvalue of the corresponding linear problem, we will now discuss the case where it is not. In this case, it is well-known from general Sturm-Liouville theory that the operator $\mathscr{L}$ is invertible and that its inverse is continuous. If there exists an $r > 0$ such that

$$\frac{\|f\|_{2r}}{r} < \frac{1}{\|\mathscr{L}^{-1}\|}$$

then there exists ε_0 such that for all $|\varepsilon| < \varepsilon_0$ we can guarantee a solution to (1)–(2).

THEOREM 1. *Suppose the following hold*

1. *There exists constants $\hat{z} > 0$ and $J > 0$ such that if $|s| > \hat{z}$ then $sf(s) > 0$ and $|f(s)| \geqslant J$.*
2. *There exists $\eta \in [0,1]$ and $r > 0$ such that $\eta > \frac{\hat{z}+\|ME\|\|f\|_{2r}}{r}$, $J(1-c\eta) > c\eta\|f\|_{2r}$.*

Then there exists a $\varepsilon_0 > 0$ such that there exists a solution to (1)–(2) *all $\varepsilon < \varepsilon_0$.*

Proof. Define the operator $H : Im(L)\times\mathbb{R} \to Im(L)\times\mathbb{R}$ by

$$H(x,\alpha) = \begin{bmatrix} ME\mathscr{F}(\alpha\psi + x) \\ \alpha - \left[\int_{-1}^{1}\psi(t)f(\alpha\psi(t)+x(t))dt + \frac{\varepsilon G_1(\alpha\psi+x)}{v_1} - \frac{\varepsilon G_2(\alpha\psi+x)}{v_2}\right] \end{bmatrix}$$

Note that H is compact as a consequence of the preceding lemma. We wish to show that $I-H$ has a nonzero Leray-Schauder degree on $\Omega_r = \{(x,\alpha)\in\mathscr{C}\times\mathbb{R} : \max\{\|x\|,|\alpha|\} \leqslant r\}$. Define the map $Q : [0,1]\times\bar{\Omega}_r \to Im(E)\times\mathbb{R}$ by

$$\begin{aligned} Q(s,(x,\alpha)) &= \begin{bmatrix} Q_1(s,(x,\alpha)) \\ Q_2(s,(x,\alpha)) \end{bmatrix} \\ &= \begin{bmatrix} x - sME\mathscr{F}(\alpha\psi+x) \\ (1-s)\alpha + s\left[\int_{-1}^{1}\psi(t)f(\alpha\psi(t)+x(t))dt + \frac{\varepsilon G_1(\alpha\psi+x)}{v_1} - \frac{\varepsilon G_2(\alpha\psi+x)}{v_2}\right] \end{bmatrix}. \end{aligned}$$

and note that Q is a homotopy between I and $I-H$. Let $(x,\alpha)\in\partial\Omega_r$ and $s\in(0,1)$. We will now show that $Q(s,(x,\alpha)) \neq 0$ for all $(s,(x,\alpha)) \in [0,1]\times\partial(\Omega_r)$. Let $(x,\alpha)\in\partial\Omega_r$ and $s\in[0,1]$. First suppose that $\|x\| = r$. Then since $|\alpha\psi(t)+x(t)| \leqslant 2r$ for all $t\in[-1,1]$ we have that

$$\|sME\mathscr{F}(\alpha\psi+x)\| \leqslant \|ME\|\|\mathscr{F}(\alpha\psi+x)\| \leqslant \|ME\|\|f\|_{2r} < r = \|x\|.$$

Therefore $Q_1(s,(x,\alpha)) \neq 0$.

Now assume that $|\alpha| = r$ holds and suppose that $x = sME\mathscr{F}(\alpha\psi + x)$ for some $s \in (0,1)$. We will first examine the case where $\alpha = r$. Let $t \in A_\eta$ with $\psi(t) \geqslant \eta$. We have that $\alpha\psi(t) > 0$, $|\alpha\psi(t)| > \|ME\|\|f\|_{2r} \geqslant |x(t)|$ and

$$\begin{aligned}|\alpha\psi(t) + x(t)| &\geqslant r\eta - \|ME\|\|f\|_{2r}\\ &> \hat{z}\end{aligned}$$

This implies then that

$$\psi(t) f(\alpha\psi(t) + x(t)) > \psi(t)J = |\psi(t)|J$$

whenever $t \in A_\eta$ with $\psi(t) \geqslant \eta$.

The same argument shows that for $t \in A_\eta$ with $\psi(t) \leqslant -\eta$,

$$\alpha\psi(t) + x(t) < -\hat{z}$$

and thus

$$\psi(t) f(\alpha\psi(t) + x(t)) > \psi(t)(-J) = |\psi(t)|J.$$

If there exists $\kappa > 0$ such that $|\psi(t)| \geqslant \kappa$ for all $t \in [a,b]$ then

$$\int_a^b \psi(t) f(\alpha\psi(t) + x(t))dt \geqslant J\kappa(b-a) > 0.$$

If not, then condition 2 implies that there exists $\alpha_0 > 0$ such that

$$\frac{(1-(C\eta+\alpha_0))}{C\eta+\alpha_0} > \frac{\|f\|_{2r}}{J}$$

For $1 \leqslant i \leqslant k$ and s close to s_i we have that $|\psi(s)| \approx |\psi'(s_i)||(s-s_i)|$ and $|\psi'(s_i)||(s-s_i)| < \eta$ implies that

$$|s - s_i| < \frac{\eta}{|\psi'(s_i)|}.$$

Based on this, it is clear that there exists $\delta > 0$ such that if $|\eta| < \delta$ then

$$\left| m(A_\eta^c) - \sum_{i=1}^{k} \frac{2}{|\psi'(s_i)|}\eta \right| < \alpha_0$$

and therefore

$$m(A_\eta^c) < \sum_{i=1}^{k} \frac{2}{|\psi'(s_i)|}\eta + \alpha_0.$$

Thus we have that

$$\begin{aligned}\frac{m(A_\eta)}{m(A_\eta^c)} &> \frac{1-(C\eta+\alpha_0)}{C\eta+\alpha_0}\\ &> \frac{\|f\|_{2r}}{J}.\end{aligned}$$

Let $M_i = \sup_{\|x\| \leqslant 2r} |G_i(x)|$ for $i = 1,2$ and

$$\varepsilon_0 = (M_1 + M_2)^{-1}(\max\{|v_1|,|v_2|\}\eta((1-c\eta) - c\eta\|f\|_{2r}).$$

Now we have that

$$\begin{aligned}\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt &= \int_{A_\eta} \psi(t)f(\alpha\psi(t)+x(t))dt \\ &\quad + \int_{A_\eta^c} \psi(t)f(\alpha\psi(t)+x(t))dt \\ &\geqslant \int_{A_\eta} |\psi(t)|Jdt + \int_{A_\eta^c} \psi(t)f(\alpha\psi(t)+x(t))dt \\ &\geqslant \int_{A_\eta} |\psi(t)|Jdt - m(A_\eta^c)\eta\|f\|_{2r} \\ &\geqslant \eta(m(A_\eta)J - m(A_\eta^c)\|f\|_{2r}) \\ &> \eta((1-c\eta) - c\eta\|f\|_{2r}).\end{aligned}$$

If $\varepsilon < \varepsilon_0$, we have that

$$\left|\frac{\varepsilon G_1(\alpha\psi+x)}{v_1} - \frac{\varepsilon G_2(\alpha\psi+x)}{v_2}\right| < \eta((1-c\eta) - c\eta\|f\|_{2r}).$$

and thus $\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt + \frac{\varepsilon G_1(\alpha\psi+x)}{v_1} - \frac{\varepsilon G_2(\alpha\psi+x)}{v_2} > 0$. In the case where $\alpha = -r$, an argument similar to the one above shows that

$$\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt < 0$$

Along with assumption 3, we have in either case that

$$\begin{aligned}Q_2(s,(x,\alpha)) &= (1-s)\alpha + s\left[\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt + \frac{\varepsilon G_1(\alpha\psi+x)}{v_1} - \frac{\varepsilon G_2(\alpha\psi+x)}{v_2}\right] \\ &\neq 0.\end{aligned}$$

This establishes the desired result due to homotopy invariance of the Leray-Schauder degree. □

The following corollary addresses the special case of this theorem where ψ is a constant function. In this case, $C = 0$ in the theorem above.

COROLLARY 1. *Suppose* $\psi(t) = k > 0$ *for all* $t \in [a,b]$. *Then if*

1. *There exists constants* $\hat{z} > 0$ *and* $J > 0$ *such that if* $|s| > \hat{z}$ *then* $sf(s) > 0$ *and* $|f(s)| \geqslant J$.
2. *There exists* $r > 0$ *such that* $k > \frac{\hat{z}+\|ME\|\|f\|_{2r}}{r}$.

Then there exists a $\varepsilon_0 > 0$ *such that there exists a solution to* (1)–(2) *all* $\varepsilon < \varepsilon_0$.

The following remark illustrates that problems analyzed in [21] and [22] can be analyzed using the framework appearing here. These papers analyze nonlinearly perturbed regular Sturm-Liouville problems and results required f to be sublinear.

REMARK 2. Note that if f is sublinear, then condition 2 is satisfied. This is because of the fact that

$$\lim_{r\to\infty} \frac{\|f\|_{2r}}{r} = 0$$

This implies that there exists r such that

$$\frac{J}{C(\|f\|_{2r}+J)} > \frac{\hat{z}+\|ME\|\|f\|_{2r}}{r}$$

Therefore, choosing η to be any value between the left hand side and the right hand side of this inequality we see that condition 2 of the theorem 2 is satisfied.

We now shift our focus to singular Sturm-Liouville problems. We then compare our results to those appearing in [11], where closely-related problems are studied.

3. Singular Sturm-Liouville problems

3.1. Case I: $p(a) = p(b) = 0$

Let $a,b \in \mathbb{R}$ with $b > a$. In the following, let $\mu \in \mathbb{R}$, $p:[a,b] \to \mathbb{R}$ and $q:[a,b] \to \mathbb{R}$, $f:\mathbb{R} \to \mathbb{R}$ be continuous with $p(t) > 0$ on (a,b).

Consider the following nonlinear boundary value problems on (a,b):

$$(p(t)x'(t))' + q(t)x(t) + \mu x(t) = f(x(t)) \tag{3}$$

subject to the condition that the following limits exist and are finite

$$\begin{aligned} &\lim_{t\to a^+} x(t), \qquad \lim_{t\to b^-} x(t) \\ &\lim_{t\to a^+} x'(t) \qquad \lim_{t\to b^-} x'(t). \end{aligned} \tag{4}$$

The framework in this section we will address the case where $p(a) = 0 = p(b)$.

In this section, we denote $\mathscr{D}$ as the collection of twice continuously differentiable functions in X satisfying the boundary conditions above. It is well-known [1] from general Sturm-Liouville theory that the closely related linear problem

$$(p(t)x'(t))' + q(t)x(t) = \lambda x(t)$$

subject to the boundary conditions above has countably many simple eigenvalues $\{\lambda_k)\}_{k=0}^{\infty}$ with corresponding eigenfunctions $\{\psi_k\}_{k=0}^{\infty}$. Define the map $L:\mathscr{D}\to X$ by

$$[Lx](t)=(p(t)x'(t))'+q(t)x(t)+\mu x(t).$$

It is straightforward to show that if μ is not an eigenvalue of the linear problem above, L is a bijection from $\mathscr{D}$ onto X. If μ is an eigenvalue, or in other words if $\mu=\lambda_k$ for some nonnegative integer k then the kernel of L is nontrivial and spanned by ψ_k. It is straightforward to show that $h\in X$ is in the image of L if and only if

$$\int_{-1}^{1}h(t)\psi(t)dt=0.$$

We define $U:X\to X$ to be

$$[Ux](t)=\psi\int_{-1}^{1}x(t)\psi(t)dt$$

Note that U is a projection onto $\ker(L)$. Define E by $E=I-U$. Also note that the map L restricted to $\mathscr{D}\cap$ Im(L) is a bijection onto Im$(L)=$Im$(I-U)$. Therefore, it follows that there exists a linear map $M:$Im$(I-E)\to\mathscr{D}\cap$ Im(L) satisfying

$$LMh=h$$

for all $h\in$ Im(L) and

$$MLx=(I-E)x$$

for all $x\in\mathscr{D}$. Define $F:X\to X$ by

$$F(x)(t)=f(x(t)).$$

We note that solving

$$Lx=F(x)$$

is equivalent to solving the system

$$\begin{cases}(I-U)x-M(I-U)F(\alpha\psi+x)=0\\ \qquad\text{and}\\ \int_{-1}^{1}\psi(s)f(\alpha\psi(s)+x(s))ds=0.\end{cases}$$

We know that the map $M:Im(L)\to\mathscr{C}$ is compact by a lemma appearing in [11].

Like in the previous section, for $\eta\in[0,1]$ we denote $A_\eta=\{t\in[-1,1]:|\psi_k(t)|\geqslant\eta\}$ and for $T\subset\mathbb{R}$ we denote $m(T)$ as the Lebesgue measure of the set T. For the sake of notation in the following lemma, define $C=0$ if ψ has no zeroes on $[a,b]$.

Otherwise, let $\{s_1,\ldots,s_l\}$ be the set of roots for ψ and let C be the constant defined by

$$C = \left(\sum_{i=1}^{l} \frac{1}{|\psi'(s_i)|}\right).$$

For $l > 0$ we will write

$$\|f\|_l = \sup_{s\in[-l,l]} |f(s)|.$$

THEOREM 2. *Suppose the following hold*

1. *There exists constants $\hat{z} > 0$ and $J > 0$ such that if $|s| > \hat{z}$ then $sf(s) > 0$ and $|f(s)| \geqslant J$.*
2. *There exists $\eta \in (0,1]$ and $r > 0$ such that $\eta > \frac{\hat{z}+\|ME\|\|f\|_{2r}}{r}$ and $J(1-C\eta) > C\eta\|f\|_{2r}$*

Then there exists a solution to (3)–(4).

Proof. Define the operator $H : Im(L)\times\mathbb{R} \to Im(L)\times\mathbb{R}$ by

$$H(x,\alpha) = \begin{bmatrix} MEF(\alpha\psi + x) \\ \alpha - \int_{-1}^{1} \psi(t) f(\alpha\psi(t)+x(t))dt \end{bmatrix}$$

Note that H is compact as a consequence of the preceding lemma. We wish to show that $I-H$ has a nonzero Leray-Schauder degree on $\Omega_r = \{(x,\alpha)\in\mathscr{C}\times\mathbb{R} : \max\{\|x\|,|\alpha|\} \leqslant r\}$. Define the map $Q : [0,1]\times\bar{\Omega}_r \to Im(E)\times\mathbb{R}$ by

$$Q(s,(x,\alpha)) = \begin{bmatrix} Q_1(s,(x,\alpha)) \\ Q_2(s,(x,\alpha)) \end{bmatrix} = \begin{bmatrix} x - sMEF(\alpha\psi + x) \\ (1-s)\alpha + s\int_{-1}^{1} \psi(t) f(\alpha\psi(t)+x(t))dt \end{bmatrix}.$$

and note that Q is a homotopy between I and $I-H$. Let $(x,\alpha)\in\partial\Omega_r$ and $s\in(0,1)$. We will now show that $Q(s,(x,\alpha)) \neq 0$ for all $(s,(x,\alpha)) \in [0,1]\times\partial(\Omega_r)$. Let $(x,\alpha)\in\partial\Omega_r$ and $s\in[0,1]$. First suppose that $\|x\| = r$. Then since $|\alpha\psi(t)+x(t)| \leqslant 2r$ for all $t\in[-1,1]$ we have that

$$\|sMEF(\alpha\psi + x)\| \leqslant \|ME\|\|F(\alpha\psi + x)\| \leqslant \|ME\|\|f\|_{2r} < r = \|x\|.$$

Therefore $Q_1(s,(x,\alpha)) \neq 0$.

Now assume that $|\alpha| = r$ holds and suppose that $x = sMEF(\alpha\psi + x)$ for some $s\in(0,1)$. We will first examine the case where $\alpha = r$. Let $t\in A_\eta$ with $\psi(t) \geqslant \eta$. We have that $\alpha\psi(t) > 0$, $|\alpha\psi(t)| > \|ME\|\|f\|_{2r} \geqslant |x(t)|$ and

$$\begin{aligned} |\alpha\psi(t)+x(t)| &\geqslant r\eta - \|ME\|\|f\|_{2r} \\ &> \hat{z} \end{aligned}$$

This implies then that

$$\psi(t)f(\alpha\psi(t)+x(t)) > \psi(t)J = |\psi(t)|J$$

whenever $t \in A_\eta$ with $\psi(t) \geqslant \eta$.

The same argument shows that for $t \in A_\eta$ with $\psi(t) \leqslant -\eta$,

$$\alpha\psi(t)+x(t) < -\hat{z}$$

and thus

$$\psi(t)f(\alpha\psi(t)+x(t)) > \psi(t)(-J) = |\psi(t)|J.$$

If there exists $\kappa > 0$ such that $|\psi(t)| \geqslant \kappa$ for all $t \in [a,b]$ then

$$\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt \geqslant J\kappa(b-a) > 0.$$

If not, then condition 2 implies that there exists $\alpha_0 > 0$ such that

$$\frac{(1-(C\eta+\alpha_0))}{C\eta+\alpha_0} > \frac{\|f\|_{2r}}{J}$$

For $1 \leqslant i \leqslant k$ and s close to s_i we have that $|\psi(s)| \approx |\psi'(s_i)||(s-s_i)|$ and $|\psi'(s_i)|$ $|(s-s_i)| < \eta$ implies that

$$|s-s_i| < \frac{\eta}{|\psi'(s_i)|}.$$

Based on this, it is clear that there exists $\delta > 0$ such that if $|\eta| < \delta$ then

$$\left| m(A_\eta^c) - \sum_{i=1}^{k} \frac{2}{|\psi'(s_i)|}\eta \right| < \alpha_0$$

and therefore

$$m(A_\eta^c) < \sum_{i=1}^{k} \frac{2}{|\psi'(s_i)|}\eta + \alpha_0.$$

Thus we have that

$$\frac{m(A_\eta)}{m(A_\eta^c)} > \frac{1-(C\eta+\alpha_0)}{C\eta+\alpha_0}$$
$$> \frac{\|f\|_{2r}}{J}.$$

Now we have that

$$\begin{aligned}\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt &= \int_{A_\eta} \psi(t)f(\alpha\psi(t)+x(t))dt \\ &\quad + \int_{A_\eta^c} \psi(t)f(\alpha\psi(t)+x(t))dt \\ &\geqslant \int_{A_\eta} |\psi(t)|Jdt + \int_{A_\eta^c} \psi(t)f(\alpha\psi(t)+x(t))dt \\ &\geqslant \int_{A_\eta} |\psi(t)|Jdt - m(A_\eta^c)\eta\|f\|_{2r} \\ &\geqslant \eta(m(A_\eta)J - m(A_\eta^c)\|f\|_{2r}) \\ &> 0\end{aligned}$$

In the case where $\alpha = -r$, the argument above shows that

$$\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt < 0$$

Then we have in either case that

$$Q_2(s,(x,\alpha)) = (1-s)\alpha + s\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt \neq 0.$$

as α and $\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt$ have the same sign.

This establishes the desired result due to homotopy invariance of the Leray-Schauder degree. □

3.2. Case II: $p(a) > 0$ and $p(b) = 0$

Let $a,b \in \mathbb{R}$ with $b > a$. We again let X denote the space $(\mathscr{C}[a,b], \|\cdot\|)$ where $\|\cdot\|$ denotes the supremum norm. In the following, let $p : [a,b] \to \mathbb{R}$, $q : [a,b] \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ be continuous with $p(t) > 0$ on $[a,b)$ and $p(b) = 0$. We use μ and ε to denote real-valued parameters, and suppose that $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$. Let $G : X \to \mathbb{R}$ be a nonlinear map that maps bounded sets of X into bounded subsets of $\mathbb{R}$ and let ε be a real-valued parameter. We denote $\mathscr{D}$ as the collection of twice continuously differentiable functions in X. Consider the following sets of nonlinear boundary value problems on (a,b):

$$(p(t)x'(t))' + q(t)x(t) + \mu x(t) = f(x(t)) \tag{5}$$

subject to

$$x(a) = \varepsilon G(x). \tag{6}$$

It is well-known [1] from general Sturm-Liouville theory that the closely related linear problem

$$(p(t)x'(t))' + q(t)x(t) = \lambda x(t)$$

subject to

$$x(a) = 0$$

has countably many simple eigenvalues $\{\lambda_k)\}_{k=0}^{\infty}$ with corresponding eigenfunctions $\{\psi_k\}_{k=0}^{\infty}$. Define the map $L : \mathscr{D} \to X$ by

$$[Lx](t) = (p(t)x'(t))' + q(t)x(t) + \mu x(t).$$

and $\mathscr{L} : \mathscr{D} \to X \times \mathbb{R}$ by

$$[\mathscr{L}x](t) = \begin{bmatrix} Lx \\ x(a) \end{bmatrix}$$

It is straightforward to show that if μ is not an eigenvalue of the linear problem above, $\mathscr{L}$ is a bijection from $\mathscr{D}$ onto X. If μ is an eigenvalue, or in other words if $\mu = \lambda_k$ for some nonnegative integer k then the kernel of L is nontrivial and spanned by ψ_k. Let $\{\psi, \phi\}$ be a basis for the solution space to $Lx = 0$ such that $\|\psi\| = \|\phi\| = 1$. Let $v = \phi(a)$. Then It is straightforward to show that $[h, w] \in X \times \mathbb{R}$ is in the image of L if and only if

$$\int_{-1}^{1} h(t)\psi(t)dt + \frac{w}{v} = 0.$$

We define $U : X \to X$ by

$$[Ux](t) = \psi \int_a^b x(t)\psi(t)dt.$$

Define $E : X \times \mathbb{R} \to X \times \mathbb{R}$ by

$$E(h,w) = m\left(\int_a^b h(t)\psi(t)dt + \frac{w}{v}\right)(h,w)$$

where $m = \frac{1}{1+v}$. Also note that the map L restricted to $\mathscr{D} \cap \operatorname{Im}(L)$ is a bijection onto $\operatorname{Im}(L) = \operatorname{Im}(I - U)$. Therefore, it follows that there exists a linear map $M : \operatorname{Im}(I - E) \to \mathscr{D} \cap \operatorname{Im}(L)$ satisfying

$$LMh = h$$

for all $h \in \operatorname{Im}(L)$ and

$$MLx = (I - E)x$$

for all $x \in \mathscr{D}$. Note that M is a compact operator from $\operatorname{Im}(I - E)$ into $\mathscr{D}$ by an argument very similar to the one appearing in the previous section. Define $F : X \to X$ by

$$F(x)(t) = f(x(t)).$$

and $\mathscr{F} : X \to X \times \mathbb{R}$ by

$$[\mathscr{F}x](t) = \begin{bmatrix} F(x) \\ \varepsilon G(x) \end{bmatrix}.$$

We note that solving

$$\mathscr{L}x = \mathscr{F}(x)$$

is equivalent to solving the system

$$\begin{cases} (I-U)x - ME\mathscr{F}(\alpha\psi + x) = 0 \\ \qquad \text{and} \\ \int_{-1}^{1} \psi(s) f(\alpha\psi(s) + x(s))ds + \frac{G(\alpha\psi+x)}{v} = 0. \end{cases}$$

Like we did in the previous section, for $\eta \in [0,1]$ we denote $A_\eta = \{t \in [a,b] : |\psi_k(t)| \geqslant \eta\}$ and for $T \subset \mathbb{R}$ we denote $m(T)$ as the Lebesgue measure of the set T. For the sake of notation in the following lemma, define $C = 0$ if ψ has no zeroes on $[a,b]$. Otherwise, let $\{s_1, \dots, s_l\}$ be the set of roots for ψ and let C be the constant defined by

$$C = \left(\sum_{i=1}^{l} \frac{1}{|\psi'(s_i)|} \right).$$

For $l > 0$ we will write

$$\|f\|_l = \sup_{s \in [-l,l]} |f(s)|.$$

THEOREM 3. *Suppose the following hold*

1. *There exists constants $\hat{z} > 0$ and $J > 0$ such that if $|s| > \hat{z}$ then $sf(s) > 0$ and $|f(s)| \geqslant J$.*
2. *There exists $\eta \in (0,1]$ and $r > 0$ such that $\eta > \frac{\hat{z} + \|ME\| \|f\|_{2r}}{r}$ and $J(1 - C\eta) > C\eta \|f\|_{2r}$.*

Then there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| \leqslant \varepsilon_0$ there exists a solution to (5)–(6).

Proof. Define the operator $H : Im(L) \times \mathbb{R} \to Im(L) \times \mathbb{R}$ by

$$H(x,\alpha) = \begin{bmatrix} ME\mathscr{F}(\alpha\psi + x) \\ \alpha - \left[\int_a^b \psi(t) f(\alpha\psi(t) + x(t))dt + \frac{\varepsilon G(\alpha\psi+x)}{v} \right] \end{bmatrix}$$

Note that H is compact as a consequence of the preceding lemma. We wish to show that $I - H$ has a nonzero Leray-Schauder degree on $\Omega_r = \{(x,\alpha) \in \mathscr{C} \times \mathbb{R} : \max\{\|x\|, |\alpha|\} \leqslant$

$r\}$. Define the map $Q : [0,1] \times \bar{\Omega}_r \to Im(E) \times \mathbb{R}$ by

$$Q(s,(x,\alpha)) = \begin{bmatrix} Q_1(s,(x,\alpha)) \\ Q_2(s,(x,\alpha)) \end{bmatrix} = \begin{bmatrix} x - sME\mathscr{F}(\alpha\psi + x) \\ (1-s)\alpha + s\left[\int_a^b \psi(t) f(\alpha\psi(t) + x(t))dt + \frac{\varepsilon G(\alpha\psi + x)}{v}\right] \end{bmatrix}.$$

and note that Q is a homotopy between I and $I - H$. Let $(x,\alpha) \in \partial\Omega_r$ and $s \in (0,1)$. We will now show that $Q(s,(x,\alpha)) \neq 0$ for all $(s,(x,\alpha)) \in [0,1] \times \partial(\Omega_r)$. Let $(x,\alpha) \in \partial\Omega_r$ and $s \in [0,1]$. First suppose that $\|x\| = r$. Then since $|\alpha\psi(t) + x(t)| \leqslant 2r$ for all $t \in [a,b]$ we have that

$$\|sME\mathscr{F}(\alpha\psi + x)\| \leqslant \|ME\|\|\mathscr{F}(\alpha\psi + x)\| \leqslant \|ME\|\|f\|_{2r} < r = \|x\|.$$

Therefore $Q_1(s,(x,\alpha)) \neq 0$.

Now assume that $|\alpha| = r$ holds and suppose that $x = sME\mathscr{F}(\alpha\psi + x)$ for some $s \in (0,1)$. We will first examine the case where $\alpha = r$. Let $t \in A_\eta$ with $\psi(t) \geqslant \eta$. We have that $\alpha\psi(t) > 0$, $|\alpha\psi(t)| > \|ME\|\|f\|_{2r} \geqslant |x(t)|$ and

$$\begin{aligned} |\alpha\psi(t) + x(t)| &\geqslant r\eta - \|ME\|\|f\|_{2r} \\ &> \hat{z} \end{aligned}$$

This implies then that

$$\psi(t) f(\alpha\psi(t) + x(t)) > \psi(t)J = |\psi(t)|J$$

whenever $t \in A_\eta$ with $\psi(t) \geqslant \eta$.

The same argument shows that for $t \in A_\eta$ with $\psi(t) \leqslant -\eta$,

$$\alpha\psi(t) + x(t) < -\hat{z}$$

and thus

$$\psi(t) f(\alpha\psi(t) + x(t)) > \psi(t)(-J) = |\psi(t)|J.$$

If there exists $\kappa > 0$ such that $|\psi(t)| \geqslant \kappa$ for all $t \in [a,b]$ then

$$\int_a^b \psi(t) f(\alpha\psi(t) + x(t))dt \geqslant J\kappa(b-a) > 0.$$

If not, then condition 2 implies that there exists $\alpha_0 > 0$ such that

$$\frac{(1 - (C\eta + \alpha_0))}{C\eta + \alpha_0} > \frac{\|f\|_{2r}}{J}$$

For $1 \leqslant i \leqslant k$ and s close to s_i we have that $|\psi(s)| \approx |\psi'(s_i)||(s - s_i)|$ and $|\psi'(s_i)||(s - s_i)| < \eta$ implies that

$$|s - s_i| < \frac{\eta}{|\psi'(s_i)|}.$$

Based on this, it is clear that there exists $\delta > 0$ such that if $|\eta| < \delta$ then

$$\left| m(A_\eta^c) - \sum_{i=1}^{k} \frac{2}{|\psi'(s_i)|}\eta \right| < \alpha_0$$

and therefore

$$m(A_\eta^c) < \sum_{i=1}^{k} \frac{2}{|\psi'(s_i)|}\eta + \alpha_0.$$

Thus we have that

$$\begin{aligned} \frac{m(A_\eta)}{m(A_\eta^c)} &> \frac{1-(C\eta+\alpha_0)}{C\eta+\alpha_0} \\ &> \frac{\|f\|_{2r}}{J}. \end{aligned}$$

Let $M = \sup_{\|x\| \leqslant 2r} |G(x)|$ and $\varepsilon_0 = M^{-1}|v|\eta((1-c\eta)-c\eta\|f\|_{2r})$.

Now we have that if $|\varepsilon| < \varepsilon_0$ that

$$\begin{aligned} &\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt + \frac{\varepsilon G(\alpha\psi+x)}{v} \\ &= \int_{A_\eta} \psi(t)f(\alpha\psi(t)+x(t))dt \\ &\quad + \int_{A_\eta^c} \psi(t)f(\alpha\psi(t)+x(t))dt + \frac{\varepsilon G(\alpha\psi+x)}{v} \\ &\geqslant \int_{A_\eta} |\psi(t)|Jdt + \int_{A_\eta^c} \psi(t)f(\alpha\psi(t)+x(t))dt + \frac{\varepsilon G(\alpha\psi+x)}{v} \\ &\geqslant \int_{A_\eta} |\psi(t)|Jdt - m(A_\eta^c)\eta\|f\|_{2r} + \frac{\varepsilon G(\alpha\psi+x)}{v} \\ &\geqslant \eta(m(A_\eta)J - m(A_\eta^c)\|f\|_{2r}) + \frac{\varepsilon G(\alpha\psi+x)}{v} \\ &> \eta((1-c\eta)-c\eta\|f\|_{2r}) - \eta((1-c\eta)-c\eta\|f\|_{2r}) \\ &= 0. \end{aligned}$$

In the case where $\alpha = -r$, an argument similar to the one above shows that

$$\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt < 0$$

Along with assumption 3, we have in either case that

$$Q_2(s,(x,\alpha)) = (1-s)\alpha + s\left[\int_a^b \psi(t)f(\alpha\psi(t)+x(t))dt + \frac{\varepsilon G(\alpha\psi+x)}{v}\right] \neq 0.$$

This establishes the desired result due to homotopy invariance of the Leray-Schauder degree. □

The following examples concern nonlinear Legendre boundary value problems, which were analyzed in [11] in cases where nonlinearities were bounded.

EXAMPLE 1. Consider the following nonlinearly perturbed Legendre boundary value problems on $(-1,1)$

$$[(1-t^2)x'(t)]' + \mu x(t) = f(x(t))$$

subject to the condition that the following limits exist and are finite

$$\lim_{t\to -1^+} x(t), \qquad \lim_{t\to 1^-} x(t)$$

$$\lim_{t\to -1^+} x'(t) \qquad \lim_{t\to 1^-} x'(t).$$

If $\mu = k(k+1)$ for some nonnegative integer k, the only solutions to

$$[(1-t^2)x'(t)]' + \mu x(t) = 0$$

for all $t \in (-1,1)$ where

$$\lim_{t\to -1^+} x(t), \qquad \lim_{t\to 1^-} x(t)$$

$$\lim_{t\to -1^+} x'(t), \qquad \lim_{t\to 1^-} x'(t).$$

all exist and are finite are constant multiples of the k^{th} Legendre polynomial which we will denote P_k. Since $\|P_k\| = 1$ for any $k \geqslant 0$ then using the general framework above we can choose $\psi = P_k$ and the constant C is straightforward to compute. The constant Legendre polynomial $P_0(t) = 1$ satisfies the first alternative in condition 2 of theorem 1 with $\kappa = 1$. The value of C for different values of k are given below.

- If $k = 1$, $C = 1$
- If $k = 2$, $C = 2\sqrt{3}$
- If $k = 3$, $C = \frac{15}{2}$

EXAMPLE 2. Consider the following nonlinearly perturbed boundary value problem on $(0,1)$

$$[(1-t^2)x'(t)]' + 12x(t) = f(x(t))$$

subject to the condition that the following limits exist and are finite

$$\lim_{t\to 1^-} x(t), \qquad \lim_{t\to 1^-} x'(t)$$

and $x(0) = 0$. The solution space to

$$[(1-t^2)x'(t)]' + \mu x(t) = 0$$

for all $t \in (-1, 1)$ where

$$\lim_{t\to 1^-} x(t), \qquad \lim_{t\to 1^-} x'(t)$$

exist and are finite and $x(0) = 0$ consists of all multiples of the 3^{rd} degree Legendre polynomial that which we will denote P_3. Further details can be found in [10]. Based on the constants computed in the previous example, the second part of condition 2 applied to this case requires $\eta \in [0, 1]$ such that

$$\frac{2-15\eta}{15\eta} > \frac{\|f\|_{2r}}{J}$$

and

$$\eta > \frac{\hat{z} + \|ME\|\|f\|_{2r}}{r}.$$

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