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COEFFICIENT INVERSE PROBLEM FOR THE DEGENERATE PARABOLIC EQUATION

NADIIA HUZYK

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Abstract. The inverse problem for the degenerate parabolic equation is considered. The minor coefficient of the equation is a polynomial of the first power with respect to the space variable with unknown time-dependent coefficients. The conditions of local in time existence and global uniqueness of the classical solution to this problem are established. The case of weak power degeneration is investigated.

1. Introduction

It is known that the inverse problems arise when under given consequences we have to find the reasons causing them. The theory of inverse problems has been extensively developed within the past decade due partly to its applications. These problems arise in geophysics, acoustics, electrodynamics, tomography, medicine, ecology, financial mathematics etc [1]–[3]. For today the coefficient inverse problems for the parabolic equation are well studied (see [4]–[14] and bibliography in them). These papers contain both the inverse problems of determination the time-dependent coefficients at the higher order derivative in the parabolic equations and the minor coefficients in them under different boundary and overdetermination conditions.

Degenerate parabolic problems arise in a lot of fields of nature and sciences. Applications of such problems include the mathematical model of the flow in a porous media, propagation of the thermal waves in plasma, population dynamics, financial mathematics and others [15]–[17]. The inverse problems for the degenerate parabolic equation are investigated quite poorly. The conditions of existence and uniqueness of the classical solution to the inverse problems of determination of the time-dependent coefficient $a = a(t)$, $a(t) > 0$, $t \in [0,T]$ in the one-dimensional degenerate parabolic equation

$$u_t = a(t)t^{\beta}u_{xx} + b(x,t)u_x + c(x,t)u + f(x,t)$$

are found for the domains with fixed boundary [18]–[19] and free boundary domains [20]–[22]. Both cases of weak $(0 < \beta < 1)$ and strong $(\beta \geqslant 1)$ degeneration are investigated. The solvability of analogous inverse problems for the multidimensional degenerate parabolic equations is studied in [23].

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The inverse problems of identification of the time-dependent minor coefficient $b = b(t)$ in the degenerate parabolic equation

$$u_t = a(t)t^{\beta}u_{xx} + b(t)u_x + c(x,t)u + f(x,t)$$

are studied in [24]–[25] for both cases of weak and strong degenerations.

The mentioned papers are united by the fact that the time-dependent coefficient is unknown. The coefficient inverse problems for the parabolic equations with degeneration with respect to space variable are studied in [26]–[27] (see also the bibliography in them).

The inverse problems of determining the coefficients from both spatial and time variables remain unexplored. In the present paper we consider an inverse problem for the one-dimensional degenerate parabolic equation with Dirichlet boundary conditions and the values of the heat moments as the overdetermination conditions. It is known that the leading coefficient of the equation is the product of the power function which caused degeneration and a known function of time. The minor coefficient is the polynomial of the first power with respect to the space variable with two unknown time-dependent coefficients. The purpose of this article is to establish the conditions of existence and uniqueness of the classical solution to this problem in the case of weak degeneration.

2. Statement of the problem

In a domain $Q_T = \{(x,t) : 0 < x < h, 0 < t < T\}$ we consider an inverse problem for simultaneous determination of the time dependent coefficients $b_1 = b_1(t)$, $b_2 = b_2(t)$ in the one-dimensional degenerate parabolic equation

$$u_t = t^{\beta}a(t)u_{xx} + (b_1(t)x + b_2(t))u_x + c(x,t)u + f(x,t) \tag{1}$$

with initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,h], \tag{2}$$

boundary conditions

$$u(0,t) = \mu_1(t),\ u(h,t) = \mu_2(t), \quad t \in [0,T] \tag{3}$$

and overdetermination conditions

$$\int_0^h u(x,t)dx = \mu_3(t), \quad t \in [0,T]. \tag{4}$$

$$\int_0^h xu(x,t)dx = \mu_4(t), \quad t \in [0,T]. \tag{5}$$

DEFINITION 1. A triplet of functions $(b_1, b_2, u) \in (C[0,T])^2 \times C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q}_T)$, is called a solution to the problem (1)–(5) if it verifies the equation (1) and conditions (2)–(5).

We will investigate the case of weak power degeneration, when $0<\beta<1$. Applying the Schauder fixed point theorem there is established conditions of existence of the solution to the named problem. The proof of the uniqueness is based on the properties of the solutions of the homogeneous integral equations of the second kind with integrable kernels.

3. Existence of the solution

THEOREM 1. *Suppose that the following conditions hold:*

A1) $\varphi\in C^2[0,h]$, $\mu_i\in C^1[0,T]$, $i\in\{\overline{1,4}\}$, $a\in C[0,T]$, $a(t)>0$, $t\in[0,T]$, $c,f\in C(\overline{Q}_T)$ *and satisfy the Hölder condition with respect to* x *uniformly to* t *with indicator* α, $0<\alpha<1$*;*

A2) $(h\mu_2(t)-\mu_3(t))^2-(\mu_2(t)-\mu_1(t))(h^2\mu_2(t)-2\mu_4(t))\neq 0$, $t\in[0,T]$;

A3) $\mu_1(0)=\varphi(0)$, $\mu_2(0)=\varphi(h)$, $\int\limits_0^h\varphi(x)dx=\mu_3(0)$, $\int\limits_0^h x\varphi(x)dx=\mu_4(0)$.

Then the problem (1)–(5) *has a solution* (b_1,b_2,u) *for* $x\in[0,h]$ *and* $t\in[0,T_0]$, *where the number* T_0, $0<T_0\leqslant T$ *is defined by the problem data.*

Proof. First of all we reduce the problem (1)–(5) to the equivalent system of equations. For this aim we make the substitution in the problem (1)–(3)

$$u(x,t)=\widetilde{u}(x,t)+\varphi(x)-\varphi(0)+\mu_1(t)+\frac{x}{h}\Big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\Big). \tag{6}$$

As a result we obtain the equation with respect to the function $\widetilde{u}=\widetilde{u}(x,t)$

$$\begin{aligned}\widetilde{u}_t=&a(t)t^\beta\widetilde{u}_{xx}+(b_1(t)x+b_2(t))\widetilde{u}_x+c(x,t)\widetilde{u}+f(x,t)-\mu_1'(t)-\frac{x}{h}(\mu_2'(t)-\mu_1'(t))\\&+a(t)t^\beta\varphi''(x)+(b_1(t)x+b_2(t))\Big(\varphi'(x)+\frac{1}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0))\Big)\\&+c(x,t)\Big(\varphi(x)-\varphi(0)+\mu_1(t)+\frac{x}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0))\Big)\end{aligned} \tag{7}$$

with homogeneous initial and boundary conditions

$$\widetilde{u}(x,0)=0,\quad x\in[0,h], \tag{8}$$

$$\widetilde{u}(0,t)=\widetilde{u}(h,t)=0,\quad t\in[0,T]. \tag{9}$$

The problem (7)–(9) is equivalent to the integro-differential equation

$$\begin{aligned}\widetilde{u}(x,t)=\int\limits_0^t\int\limits_0^h G_1(x,t,\xi,\tau)\Big(&(b_1(\tau)\xi+b_2(\tau))\widetilde{u}_\xi(\xi,\tau)+c(\xi,\tau)\widetilde{u}(\xi,\tau)\\&+f(\xi,\tau)-\mu_1'(\tau)+a(\tau)\tau^\beta\varphi''(\xi)-\frac{\xi}{h}(\mu_2'(\tau)-\mu_1'(\tau))\end{aligned}$$

$$+\,(b_1(\tau)\xi+b_2(\tau))\big(\varphi'(\xi)+\frac{1}{h}(\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0))\big)$$
$$+c(\xi,\tau)(\varphi(\xi)-\varphi(0)+\mu_1(\tau)+\frac{\xi}{h}(\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)))\Big)\,d\xi\,d\tau, \tag{10}$$

where $G_1=G_1(x,t,\xi,\tau)$ denotes the Green function for the first boundary value problem for the heat equation

$$u_t=a(t)t^{\beta}u_{xx}.$$

It is known [28] that the Green functions for the first $(k=1)$ or the second $(k=2)$ boundary value problem can be written in the form

$$G_k(x,t,\xi,\tau)=\frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}}\sum_{n=-\infty}^{+\infty}\Bigg(\exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\right)$$
$$+(-1)^k\exp\left(-\frac{(x+\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\right)\Bigg),\ k=1,2,$$

where $\theta(t)=\int\limits_0^t a(\tau)\tau^{\beta}d\tau$. These functions possess the properties

$$\int\limits_0^h G_1(x,t,\xi,\tau)d\xi d\tau\leqslant 1,\ \int\limits_0^h|G_{1x}(x,t,\xi,\tau)|\,d\xi\leqslant\frac{C_1}{\sqrt{\theta(t)-\theta(\tau)}}. \tag{11}$$

Put $v(x,t)\equiv u_x(x,t)$. Using (6), (10), the direct problem (1)–(3) we reduce to the system of equations with respect to functions $u=u(x,t)$, $v=v(x,t)$:

$$u(x,t)=\int\limits_0^t\int\limits_0^h G_1(x,t,\xi,\tau)\Big((b_1(\tau)\xi+b_2(\tau))v(\xi,\tau)+c(\xi,\tau)u(\xi,\tau)-\mu_1'(\tau)$$
$$-\frac{\xi}{h}(\mu_2'(\tau)-\mu_1'(\tau))+a(\tau)\tau^{\beta}\varphi''(\xi)+f(\xi,\tau)\Big)\,d\xi\,d\tau+\varphi(x)-\varphi(0)$$
$$+\mu_1(t)+\frac{x}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)),\ (x,t)\in\overline{Q}_T, \tag{12}$$

$$v(x,t)=\int\limits_0^t\int\limits_0^h G_{1x}(x,t,\xi,\tau)\Big((b_1(\tau)\xi+b_2(\tau))v(\xi,\tau)+c(\xi,\tau)u(\xi,\tau)-\mu_1'(\tau)$$
$$-\frac{\xi}{h}(\mu_2'(\tau)-\mu_1'(\tau))+a(\tau)\tau^{\beta}\varphi''(\xi)+f(\xi,\tau)\Big)\,d\xi\,d\tau+\varphi'(x)$$
$$+\frac{1}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)),\ (x,t)\in\overline{Q}_T. \tag{13}$$

Note, that we obtain the equation (13) differentiating (12) with respect to the space variable. Taking into account the estimates (11) we establish the behavior of the integrals in the right-hand sides of the formulas (12), (13):

$$I_1 \equiv \left| \int_0^t \int_0^h G_1(x,t,\xi,\tau)\Big((b_1(\tau)\xi + b_2(\tau))v(\xi,\tau) + c(\xi,\tau)u(\xi,\tau) - \mu_1'(\tau) - \frac{\xi}{h}(\mu_2'(\tau) - \mu_1'(\tau)) + a(\tau)\tau^\beta \varphi''(\xi) + f(\xi,\tau)\Big)\, d\xi\, d\tau \right| \leqslant C_2 t,$$

$$I_2 \equiv \left| \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau)\Big((b_1(\tau)\xi + b_2(\tau))v(\xi,\tau) + c(\xi,\tau)u(\xi,\tau) - \mu_1'(\tau) - \frac{\xi}{h}(\mu_2'(\tau) - \mu_1'(\tau)) + a(\tau)\tau^\beta \varphi''(\xi) + f(\xi,\tau)\Big)\, d\xi\, d\tau \right|$$
$$\leqslant C_3 \int_0^t \frac{d\tau}{\sqrt{\theta(t)-\theta(\tau)}} \leqslant C_4 \int_0^t \frac{d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}} \leqslant C_5 t^{\frac{1-\beta}{2}}.$$

Using the definition of the weak degeneration $(0<\beta<1)$ we conclude that the integrals in the right hand sides of the formulas (12), (13) tend to zero when $t \to 0$.

To obtain the equations for functions $b_1 = b_1(t)$, $b_2 = b_2(t)$ we multiply the equation (1) by x^k, $k = 0,1$ and integrate with respect to x from 0 to h:

$$b_1(t) = \triangle^{-1}\Bigg(\Big(\mu_3'(t) - a(t)t^\beta (v(h,t) - v(0,t)) - \int_0^h (c(x,t)u(x,t) + f(x,t))dx\Big) \times \big(h\mu_2(t) - \mu_3(t)\big) - \Big(\mu_4'(t) - a(t)t^\beta (hv(h,t) - \mu_2(t) + \mu_1(t)) - \int_0^h x(c(x,t)u(x,t) + f(x,t))dx\Big)\big(\mu_2(t) - \mu_1(t)\big)\Bigg),\ t \in [0,T], \tag{14}$$

$$b_2(t) = \triangle^{-1}\Bigg(\Big(\mu_4'(t) - a(t)t^\beta (hv(h,t) - \mu_2(t) + \mu_1(t)) - \int_0^h x(c(x,t)u(x,t) + f(x,t))dx\Big)\big(h\mu_2(t) - \mu_3(t)\big) - \Big(\mu_3'(t) - a(t)t^\beta (v(h,t) - v(0,t)) - \int_0^h (c(x,t)u(x,t) + f(x,t))dx\Big)\big(h^2\mu_2(t) - 2\mu_4(t)\big)\Bigg),\ t \in [0,T], \tag{15}$$

where the expression

$$\triangle = (h\mu_2(t) - \mu_3(t))^2 - (\mu_2(t) - \mu_1(t))(h^2\mu_2(t) - 2\mu_4(t)) \tag{16}$$

is different from zero according to the condition $(A2)$ of the Theorem 1.

Thus, the inverse problem (1)–(5) we reduce to the equivalent system of equations (12)–(15). The equivalence we understand in the following sense: if a triplet of functions (b_1, b_2, u) is a solution to the problem (1)–(5), then (u, v, b_1, b_2) is the continuous solution to the system (12)–(15), and, contrary, if $(u, v, b_1, b_2) \in (C(\overline{Q}_T))^2 \times (C[0,T])^2$ is a solution to the system of equations (12)–(15), then (b_1, b_2, u) is the solution to (1)–(5).

The first part of this statement follows from the way of obtaining of the system of equations (12)–(15). Let us show that the contrary statement is true. Assume that (u, v, b_1, b_2) is the continuous solution to the system of equation (12)–(15). The condition $(A1)$ of the Theorem 1 allows us to differentiate the equation (12) with respect to x. Using the uniqueness properties of the solutions to the system of Voltera integral equations of the second kind it is easy to see that $v(x,t) \equiv u_x(x,t)$. This means that $u \in C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q}_T)$ is a solution to the problem (1)–(3). Then multiplying the equation (14) by $h\mu_2(t) - \mu_3(t)$, the equation (15) by $\mu_2(t) - \mu_1(t)$ and adding them we obtain

$$\begin{aligned} &b_1(t)(h\mu_2(t) - \mu_3(t)) + b_2(t)(\mu_2(t) - \mu_1(t)) \\ &= \mu_3'(t) - a(t)t^\beta(u_x(h,t) - u_x(0,t)) - \int\limits_0^h (c(x,t)u(x,t) + f(x,t))dx. \end{aligned}$$

Using (1)–(3) we rewrite this equality in the form

$$b_1(t)\left(\int\limits_0^h u(x,t)dx - \mu_3(t)\right) = -\left(\int\limits_0^h u_t(x,t)dx - \mu_3'(t)\right).$$

Now, taking into account the compatibility conditions, we can state that the condition (4) holds true.

Similarly, multiplying the equation (14) by $h^2\mu_2(t) - 2\mu_4(t)$, the equation (15) – by $h\mu_2(t) - \mu_3(t)$ and adding them we conclude

$$\begin{aligned} &b_1(t)(h^2\mu_2(t) - 2\mu_4(t)) + b_2(t)(h\mu_2(t) - \mu_3(t)) \\ &= \mu_4'(t) - a(t)t^\beta(hu_x(h,t) - \mu_2(t) + \mu_1(t)) - \int\limits_0^h x(c(x,t)u(x,t) + f(x,t))dx. \end{aligned}$$

Then (1)–(3) and compatibility condition yield (5). It means that the equivalence of the inverse problem (1)–(5) and the system of equations (12)–(15) is proved.

Now we start studying the system of equations (12)–(15). We consider this system as an operator equation

$$\omega = P\omega, \tag{17}$$

where $\omega = (u, v, b_1, b_2)$ and the operator $P = (P_1, P_2, P_3, P_4)$ is defined by the right-hand sides of these equations.

Assume that $|u(x,t)| \leqslant M_1$, $|v(x,t)| \leqslant M_2$, where M_1, M_2 are some positive constants. We will define them below. Using these estimates in (14), (15) we find

$$|b_1(t)| < \frac{C_6(1+M_1+M_2)}{\min\limits_{t\in[0,T]} \triangle} \equiv M_3,\ t\in[0,T], \tag{18}$$

$$|b_2(t)| < \frac{C_7(1+M_1+M_2)}{\min\limits_{t\in[0,T]} \triangle} \equiv M_4,\ t\in[0,T], \tag{19}$$

where the constants C_6, C_7 depend on the problem data.

Then from the (12), (13) we deduce

$$\begin{aligned} |P_1\omega| \leqslant & \int_0^t \int_0^h \left|G_1(x,t,\xi,\tau)\right| \Big((M_3\xi+M_4)M_2 + |c(\xi,\tau)|M_1 \\ & + \left| -\mu_1'(\tau) - \frac{\xi}{h}(\mu_2'(\tau)-\mu_1'(\tau)) + a(\tau)\tau^\beta \varphi''(\xi) + f(\xi,\tau) \right| \Big) d\xi\, d\tau \\ & + \left| \varphi(x) - \varphi(0) + \mu_1(t) + \frac{x}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)) \right| \\ & < C_8 t + \max_{(x,t)\in\overline{Q}_T} \left| \varphi(x) - \varphi(0) + \mu_1(t) + \frac{x}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)) \right|, \end{aligned} \tag{20}$$

$$\begin{aligned} |P_2\omega| \leqslant & \int_0^t \int_0^h \left|G_{1x}(x,t,\xi,\tau)\right| \Big((M_3\xi+M_4)M_2 + |c(\xi,\tau)|M_1 \\ & + \left| -\mu_1'(\tau) - \frac{\xi}{h}(\mu_2'(\tau)-\mu_1'(\tau)) + a(\tau)\tau^\beta \varphi''(\xi) + f(\xi,\tau) \right| \Big) d\xi\, d\tau \\ & + \left| \varphi'(x) + \frac{1}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)) \right| \\ & < C_9 t^{\frac{\beta-1}{2}} + \max_{(x,t)\in\overline{Q}_T} \left| \varphi'(x) + \frac{1}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)) \right|. \end{aligned} \tag{21}$$

Let's choose the constants M_1, M_2 such that

$$M_1 > \max_{(x,t)\in\overline{Q}_T} \left| \varphi(x) - \varphi(0) + \mu_1(t) + \frac{x}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)) \right|,$$

$$M_2 > \max_{(x,t)\in\overline{Q}_T} \left| \varphi'(x) + \frac{1}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)) \right|.$$

Then we fix such number T_0, $0 < T_0 \leqslant T$ that

$$C_8 T_0 + \max_{(x,t)\in\overline{Q}_T} \left| \varphi(x) - \varphi(0) + \mu_1(t) + \frac{x}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)) \right| \leqslant M_1, \tag{22}$$

$$C_9 T_0^{\frac{\beta-1}{2}} + \max_{(x,t)\in\overline{Q}_T} \left| \varphi'(x) + \frac{1}{h}(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)) \right| \leqslant M_2. \tag{23}$$

We consider an operator equation (17) on the closed and convex set $\mathscr{N} \equiv \{(u,v,b_1, b_2) \in (C(\overline{Q}_{T_0}))^2 \times (C[0,T_0])^2 : |u(x,t)| \leqslant M_1,\ |v(x,t)| \leqslant M_2,\ |b_1(t)| \leqslant M_3,\ |b_2(t)| \leqslant M_4\}$ in a Banach space $\mathscr{B} \equiv (C(\overline{Q}_{T_0}))^2 \times (C[0,T_0])^2$. The estimates (20)–(23) guarantee that P maps $\mathscr{N}$ into $\mathscr{N}$. The compactness of operator P can be established as in non degenerate case [28]. Now the Schauder fixed-point theorem yields the existence of the continuous solution to the system (12)–(15) on $[0,h] \times [0,T_0]$. It means that exists the solution (b_1,b_2,u) to the inverse problem (1)–(5) on $[0,h] \times [0,T_0]$. The proof of the Theorem 1 is completed. □

4. Uniqueness of the solution

THEOREM 2. *Assume that the condition* $(A2)$ *of the Theorem* 1 *holds. Then the solution to the problem* (1)–(5) *is unique.*

Proof. Suppose that the problem (1)–(5) has two solutions (b_{1i},b_{2i},u_i), $i=1,2$. Denote $b_1(t)=b_{11}(t)-b_{12}(t)$, $b_2(t)=b_{21}(t)-b_{22}(t)$, $u(x,t)=u_1(x,t)-u_2(x,t)$. We obtain from (1)–(5) the equation

$$u_t = t^\beta a(t)u_{xx} + (b_{11}(t)x + b_{21}(t))u_x + c(x,t)u + (b_1(t)x + b_2(t))u_{2x} \tag{24}$$

with homogeneous initial, boundary and overdetermination conditions:

$$u(x,0)=0, \quad x \in [0,h], \tag{25}$$

$$u(0,t)=0, \quad u(h,t)=0, \quad t \in [0,T], \tag{26}$$

$$\int_0^h u(x,t)dx = 0, \quad t \in [0,T]. \tag{27}$$

$$\int_0^h xu(x,t)dx = 0, \quad t \in [0,T]. \tag{28}$$

With the aid of Green function $G^*(x,t,\xi,\tau)$ of the first value-boundary problem for the equation

$$u_t = t^\beta a(t)u_{xx} + (b_{11}(t)x + b_{21}(t))u_x + c(x,t)u$$

we represent the solution to the problem (24)–(26) in the form

$$u(x,t) = \int_0^t \int_0^h G^*(x,t,\xi,\tau)(b_1(\tau)\xi + b_2(\tau))u_{2\xi}(\xi,\tau)d\xi d\tau,\ (x,t) \in \overline{Q}_T. \tag{29}$$

Differentiating (29) with respect to x, we find

$$u_x(x,t) = \int_0^t \int_0^h G^*_x(x,t,\xi,\tau)(b_1(\tau)\xi + b_2(\tau))u_{2\xi}(\xi,\tau)d\xi d\tau,\ (x,t) \in \overline{Q}_T. \tag{30}$$

Multiplying the equation (24) by $x^k, k=0,1$ and integrating with respect to x from 0 to h we deduce

$$b_1(t)=\triangle^{-1}\Big(\Big(a(t)t^{\beta}hu_x(h,t)+\int_0^h xc(x,t)u(x,t)dx\Big)\big(\mu_2(t)-\mu_1(t)\big)$$
$$-\Big(a(t)t^{\beta}(u_x(h,t)-u_x(0,t))+\int_0^h c(x,t)u(x,t)dx\Big)\big(h\mu_2(t)-\mu_3(t)\big)\Big), \tag{31}$$

$$b_2(t)=\triangle^{-1}\Big(\Big(a(t)t^{\beta}hu_x(h,t)+\int_0^h xc(x,t)u(x,t)dx\Big)\big(h\mu_2(t)-\mu_3(t)\big)$$
$$-\Big(a(t)t^{\beta}(u_x(h,t)-u_x(0,t))+\int_0^h c(x,t)u(x,t)dx\Big)\big(h^2\mu_2(t)-2\mu_4(t)\big)\Big), \tag{32}$$

where $\triangle$ is defined by the formula (16).

Substituting (29), (30) into (31), (32) we obtain the system of homogeneous integral Voltera equations of the second kind with unknowns $b_1=b_1(t), b_2=b_2(t)$:

$$b_1(t)=\int_0^t (K_{11}(t,\tau)b_1(\tau)+K_{12}(t,\tau)b_2(\tau))d\tau, \tag{33}$$

$$b_2(t)=\int_0^t (K_{21}(t,\tau)b_1(\tau)+K_{22}(t,\tau)b_2(\tau))d\tau. \tag{34}$$

where

$$K_{11}(t,\tau)=\triangle^{-1}\Big(\big(\mu_2(t)-\mu_1(t)\big)\Big(a(t)t^{\beta}h\int_0^h G_x^*(h,t,\xi,\tau)\xi u_{2\xi}(\xi,\tau)d\xi$$
$$+\int_0^h\int_0^h G^*(x,t,\xi,\tau)x\xi c(x,t)u_{2\xi}(\xi,\tau)d\xi dx\Big)-\big(h^2\mu_2(t)-2\mu_4(t)\big)$$
$$\times\Big(a(t)t^{\beta}\int_0^h (G_x^*(h,t,\xi,\tau)-G_x^*(0,t,\xi,\tau))\xi u_{2\xi}(\xi,\tau)d\xi$$
$$+\int_0^h\int_0^h G^*(x,t,\xi,\tau)\xi c(x,t)u_{2\xi}(\xi,\tau)d\xi dx\Big)\Big),$$

$$K_{12}(t,\tau)=\triangle^{-1}\Bigg(\big(\mu_2(t)-\mu_1(t)\big)\Bigg(a(t)t^{\beta}h\int_0^h G_x^*(h,t,\xi,\tau)u_{2\xi}(\xi,\tau)d\xi$$

$$+\int_0^h\int_0^h G^*(x,t,\xi,\tau)xc(x,t)u_{2\xi}(\xi,\tau)d\xi dx\Bigg)-\big(h^2\mu_2(t)-2\mu_4(t)\big)$$

$$\times\Bigg(a(t)t^{\beta}\int_0^h (G_x^*(h,t,\xi,\tau)-G_x^*(0,t,\xi,\tau))u_{2\xi}(\xi,\tau)d\xi$$

$$+\int_0^h\int_0^h G^*(x,t,\xi,\tau)c(x,t)u_{2\xi}(\xi,\tau)d\xi dx\Bigg)\Bigg),$$

$$K_{21}(t,\tau)=\triangle^{-1}\Bigg(\big(h\mu_2(t)-\mu_3(t)\big)\Bigg(a(t)t^{\beta}h\int_0^h G_x^*(h,t,\xi,\tau)\xi u_{2\xi}(\xi,\tau)d\xi$$

$$+\int_0^h\int_0^h G^*(x,t,\xi,\tau)x\xi c(x,t)u_{2\xi}(\xi,\tau)d\xi dx\Bigg)-\big(h^2\mu_2(t)-2\mu_4(t)\big)$$

$$\times\Bigg(a(t)t^{\beta}\int_0^h (G_x^*(h,t,\xi,\tau)-G_x^*(0,t,\xi,\tau))\xi u_{2\xi}(\xi,\tau)d\xi$$

$$+\int_0^h\int_0^h G^*(x,t,\xi,\tau)\xi c(x,t)u_{2\xi}(\xi,\tau)d\xi dx\Bigg)\Bigg),$$

$$K_{22}(t,\tau)=\triangle^{-1}\Bigg(\big(h\mu_2(t)-\mu_3(t)\big)\Bigg(a(t)t^{\beta}h\int_0^h G_x^*(h,t,\xi,\tau)u_{2\xi}(\xi,\tau)d\xi$$

$$+\int_0^h\int_0^h G^*(x,t,\xi,\tau)xc(x,t)u_{2\xi}(\xi,\tau)d\xi dx\Bigg)-\big(h^2\mu_2(t)-2\mu_4(t)\big)$$

$$\times\Bigg(a(t)t^{\beta}\int_0^h (G_x^*(h,t,\xi,\tau)-G_x^*(0,t,\xi,\tau))u_{2\xi}(\xi,\tau)d\xi$$

$$+\int_0^h\int_0^h G^*(x,t,\xi,\tau)c(x,t)u_{2\xi}(\xi,\tau)d\xi dx\Bigg)\Bigg),$$

Applying the known estimates of the Green function [29, p. 469]

$$|D_t^r D_y^s G^*(y,t,\eta,\tau)|\leqslant C_8(\theta_0(t)-\theta_0(\tau))^{-\frac{1+2r+s}{2}}\exp\Bigg(-C_9\frac{(y-\eta)^2}{\theta_0(t)-\theta_0(\tau)}\Bigg),$$

$$\theta_0(t)=\int\limits_0^t \tau^{\beta}d\tau,\ r\in\{0,1\},\ s\in\{0,1,2\},\ 2r+s=1 \text{ or } 2r+s=2,\ \tau<t$$

and definition of weak degeneration we conclude that the kernels $K_{11}(t,\tau)$, $K_{12}(t,\tau)$, $K_{21}(t,\tau)$, $K_{22}(t,\tau)$ have the integrable singularities. It means that the system (31), (32) has only trivial solution

$$b_1(t)\equiv 0,\ b_2(t)\equiv 0,\ t\in[0,T].$$

Using this fact in the problem (24)–(28) we obtain that

$$u(x,t)\equiv 0,\ (x,t)\in\overline{Q}_T.$$

The Theorem 2 is proved. □

5. Example

Let us consider the problem

$$\begin{aligned}
&u_t=t^{\beta}u_{xx}+(b_1(t)x+b_2(t))u_x,\ 0<x<1,\ 0<t<T,\\
&u(x,0)=x+1,\ 0\leqslant x\leqslant 1,\\
&u(0,t)=e^t,\ u(1,t)=2e^t,\ 0\leqslant t\leqslant T,\\
&\int\limits_0^1 u(x,t)dx=\frac{3}{2}e^t,\ 0\leqslant t\leqslant T,\\
&\int\limits_0^1 xu(x,t)dx=\frac{5}{6}e^t,\ 0\leqslant t\leqslant T.
\end{aligned}$$

It is easy to verify that the triplet $\left(1,1,(x+1)e^t\right)$ is the solution to this problem. The condition $(A2)$ in this case can be written in the form

$$\left(2e^t-\frac{3}{2}e^t\right)^2-(2e^t-e^t)\left(2e^t-\frac{5}{3}e^t\right)=\frac{e^{2t}}{4}-\frac{e^{2t}}{3}=-\frac{e^{2t}}{12}$$

and it is fulfilled.

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Nadiia Huzyk
32 Herous of Maidan street
Lviv, Ukraine 79026
e-mail: hryntsiv@ukr.net