

EXISTENCE AND CONTINUATION OF SOLUTIONS OF HILFER–KATUGAMPOLA–TYPE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we discuss the existence and continuation of the solution to general-ized Cauchy-type problem involving Hilfer-Katugampola-type fractional derivative. Firstly, we establish new theorems of local existence by using Schauder fixed point theorem. Then, we deduce continuation theorems for general fractional differential equations. By applying continuation theorems, we present several global existence results. Moreover, the examples are given to illustrate our main results.

1. Introduction

During last years, the operators number of fractional integration and differentiation has been increasing with new definitions for them, which correspond to the Riemann-Liouville, Caputo, Hadamard, Hilfer, etc. [13, 15]. Actually, there exists more than one new definition for fractional derivatives, which are Katugampola, Caputo-Katugampola and Hilfer-Katugampola for more details see [2, 14, 24].

Here, we mention that the authors have interest to solve fractional differential equations, where they have studied the existence and uniqueness theorems of a solution for fractional differential equations by applying a fixed point theory on a finite interval $[0, T]$ and they also obtained a global existence of solutions by establishing local existence theorems and continuation theorems on the half axis $[0, +\infty)$, [1, 3, 4, 5, 7, 8, 9, 16, 17, 19, 20, 21, 22, 23, 26, 27, 28, 29, 30] and references therein.

C. Kau et al. [18], have obtained the existence and continuation theorems for the following Riemann-Liouville type fractional differential equations

$$\begin{cases} {}_{RL}D_{0,t}^{\alpha}x(t) = \varphi(t, x(t)), & t \in (0, +\infty), \quad 0 < \alpha < 1, \\ {}_{RL}I_{0+}^{1-\alpha}x(t)|_{t=0} = x_0, \end{cases} \quad (1.1)$$

where ${}_{RL}D_{0,t}^{\alpha}$ is the Riemann-Liouville-type fractional derivative of order α .

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Moreover, S. P. Bhairat [6], found the existence and continuation of solutions for the following Hilfer fractional differential equations

$$\begin{cases} D_{0+}^{\alpha,\beta} x(t) = \varphi(t, x(t)), & t \in (0, +\infty), \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1, \\ I_{0+}^{1-\gamma} x(t)|_{t=0} = x_0, & \gamma = \alpha + \beta(1 - \alpha), \end{cases} \quad (1.2)$$

where $D_{0+}^{\alpha,\beta}$ is the Hilfer-type fractional derivative of order α and type β .

In this paper, we consider the Cauchy-type problem involving Hilfer-Katugampola-type fractional derivative with initial value problem

$$\begin{cases} \rho D_{0+}^{\alpha,\beta} x(t) = \varphi(t, x(t)), & t \in (0, +\infty), \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1, \\ \rho I_{0+}^{1-\gamma} x(t)|_{t=0} = x_0, & \gamma = \alpha + \beta(1 - \alpha), \end{cases} \quad (1.3)$$

where $\rho > 0$ and $\rho D_{0+}^{\alpha,\beta}$ is the Hilfer-Katugampola-type fractional derivative of order α and type β [24] and $\varphi : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ has a weak singularity with respect to t and here φ satisfies a Lipschitz condition

$$|\varphi(t, x(t)) - \varphi(t, y(t))| \leq A|x(t) - y(t)|,$$

where $A > 0$ is Lipschitz constant.

Also, we consider a system of fractional differential equations with general initial value problems

$$\begin{cases} \rho D_{0+}^{\alpha,\beta} x_1(t) = \varphi_1(t, x_1(t), x_2(t), \dots, x_n(t)), \\ \rho D_{0+}^{\alpha,\beta} x_2(t) = \varphi_2(t, x_1(t), x_2(t), \dots, x_n(t)), \\ \vdots \\ \rho D_{0+}^{\alpha,\beta} x_n(t) = \varphi_n(t, x_1(t), x_2(t), \dots, x_n(t)), \\ \rho I_{0+}^{1-\gamma} x_\ell(t)|_{t=0} = x_0, \quad \gamma = \alpha + \beta(1 - \alpha), \quad \ell = 1, 2, \dots, n, \end{cases} \quad (1.4)$$

where $0 < \alpha < 1, 0 \leq \beta \leq 1, \gamma = \alpha + \beta(1 - \alpha), \rho > 0$ and $\varphi_\ell : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ also has a weak singularity with respect to t and here $\varphi_n(t, x_1(t), x_2(t), \dots, x_n(t))$ satisfy a Lipschitz condition

$$|\varphi_j(t, x_1(t), x_2(t), \dots, x_n(t)) - \varphi_j(t, y_1(t), y_2(t), \dots, y_n(t))| \leq \sum_{j=1}^n A_j |x_j(t) - y_j(t)|,$$

where $A_j > 0, j = 1, 2, \dots, n$ are Lipschitz constants.

The aim of this article is to develop the existence and uniqueness theory. Firstly, we establish the local existence of solutions of Hilfer-Katugampola fractional differential equations as well as the system of Hilfer-Katugampola fractional differential equations, then we study continuation theorems of Hilfer-Katugampola fractional differential equations to extend the existence of global solutions.

The remaining parts of this paper is ordered as below:

In Section 2, we present some basic notations, definitions and lemmas used in our main results. Section 3, includes the study of a local existence of solutions, in which we obtain the new local existence theorems for the initial value problems (1.3) and (1.4). Two continuation theorems with global existence theorems for the initial value problems (1.3) are given in Section 4. The last section contains concluding remarks.

2. Preliminaries

In this section, we introduce some notations, definitions and lemmas from theory of fractional calculus which will be used later.

DEFINITION 1. [14] Let $\Omega = [0, T]$ be a finite interval and $\rho > 0$, the weighted space $C_{1-\gamma, \rho}[0, T]$ of continuous functions φ on $(0, T]$ is defined by

$$C_{1-\gamma, \rho}[0, T] = \{ \varphi : (0, T] \rightarrow \mathbb{R} : [(t^\rho / \rho)]^{1-\gamma} \varphi(t) \in C[0, T] \}$$

with the norm

$$\| \varphi \|_{C_{1-\gamma, \rho}} = \left\| \left[(t^\rho / \rho) \right]^{1-\gamma} \varphi(t) \right\|_C, \quad C_{0, \rho}[0, T] = C[0, T].$$

The space $C_{1-\gamma, \rho}[0, T]$ is the complete metric space defined with the distance d as

$$d(x_1, x_2) = \|x_1 - x_2\|_{C_{1-\gamma, \rho}[0, T]} := \max_{t \in [0, T]} \left| \left[(t^\rho / \rho) \right]^{1-\gamma} [x_1(t) - x_2(t)] \right|.$$

DEFINITION 2. [14] Let $\Omega = [0, T]$ and $\phi : (0, \infty) \rightarrow \mathbb{R}$, the Katugampola fractional integral ${}_\rho I_{0+}^\alpha \phi$ of order $\gamma \in \mathbb{C}(\Re(\alpha) > 0)$ is defined for $\rho > 0$ as

$$({}_\rho I_{0+}^\alpha \phi)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \phi(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau, \quad (t > 0), \tag{2.1}$$

and the corresponding Katugampola fractional derivative ${}_\rho D_{0+}^\alpha \phi$ is defined as

$$\begin{aligned} ({}_\rho D_{0+}^\alpha \phi)(t) &:= (t^{1-\rho} \frac{d}{dt})^n ({}_\rho I_{0+}^{n-\alpha} \phi)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} (t^{1-\rho} \frac{d}{dt})^n \int_0^t \frac{\tau^{\rho-1} \phi(\tau)}{(t^\rho - \tau^\rho)^{\alpha-n+1}} d\tau, \quad (t > 0). \end{aligned} \tag{2.2}$$

DEFINITION 3. [24] Let $0 < \alpha < 1, 0 \leq \beta \leq 1, \varphi \in C_{1-\gamma, \rho}[0, T]$. The Hilfer-Katugampola fractional derivative ${}_\rho D^{\alpha, \beta}$ of order α and type β of φ is defined as

$$\begin{aligned} ({}_\rho D^{\alpha, \beta} \varphi)(t) &= \left({}_\rho I^{\beta(1-\alpha)} (t^{1-\rho} \frac{d}{dt}) {}_\rho I^{(1-\alpha)(1-\beta)} \varphi \right)(t) \\ &= \left({}_\rho I^{\beta(1-\alpha)} (\delta_\rho) {}_\rho I^{(1-\alpha)(1-\beta)} \varphi \right)(t); \quad \gamma = \alpha + \beta(1-\alpha). \end{aligned} \tag{2.3}$$

Where ${}_\rho I^{(\cdot)}$ is the Katugampola fractional integral defined in (2.1).

LEMMA 1. [17] Let $a < b < c$, $0 \leq v < 1$, $x \in C_v[a, b]$, $y \in C[b, c]$ and $x(b) = y(b)$. Define

$$z(t) = \begin{cases} x(t) & \text{if } t \in (a, b), \\ y(t) & \text{if } t \in [b, c]. \end{cases} \tag{2.4}$$

Then, $z \in C_v[a, c]$.

LEMMA 2. [10] (Schauder fixed point Theorem) Let U be a closed bounded convex subset of a Banach space E and Suppose that $T : U \rightarrow U$ is completely continuous operator. Then, T has a fixed point in U .

LEMMA 3. [24] Let $\Omega = [0, T]$ be a finite interval, $\alpha > 0$ and $0 \leq v < 1$.

(a) If $v > \alpha$, then the fractional integration operator ρI_{0+}^α is bounded from $C_{v,\rho}[0, T]$ into $C_{v-\alpha,\rho}[0, T]$.

(b) If $v \leq \alpha$, then the fractional integration operator ρI_{0+}^α is bounded from $C_{v,\rho}[0, T]$ into $C[0, T]$.

LEMMA 4. [24] Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta(1 - \alpha)$, and assume that $\varphi(t, x(t)) \in C_{1-\gamma,\rho}[0, T]$ where $\varphi : (0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function for any $x \in C_{1-\gamma,\rho}[0, T]$. If $x \in C_{1-\gamma,\rho}^\gamma[0, T]$, then x satisfies (1.3) if and only if x satisfies the second kind Volterra fractional integral equation

$$x(t) = \frac{x_0}{\Gamma(\gamma)} (t^\rho/\rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \varphi(\tau, x(\tau))}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau, \quad (t > 0). \tag{2.5}$$

In the light of the Lemma 2.3 (see[18]), we have the following Lemma:

LEMMA 5. Let \mathcal{A} be the subset of $C_{1-\gamma,\rho}[0, T]$. Then, \mathcal{A} is precompact if and only if the following conditions are satisfied:

- (1) $\{(t^\rho/\rho)^{1-\gamma}x(t) : x \in \mathcal{A}\}$ is uniformly bounded,
- (2) $\{(t^\rho/\rho)^{1-\gamma}x(t) : x \in \mathcal{A}\}$ is equicontinuous on $[0, T]$.

3. The local existence

In this section, we study the local existence of solutions for the initial value problems (1.3) and (1.4). Assume that $\varphi(t, x(t))$ in (1.3) and $\varphi_\ell(t, x_\ell(t))$, ($\ell = 1, 2, \dots, n$) in (1.4) have some weak singularities with respect to t respectively. By using Schauder fixed point theorem, we obtain new local existence theorems.

For convenience, we create the following two hypothesis:

(\mathcal{H}_1) Assume that $\varphi : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ in (1.3) is a continuous function and there exists a constant $0 \leq \lambda < 1$ such that $(\mathcal{M}x)(t) = t^\lambda \varphi(t, x(t))$ is a continuous bounded map from $C_{1-\gamma,\rho}[0, T]$ into $C[0, T]$, where T is a positive constant.

(\mathcal{H}_2) Assume that $\varphi_\ell : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ in (1.4) are continuous functions and there exist constants $0 \leq \lambda_\ell < 1$ such that $(\mathcal{M}_\ell x_\ell)(t) = t^{\lambda_\ell} \varphi_\ell(t, x_1(t), x_2(t), \dots, x_n(t))$, ($\ell = 1, 2, \dots, n$) are continuous bounded maps from $C_{1-\gamma,\rho}[0, T]$ into $C[0, T]$, where T is a positive constant.

THEOREM 1. Assume that condition (\mathcal{H}_1) is satisfied. Then, the initial value problem (1.3) has at least one solution $x \in C_{1-\gamma,\rho}[0, h]$ for some $(T \geqslant)h > 0$.

Proof. Let

$$D = \left\{ x \in C_{1-\gamma,\rho}[0, T] : \left\| x - \frac{x_0}{\Gamma(\gamma)} (t^\rho / \rho)^{\gamma-1} \right\|_{C_{1-\gamma,\rho}[0, T]} = \sup_{0 \leqslant t \leqslant T} \left| (t^\rho / \rho)^{1-\gamma} x(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leqslant k \right\}, \quad (3.1)$$

where $k > 0$ is a constant. Since the operator \mathcal{M} is bounded, there exists a constant $L > 0$ such that

$$\sup \{ |(\mathcal{M}x)(t)| : t \in [0, T], x \in D \} \leqslant L.$$

Again, let

$$E_h = \left\{ x : x \in C_{1-\gamma,\rho}[0, h], \sup_{0 \leqslant t \leqslant h} \left| (t^\rho / \rho)^{1-\gamma} x(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leqslant k \right\}, \quad (3.2)$$

where $h = \min \left\{ \left(\frac{k \rho^{\alpha-\gamma+1} \Gamma(\alpha-\lambda/\rho+1)}{L \Gamma(1-\lambda/\rho)} \right)^{\frac{1}{\rho(\alpha-\gamma-\lambda/\rho+1)}}, T \right\}$. Obviously, $E_h \subseteq C_{1-\gamma,\rho}[0, T]$ is a nonempty, bounded closed and convex subset.

Note that $h \leqslant T$, we can regard E_h and $C_{1-\gamma,\rho}[0, T]$ as the restrictions of D and $C_{1-\gamma,\rho}[0, T]$, respectively. Define the operator \mathcal{N} as follows:

$$(\mathcal{N}x)(t) = \frac{x_0}{\Gamma(\gamma)} (t^\rho / \rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau) d\tau, \quad t \in [0, h]. \quad (3.3)$$

Observe that from (\mathcal{H}_1) and Lemma 3 we have $\mathcal{N}(C_{1-\gamma,\rho}[0, h]) \subset C_{1-\gamma,\rho}[0, h]$.

By relation (3.3), for any $x \in C_{1-\gamma,\rho}[0, h]$, we obtain

$$\begin{aligned} & \left| (t^\rho / \rho)^{1-\gamma} (\mathcal{N}x)(t) - \frac{x_0}{\Gamma(\gamma)} \right| \\ &= \left| (t^\rho / \rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda-1} (t^\rho - \tau^\rho)^{\alpha-1} [\tau^\lambda \varphi(\tau, x(\tau))] d\tau \right| \\ &\leqslant \frac{(t^\rho / \rho)^{1-\gamma} \rho^{1-\alpha} L}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau \\ &\leqslant \frac{L h^{\rho(\alpha-\gamma-\lambda/\rho+1)} \Gamma(1-\lambda/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha-\lambda/\rho+1)} \leqslant k, \end{aligned}$$

which yields that $\mathcal{N}E_h \subset E_h$.

Next, we will show that \mathcal{N} is continuous. For that let $x_n, x \in E_h$, $\|x_n - x\|_{C_{1-\gamma,\rho}[0, h]} \rightarrow 0$ as $n \rightarrow +\infty$. In the light of a continuity of \mathcal{M} , we have $\|\mathcal{M}x_n - \mathcal{M}x\|_{C_{1-\gamma,\rho}[0, h]} \rightarrow 0$ as $n \rightarrow +\infty$.

Now, noticing that

$$\begin{aligned}
 & \left| (t^\rho/\rho)^{1-\gamma}(\mathcal{N}x_n)(t) - (t^\rho/\rho)^{1-\gamma}(\mathcal{N}x)(t) \right| \\
 &= \left| (t^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x_n(\tau)) d\tau \right. \\
 &\quad \left. - (t^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \\
 &\leq \frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda-1} (t^\rho - \tau^\rho)^{\alpha-1} \left| \tau^\lambda [\varphi(\tau, x_n(\tau)) - \varphi(\tau, x(\tau))] \right| d\tau \\
 &\leq \frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau \| \mathcal{M}x_n - \mathcal{M}x \|_{[0, h]}.
 \end{aligned}$$

Then, we have

$$\| \mathcal{N}x_n - \mathcal{N}x \|_{C_{1-\gamma, \rho}[0, h]} \leq \frac{h^{\rho(\alpha-\gamma-\lambda/\rho+1)} \Gamma(1-\lambda/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha-\lambda/\rho+1)} \| \mathcal{M}x_n - \mathcal{M}x \|_{[0, h]}. \tag{3.4}$$

Thus, $\| \mathcal{N}x_n - \mathcal{N}x \|_{C_{1-\gamma, \rho}[0, h]} \rightarrow 0$ as $n \rightarrow +\infty$. Therefore, \mathcal{N} is continuous. Moreover, we shall prove that the operator $\mathcal{N}E_h$ is equicontinuous. Let $x \in E_h$ and $0 \leq t_1 < t_2 \leq h$, for any $\delta > 0$, note that

$$\frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau = \frac{t^{\rho(\alpha-\gamma-\lambda/\rho+1)} \Gamma(1-\lambda/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha-\lambda/\rho+1)} \rightarrow 0 \quad \text{as } t \rightarrow 0^+,$$

where $0 \leq \lambda < 1$, there exists a $(h >) \varepsilon_1 > 0$ such that, for $t \in [0, \varepsilon_1]$, we have

$$\frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau < \frac{\delta}{2}. \tag{3.5}$$

In the case, for $t_1, t_2 \in [0, \varepsilon_1]$, we get

$$\begin{aligned}
 & \left| (t_1^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} (t_1^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right. \\
 &\quad \left. - (t_2^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \\
 &\leq \frac{(t_1^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-\lambda-1} (t_1^\rho - \tau^\rho)^{\alpha-1} d\tau \\
 &\quad + \frac{(t_2^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L}{\Gamma(\alpha)} \int_0^{t_2} \tau^{\rho-\lambda-1} (t_2^\rho - \tau^\rho)^{\alpha-1} d\tau \tag{3.6} \\
 &< \frac{\delta}{2} + \frac{\delta}{2} = \delta.
 \end{aligned}$$

In the case, for $t_1, t_2 \in [\frac{\varepsilon_1}{2}, h]$, we have

$$\begin{aligned} & \left| (t_1^\rho/\rho)^{1-\gamma}(\mathcal{N}x)(t_1) - (t_2^\rho/\rho)^{1-\gamma}(\mathcal{N}x)(t_2) \right| \\ &= \left| (t_1^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} (t_1^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right. \\ &\quad \left. - (t_2^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \\ &\leq \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \varphi(\tau, x(\tau)) d\tau \right| \\ &\quad + \left| \frac{(t_2^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right|, \tag{3.7} \end{aligned}$$

its easy to see form the fact that if $0 \leq v_1 < v_2 \leq h$, then

$$(v_1^\rho/\rho)^{1-\gamma} (v_1^\rho - \tau^\rho)^{\alpha-1} > (v_2^\rho/\rho)^{1-\gamma} (v_2^\rho - \tau^\rho)^{\alpha-1}$$

for $0 \leq \tau < v_1$, we get

$$\begin{aligned} & \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \varphi(\tau, x(\tau)) d\tau \right| \\ &\leq \frac{\rho^{1-\alpha} L}{\Gamma(\alpha)} \int_0^{t_1} \left| \tau^{\rho-\lambda-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \right| d\tau \\ &\leq \frac{\rho^{1-\alpha} L}{\Gamma(\alpha)} \int_0^{\frac{\varepsilon_1}{2}} \left| \tau^{\rho-\lambda-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \right| d\tau \\ &\quad + \frac{(\frac{\varepsilon_1}{2})^{-\lambda} \rho^{1-\alpha} L}{\Gamma(\alpha)} \int_{\frac{\varepsilon_1}{2}}^{t_1} \left| \tau^{\rho-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \right| d\tau \\ &\leq \frac{2((\frac{\varepsilon_1}{2})^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L}{\Gamma(\alpha)} \int_0^{\frac{\varepsilon_1}{2}} \tau^{\rho-\lambda-1} \left(\left(\frac{\varepsilon_1}{2} \right)^\rho - \tau^\rho \right)^{\alpha-1} d\tau + \frac{(\frac{\varepsilon_1}{2})^{-\lambda} L}{\rho^{\alpha-\gamma+1} \Gamma(\alpha+1)} \\ &\quad \times \left[t_2^{\rho(1-\gamma)} (t_2^\rho - t_1^\rho)^\alpha - t_2^{\rho(1-\gamma)} \left(t_2^\rho - \left(\frac{\varepsilon_1}{2} \right)^\rho \right)^\alpha + t_1^{\rho(1-\gamma)} \left(t_1^\rho - \left(\frac{\varepsilon_1}{2} \right)^\rho \right)^\alpha \right] \\ &\leq \delta + \frac{(\frac{\varepsilon_1}{2})^{-\lambda} L}{\rho^{\alpha-\gamma+1} \Gamma(\alpha+1)} \\ &\quad \times \left[h^{\rho(1-\gamma)} (t_2^\rho - t_1^\rho)^\alpha + \left| t_2^{\rho(1-\gamma)} \left(t_2^\rho - \left(\frac{\varepsilon_1}{2} \right)^\rho \right)^\alpha - t_1^{\rho(1-\gamma)} \left(t_1^\rho - \left(\frac{\varepsilon_1}{2} \right)^\rho \right)^\alpha \right| \right]. \tag{3.8} \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \left| \frac{(t_2^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \\
 & \leq \frac{(\frac{\varepsilon_1}{2})^{-\lambda} L}{\rho^{\alpha-\gamma} \Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} d\tau \\
 & = \frac{(\frac{\varepsilon_1}{2})^{-\lambda} L}{\rho^{\alpha-\gamma+1} \Gamma(\alpha+1)} \left[t_2^{\rho(1-\gamma)} (t_2^\rho - t_1^\rho)^\alpha \right] \\
 & \leq \frac{(\frac{\varepsilon_1}{2})^{-\lambda} L}{\rho^{\alpha-\gamma+1} \Gamma(\alpha+1)} \left[h^{\rho(1-\gamma)} (t_2^\rho - t_1^\rho)^\alpha \right]. \tag{3.9}
 \end{aligned}$$

Obviously, there exists a $(\frac{\varepsilon_1}{2} >) \varepsilon > 0$ such that, for $t_1, t_2 \in [\frac{\varepsilon_1}{2}, h]$, $|t_1 - t_2| < \varepsilon$ implies

$$\left| (t_1^\rho/\rho)^{1-\gamma} (\mathcal{N}x)(t_1) - (t_2^\rho/\rho)^{1-\gamma} (\mathcal{N}x)(t_2) \right| < 2\delta. \tag{3.10}$$

Finally, it observe from (3.6) and (3.10) that $\{(t^\rho/\rho)^{1-\gamma} \mathcal{N} : x \in E_h\}$ is equicontinuous. Evidently, $\{(t^\rho/\rho)^{1-\gamma} \mathcal{N} : x \in E_h\}$ is uniformly bounded, due to $\mathcal{N}E_h \subset E_h$. Then, by Lemma 5, $\mathcal{N}E_h$ is precompact. Thus, \mathcal{N} is completely continuous. Therefore, by Lemma 2 (Schauder fixed point theorem) and Lemma 4, the initial value problem (1.3) has a local solution. \square

THEOREM 2. *Assume that condition (\mathcal{A}_2) is satisfied. Then, the initial value problem (1.4) has at least one solution $x_\ell \in C_{1-\gamma, \rho}[0, h]$, $(\ell = 1, 2, \dots, n)$ for some $(T \geq) h > 0$.*

Proof. Let

$$\begin{aligned}
 D & = \left\{ x_\ell \in C_{1-\gamma, \rho}[0, T] : \left\| x_\ell - \frac{x_0}{\Gamma(\gamma)} (t^\rho/\rho)^{\gamma-1} \right\|_{C_{1-\gamma, \rho}[0, T]} \right. \\
 & \quad \left. = \sup_{0 \leq t \leq T} \left| (t^\rho/\rho)^{1-\gamma} x_\ell(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leq k_\ell \right\}, \tag{3.11}
 \end{aligned}$$

where $k_\ell > 0$, $(\ell = 1, 2, \dots, n)$ are constants. Since the operators \mathcal{M}_ℓ , $(\ell = 1, 2, \dots, n)$ is bounded, there exists a constant $L_\ell > 0$, $(\ell = 1, 2, \dots, n)$ such that

$$\sup \left\{ |(\mathcal{M}_\ell x_\ell)(t)| : t \in [0, T], x_\ell \in D \right\} \leq L_\ell, \quad (\ell = 1, 2, \dots, n).$$

Again, let

$$E_{\ell h} = \left\{ x_\ell : x_\ell \in C_{1-\gamma, \rho}[0, h], \sup_{0 \leq t \leq h} \left| (t^\rho/\rho)^{1-\gamma} x_\ell(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leq k_\ell, \ell = 1, 2, \dots, n \right\}, \tag{3.12}$$

where

$$h = \min \left\{ \left(\frac{k_1 \rho^{\alpha-\gamma+1} \Gamma(\alpha - \lambda_1/\rho + 1)}{L_1 \Gamma(1 - \lambda_1/\rho)} \right)^{\frac{1}{\rho(\alpha-\gamma-\lambda_1/\rho+1)}}, \dots, \left(\frac{k_n \rho^{\alpha-\gamma+1} \Gamma(\alpha - \lambda_n/\rho + 1)}{L_n \Gamma(1 - \lambda_n/\rho)} \right)^{\frac{1}{\rho(\alpha-\gamma-\lambda_n/\rho+1)}}, T \right\}.$$

Obviously, $E_{\ell h} \subseteq C_{1-\gamma,\rho}[0, T]$ be a nonempty, bounded closed and convex subset.

Note that $h \leq T$, we can regard $E_{\ell h}$ and $C_{1-\gamma,\rho}[0, T]$ as the restrictions of D and $C_{1-\gamma,\rho}[0, T]$, respectively. Define the operators \mathcal{N}_ℓ as follows

$$\begin{cases} (\mathcal{N}_1 x_1)(t) = \frac{x_0}{\Gamma(\gamma)} (t^\rho/\rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi_1(\tau, x_1(\tau), x_2(\tau), \dots, x_n(\tau)) d\tau, \\ (\mathcal{N}_2 x_2)(t) = \frac{x_0}{\Gamma(\gamma)} (t^\rho/\rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi_2(\tau, x_1(\tau), x_2(\tau), \dots, x_n(\tau)) d\tau, \\ \vdots \\ (\mathcal{N}_n x_n)(t) = \frac{x_0}{\Gamma(\gamma)} (t^\rho/\rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi_n(\tau, x_1(\tau), x_2(\tau), \dots, x_n(\tau)) d\tau, \end{cases} \tag{3.13}$$

for $t \in [0, h]$. Observe that from (\mathcal{A}_2) and Lemma 3, we have

$$\mathcal{N}_\ell(C_{1-\gamma,\rho}[0, h]) \subset C_{1-\gamma,\rho}[0, h], \quad (\ell = 1, 2, \dots, n).$$

By relation (3.13), for any $x \in C_{1-\gamma,\rho}[0, h]$, we obtain

$$\begin{aligned} & \left| (t^\rho/\rho)^{1-\gamma} (\mathcal{N}_1 x_1)(t) - \frac{x_0}{\Gamma(\gamma)} \right| \\ &= \left| (t^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda_1-1} (t^\rho - \tau^\rho)^{\alpha-1} [\tau^{\lambda_1} \varphi_1(\tau, x_1(\tau), x_2(\tau), \dots, x_n(\tau))] d\tau \right| \\ &\leq \frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L_1}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda_1-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau \\ &\leq \frac{L_1 \Gamma(1 - \lambda_1/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha - \lambda_1/\rho + 1)} t^{\rho(\alpha-\gamma-\lambda_1/\rho+1)}, \\ & \left| (t^\rho/\rho)^{1-\gamma} (\mathcal{N}_1 x_1)(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leq \frac{L_1 \Gamma(1 - \lambda_1/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha - \lambda_1/\rho + 1)} h^{\rho(\alpha-\gamma-\lambda_1/\rho+1)} \leq k_1, \\ & \left| (t^\rho/\rho)^{1-\gamma} (\mathcal{N}_2 x_2)(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leq \frac{L_2 \Gamma(1 - \lambda_2/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha - \lambda_2/\rho + 1)} h^{\rho(\alpha-\gamma-\lambda_2/\rho+1)} \leq k_2, \\ & \vdots \\ & \left| (t^\rho/\rho)^{1-\gamma} (\mathcal{N}_n x_n)(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leq \frac{L_n \Gamma(1 - \lambda_n/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha - \lambda_n/\rho + 1)} h^{\rho(\alpha-\gamma-\lambda_n/\rho+1)} \leq k_n, \end{aligned}$$

which yields that $\mathcal{N}_\ell E_{\ell h} \subset E_{\ell h}$, $\ell = 1, 2, \dots, n$.

Next, we will show that \mathcal{N}_ℓ are continuous. For that let $x_m, x_\ell \in E_{\ell h}$, $m > n$, $\ell = 1, 2, \dots, n$ such that $\|x_m - x_\ell\|_{C_{1-\gamma, \rho}[0, h]} \rightarrow 0$ as $m \rightarrow +\infty$. In the light of a continuity of \mathcal{M}_ℓ , we have $\|\mathcal{M}_\ell x_m - \mathcal{M}_\ell x_\ell\|_{C_{1-\gamma, \rho}[0, h]} \rightarrow 0$ as $m \rightarrow +\infty$.

Now, noticing that

$$\begin{aligned} & \left| (t^\rho/\rho)^{1-\gamma}(\mathcal{N}_\ell x_m)(t) - (t^\rho/\rho)^{1-\gamma}(\mathcal{N}_\ell x_\ell)(t) \right| \\ &= \left| (t^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi_\ell(\tau, x_m(\tau)) d\tau \right. \\ & \quad \left. - (t^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right| \\ &\leq \frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda_\ell-1} (t^\rho - \tau^\rho)^{\alpha-1} \left| \tau^{\lambda_\ell} [\varphi_\ell(\tau, x_m(\tau)) - \varphi_\ell(\tau, x_\ell(\tau))] \right| d\tau \\ &\leq \frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda_\ell-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau \|\mathcal{M}_\ell x_m - \mathcal{M}_\ell x_\ell\|_{[0, h]}. \end{aligned}$$

Then, we have

$$\|\mathcal{N}_\ell x_m - \mathcal{N}_\ell x_\ell\|_{C_{1-\gamma, \rho}[0, h]} \leq \frac{h^{\rho(\alpha-\gamma-\lambda_\ell/\rho+1)} \Gamma(1-\lambda_\ell/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha-\lambda_\ell/\rho+1)} \|\mathcal{M}_\ell x_m - \mathcal{M}_\ell x_\ell\|_{[0, h]}. \tag{3.14}$$

Thus, $\|\mathcal{N}_\ell x_m - \mathcal{N}_\ell x_\ell\|_{C_{1-\gamma, \rho}[0, h]} \rightarrow 0$ as $m \rightarrow +\infty$. Therefore, \mathcal{N}_ℓ is continuous. Moreover, we shall prove that the operators $\mathcal{N}_\ell E_{\ell h}$ are equicontinuous. Let $x_\ell \in E_{\ell h}$ and $0 \leq t_1 < t_2 \leq h$, for any $\delta > 0$, note that

$$\frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda_\ell-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau = \frac{t^{\rho(\alpha-\gamma-\lambda_\ell/\rho+1)} \Gamma(1-\lambda_\ell/\rho)}{\rho^{\alpha-\gamma+1} \Gamma(\alpha-\lambda_\ell/\rho+1)} \rightarrow 0 \text{ as } t \rightarrow 0^+,$$

where $0 \leq \lambda_\ell < 1$, there exists a $(h >) \varepsilon_\ell > 0$ such that, for $t \in [0, \varepsilon_\ell]$, we have

$$\frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L_\ell}{\Gamma(\alpha)} \int_0^t \tau^{\rho-\lambda_\ell-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau < \frac{\delta}{2}. \tag{3.15}$$

In the case, for $t_1, t_2 \in [0, \varepsilon_\ell]$, we get

$$\begin{aligned} & \left| (t_1^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} (t_1^\rho - \tau^\rho)^{\alpha-1} \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right. \\ & \quad \left. - (t_2^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right| \\ &\leq \frac{(t_1^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L_\ell}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-\lambda_\ell-1} (t_1^\rho - \tau^\rho)^{\alpha-1} d\tau \\ & \quad + \frac{(t_2^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L_\ell}{\Gamma(\alpha)} \int_0^{t_2} \tau^{\rho-\lambda_\ell-1} (t_2^\rho - \tau^\rho)^{\alpha-1} d\tau \tag{3.16} \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

In the case, for $t_1, t_2 \in [\frac{\varepsilon_\ell}{2}, h]$, we have

$$\begin{aligned} & \left| (t_1^\rho/\rho)^{1-\gamma}(\mathcal{N}_\ell x_\ell)(t_1) - (t_2^\rho/\rho)^{1-\gamma}(\mathcal{N}_\ell x_\ell)(t_2) \right| \\ &= \left| (t_1^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} (t_1^\rho - \tau^\rho)^{\alpha-1} \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right. \\ & \quad \left. - (t_2^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right| \\ &\leq \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right| \\ & \quad + \left| \frac{(t_2^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right|, \end{aligned} \tag{3.17}$$

its easy to see form the fact that if $0 \leq v_1 < v_2 \leq h$, then

$$(v_1^\rho/\rho)^{1-\gamma} (v_1^\rho - \tau^\rho)^{\alpha-1} > (v_2^\rho/\rho)^{1-\gamma} (v_2^\rho - \tau^\rho)^{\alpha-1}$$

for $0 \leq \tau < v_1$, we get

$$\begin{aligned} & \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right| \\ &\leq \frac{\rho^{1-\alpha} L_\ell}{\Gamma(\alpha)} \int_0^{t_1} \left| \tau^{\rho-\lambda_\ell-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \right| d\tau \\ &\leq \frac{\rho^{1-\alpha} L_\ell}{\Gamma(\alpha)} \int_0^{\frac{\varepsilon_\ell}{2}} \left| \tau^{\rho-\lambda_\ell-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \right| d\tau \\ & \quad + \frac{(\frac{\varepsilon_\ell}{2})^{-\lambda_\ell} \rho^{1-\alpha} L_\ell}{\Gamma(\alpha)} \int_{\frac{\varepsilon_\ell}{2}}^{t_1} \left| \tau^{\rho-1} \left[(t_1^\rho/\rho)^{1-\gamma} (t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho/\rho)^{1-\gamma} (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \right| d\tau \\ &\leq \frac{2((\frac{\varepsilon_\ell}{2})^\rho/\rho)^{1-\gamma} \rho^{1-\alpha} L_\ell}{\Gamma(\alpha)} \int_0^{\frac{\varepsilon_\ell}{2}} \tau^{\rho-\lambda_\ell-1} \left(\left(\frac{\varepsilon_\ell}{2}\right)^\rho - \tau^\rho \right)^{\alpha-1} d\tau + \frac{(\frac{\varepsilon_\ell}{2})^{-\lambda_\ell} L_\ell}{\rho^{\alpha-\gamma+1} \Gamma(\alpha+1)} \\ & \quad \times \left[t_2^{\rho(1-\gamma)} (t_2^\rho - t_1^\rho)^\alpha - t_2^{\rho(1-\gamma)} \left(t_2^\rho - \left(\frac{\varepsilon_\ell}{2}\right)^\rho \right)^\alpha + t_1^{\rho(1-\gamma)} \left(t_1^\rho - \left(\frac{\varepsilon_\ell}{2}\right)^\rho \right)^\alpha \right] \\ &\leq \delta + \frac{(\frac{\varepsilon_\ell}{2})^{-\lambda_\ell} L_\ell}{\rho^{\alpha-\gamma+1} \Gamma(\alpha+1)} \\ & \quad \times \left[h^{\rho(1-\gamma)} (t_2^\rho - t_1^\rho)^\alpha + \left| t_2^{\rho(1-\gamma)} \left(t_2^\rho - \left(\frac{\varepsilon_\ell}{2}\right)^\rho \right)^\alpha - t_1^{\rho(1-\gamma)} \left(t_1^\rho - \left(\frac{\varepsilon_\ell}{2}\right)^\rho \right)^\alpha \right| \right]. \end{aligned} \tag{3.18}$$

On the other hand,

$$\begin{aligned}
 & \left| \frac{(t_2^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi_\ell(\tau, x_\ell(\tau)) d\tau \right| \\
 & \leq \frac{(\frac{\varepsilon_\ell}{2})^{-\lambda_\ell} L_\ell}{\rho^{\alpha-\gamma} \Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} d\tau \\
 & = \frac{(\frac{\varepsilon_\ell}{2})^{-\lambda_\ell} L_\ell}{\rho^{\alpha-\gamma+1} \Gamma(\alpha+1)} \left[t_2^{\rho(1-\gamma)} (t_2^\rho - t_1^\rho)^\alpha \right] \\
 & \leq \frac{(\frac{\varepsilon_\ell}{2})^{-\lambda_\ell} L_\ell}{\rho^{\alpha-\gamma+1} \Gamma(\alpha+1)} \left[h^{\rho(1-\gamma)} (t_2^\rho - t_1^\rho)^\alpha \right]. \tag{3.19}
 \end{aligned}$$

Obviously, there exists a σ , $(\frac{\varepsilon_\ell}{2})^\sigma > 0$ such that, for $t_1, t_2 \in [\frac{\varepsilon_\ell}{2}, h]$, $|t_1 - t_2| < \sigma$ implies

$$\left| (t_1^\rho/\rho)^{1-\gamma} (\mathcal{N}_\ell x_\ell)(t_1) - (t_2^\rho/\rho)^{1-\gamma} (\mathcal{N}_\ell x_\ell)(t_2) \right| < 2\delta. \tag{3.20}$$

Finally, it observe from (3.16) and (3.20) that $\{(t^\rho/\rho)^{1-\gamma} \mathcal{N}_\ell : x_\ell \in E_{\ell h}\}$ is equicontinuous. Evidently, $\{(t^\rho/\rho)^{1-\gamma} \mathcal{N}_\ell : x_\ell \in E_{\ell h}\}$ is uniformly bounded, due to $\mathcal{N}_\ell E_{\ell h} \subset E_{\ell h}$. Then, by Lemma 5, $\mathcal{N}_\ell E_{\ell h}$ is precompact. Thus, \mathcal{N}_ℓ is completely continuous. Therefore, by Lemma 2 (Schauder fixed point theorem) and Lemma 4, the initial value problem (1.4) has a local solution. \square

EXAMPLE 1. Consider the initial value problem

$$\begin{cases} \frac{1}{2} D_{0+}^{\frac{1}{2}, \frac{1}{3}} x(t) = \varphi(t, x(t)), & t \in (0, +\infty), \\ \frac{1}{2} I_{0+}^{\frac{1}{3}} x(t)|_{t=0} = \frac{\sqrt{\pi}}{2}, \end{cases} \tag{3.21}$$

here, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{2}{3}$, $\rho = \frac{1}{2}$ and $\varphi(t, x(t)) = \frac{\sin(1 + \sqrt[3]{t^2} x^2(t))}{\sqrt[3]{t}}$, $t \in (0, +\infty)$.

Easily, we can verify that the operator $(\mathcal{M}x)(t) = t^\lambda \varphi(t, x(t)) = \sin(1 + \sqrt[3]{t^2} x^2(t))$, (where $\lambda = \frac{1}{3}$, in special case), be the continuous bounded map from $C_{\frac{1}{3}, \frac{1}{2}}[0, T]$ into $C[0, T]$, where T is a positive constant. Then, by Theorem 1, the initial value problem (3.21) has a local solution.

4. The continuation and global existence

This section contains two parts, in the first part we discuss the continuation of solution for the initial value problems (1.3) and in the second part we present some results of the global existence. Firstly, we present the following definition:

DEFINITION 4. [18] Assume that $x(t)$ and $\hat{x}(t)$ are solutions of the initial value problems (1.3) on $(0, \mu)$ and $(0, \hat{\mu})$, respectively. If $\mu < \hat{\mu}$ and $x(t) = \hat{x}(t)$ for $t \in (0, \mu)$, then we say $\hat{x}(t)$ is the continuation of $x(t)$, or $x(t)$ can be continued to $(0, \hat{\mu})$.

The solution $x(t)$ is non-continuable if, it has no continuation. The existing interval of non-continuable solution $x(t)$ is called a maximum existing interval of $x(t)$.

In the light of the Lemma 4.1 (see[18]), we have the following Lemma:

LEMMA 6. Let $\rho > 0, h, \mu > 0, 0 < \alpha < 1, 0 \leq \nu < 1, \phi_1 \in C_\nu[0, \frac{\mu}{2}]$ and $\phi_2 \in C[\frac{\mu}{2}, \mu]$. Then, we have

$$\mathcal{I}_1 = \int_0^{\frac{\mu}{2}} \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \phi_1(\tau) d\tau$$

and

$$\mathcal{I}_2 = \int_{\frac{\mu}{2}}^\mu \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \phi_2(\tau) d\tau$$

are continuous on $[\mu, \mu + h]$.

Now, we present the first theorem of continuation as follows:

THEOREM 3. Suppose that (\mathcal{H}_1) is satisfied. Then, $x = x(t), t \in (0, \mu)$ is non-continuable if and only if for some $\zeta \in (0, \frac{\mu}{2})$ and any bounded closed subset $E \subset [\zeta, +\infty) \times \mathbb{R}$, there exists a $t^* \in [\zeta, \mu)$ such that, $(t^*, x(t^*)) \notin E$.

Proof. Firstly, assume that $x = x(t)$ is continuable. Then, there exists solution $\hat{x}(t)$ of initial value problems (1.3) defined on $(0, \hat{\mu})$ such that, $x(t) = \hat{x}(t)$ for $t \in (0, \mu)$, which yields that $\lim_{t \rightarrow \mu^-} x(t) = \hat{x}(\mu)$. Now, define $x(\mu) = \hat{x}(\mu)$. Obviously, $D = \{(t, x(t)) : t \in [\zeta, \mu)\}$ is a compact subset of $[\zeta, +\infty) \times \mathbb{R}$. Moreover, there exists no $t^* \in [\zeta, \mu)$ such that, $(t^*, x(t^*)) \notin D$. This contradiction gives that $x(t)$ is non-continuable.

Secondly, assume that there exists a compact subset $E \subset [\zeta, +\infty) \times \mathbb{R}$ such that, $\{(t, x(t)) : t \in [\zeta, \mu)\} \subset E$. Then, the compactness of E yields that $\mu < +\infty$. By (\mathcal{H}_1) , there exists $\Lambda > 0$ such that, $\sup_{(t, x(t)) \in E} |\varphi(t, x(t))| \leq \Lambda$.

Step 1. Now, we show that $\lim_{t \rightarrow \mu^-} x(t)$ exists. For that we put

$$\Psi(\tau, t) = \left| \frac{x_0}{\Gamma(\gamma)} (\tau^\rho / \rho)^{\gamma-1} - \frac{x_0}{\Gamma(\gamma)} (t^\rho / \rho)^{\gamma-1} \right|, \quad (\tau, t) \in [2\zeta, \mu] \times [2\zeta, \mu] \quad (4.1)$$

$$\mathcal{J} = \int_0^\zeta \tau^{\rho-\lambda-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau, \quad t \in [2\zeta, \mu]. \quad (4.2)$$

Easily, we can verify that $\Psi(\tau, t)$ and \mathcal{J} are uniformly continuous on $[2\zeta, \mu] \times [2\zeta, \mu]$ and $[2\zeta, \mu]$, respectively.

Next, $\forall t_1, t_2 \in [2\zeta, \mu]$, $t_1 < t_2$, by using equation (4.1) we have

$$\begin{aligned}
 & |x(t_1) - x(t_2)| \\
 &= \left| \frac{x_0}{\Gamma(\gamma)} (t_1^\rho / \rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} (t_1^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right. \\
 &\quad \left. - \left[\frac{x_0}{\Gamma(\gamma)} (t_2^\rho / \rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right] \right| \\
 &\leq \Psi(t_1, t_2) + \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \tau^{\rho-1} (t_1^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right. \\
 &\quad \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \\
 &\leq \Psi(t_1, t_2) + \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\zeta \tau^{\rho-\lambda-1} \left[(t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho - \tau^\rho)^{\alpha-1} \right] (\mathcal{M}x)(\tau) d\tau \right| \\
 &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\zeta^{t_1} \tau^{\rho-1} \left[(t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho - \tau^\rho)^{\alpha-1} \right] |\varphi(\tau, x(\tau))| d\tau \\
 &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} |\varphi(\tau, x(\tau))| d\tau \\
 &\leq \Psi(t_1, t_2) + \frac{\|\mathcal{M}x\|_{[0, \zeta]} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\zeta \tau^{\rho-\lambda-1} \left| \left[(t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \right| d\tau \\
 &\quad + \frac{\Lambda \rho^{1-\alpha}}{\Gamma(\alpha)} \int_\zeta^{t_1} \tau^{\rho-1} \left[(t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho - \tau^\rho)^{\alpha-1} \right] d\tau \\
 &\quad + \frac{\Lambda \rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} d\tau \\
 &\leq \Psi(t_1, t_2) + \frac{\|\mathcal{M}x\|_{[0, \zeta]} \rho^{1-\alpha}}{\Gamma(\alpha)} |\mathcal{J}(t_1) - \mathcal{J}(t_2)| \\
 &\quad + \frac{\Lambda}{\rho^\alpha \Gamma(\alpha + 1)} \left[2(t_2^\rho - t_1^\rho)^\alpha + (t_1^\rho - \zeta^\rho)^\alpha - (t_2^\rho - \zeta^\rho)^\alpha \right]. \tag{4.3}
 \end{aligned}$$

By uniform continuity of $\Psi(\tau, t)$ and $\mathcal{J}(t)$, together with the Cauchy convergence criterion, we get $\lim_{t \rightarrow \mu^-} x(t) = x^*$.

Step 2. In this part we show that $x(t)$ is continuable. Since E is a closed subset, we have $(\mu, x^*) \in E$. Define $x(\mu) = x^*$, then $x(t) \in C_{1-\gamma, \rho}[0, \mu]$. We denote

$$x_1(t) = \frac{x_0}{\Gamma(\gamma)} (t^\rho / \rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau, \quad t \in [\mu, \mu + 1], \tag{4.4}$$

and we define the operator \mathcal{K} as follows

$$(\mathcal{K}y)(t) = x_1(t) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, y(\tau)) d\tau, \quad t \in [\mu, \mu + 1], \tag{4.5}$$

where $y \in C[\mu, \mu + 1]$. In the light of Lemmas 3 and 6, we get $\mathcal{K}(C[\mu, \mu + 1]) \subset C[\mu, \mu + 1]$.

Now, assume that

$$S_k = \left\{ (t, y) : \mu \leq t \leq \mu + 1, |y| \leq \max_{\mu \leq t \leq \mu + 1} |x(t)| + k \right\}, \quad k > 0. \tag{4.6}$$

In the view of a continuity of φ on S_k , we can denote $\Theta = \max_{(t,y) \in S_k} |\varphi(t, y)|$.

Again, assume that

$$S_h = \left\{ y \in [\mu, \mu + h] : \max_{t \in [\mu, \mu + h]} |y(t) - x_1(t)| \leq k, \quad y(\mu) = x_1(\mu) \right\}, \tag{4.7}$$

where $h = \min \left\{ \left(\frac{k \rho^\alpha \Gamma(\alpha + 1)}{\Theta} \right)^{\frac{1}{\rho\alpha}}, 1 \right\}$. We can claim that the operator \mathcal{K} is a completely continuous on S_h . Firstly, we will show that \mathcal{K} is continuous. Put $\{y_n\} \subseteq C[\mu, \mu + h]$, $\|y_n - y\| \rightarrow 0$ as $n \rightarrow +\infty$. So, we have

$$\begin{aligned} & |(\mathcal{K}y_n)(t) - (\mathcal{K}y)(t)| \\ &= \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} [\varphi(\tau, y_n(\tau)) - \varphi(\tau, y(\tau))] d\tau \right| \\ &\leq \|\varphi(\tau, y_n(\tau)) - \varphi(\tau, y(\tau))\|_{[\mu, \mu + h]} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} d\tau \\ &\leq \|\varphi(\tau, y_n(\tau)) - \varphi(\tau, y(\tau))\|_{[\mu, \mu + h]} \frac{h^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha + 1)}. \end{aligned} \tag{4.8}$$

By a continuity of φ on S_k , we get $\|\varphi(\tau, y_n(\tau)) - \varphi(\tau, y(\tau))\|_{[\mu, \mu + h]} \rightarrow 0$ as $n \rightarrow +\infty$. Thus, $\|\mathcal{K}y_n - \mathcal{K}y\|_{[\mu, \mu + h]} \rightarrow 0$ as $n \rightarrow +\infty$, which yields that the \mathcal{K} is a continuous.

Secondly, we will prove that $\mathcal{K}S_h$ is equicontinuous. For any $y \in S_h$, we have $(\mathcal{K}y)(\mu) = x_1(\mu)$ and

$$\begin{aligned} |(\mathcal{K}y)(t) - x_1(t)| &= \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, y(\tau)) d\tau \right| \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} |\varphi(\tau, y(\tau))| d\tau \\ &\leq \frac{\Theta(t^\rho - \mu^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha + 1)} \leq \frac{h^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha + 1)} \leq k. \end{aligned} \tag{4.9}$$

Therefore, $\mathcal{K}S_h \subset S_h$.

Now, put

$$\mathcal{I}(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau.$$

By using Lemma 6, $\mathcal{I}(t)$ is a continuous on $[\mu, \mu + h]$. $\forall y \in S_h, \mu \leq t_1 < t_2 \leq \mu + h$, we have

$$\begin{aligned}
 & |(\mathcal{K}y)(t_1) - (\mathcal{K}y)(t_2)| \\
 & \leq \Psi(t_1, t_2) + \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \tau^{\rho-1} \left[(t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \varphi(\tau, x(\tau)) d\tau \right| \\
 & \quad + \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^{t_1} \tau^{\rho-1} \left[(t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho - \tau^\rho)^{\alpha-1} \right] \varphi(\tau, x(\tau)) d\tau \right| \\
 & \quad + \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \\
 & \leq \Psi(t_1, t_2) + |\mathcal{I}(t_1) - \mathcal{I}(t_2)| \\
 & \quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^{t_1} \tau^{\rho-1} \left[(t_1^\rho - \tau^\rho)^{\alpha-1} - (t_2^\rho - \tau^\rho)^{\alpha-1} \right] |\varphi(\tau, x(\tau))| d\tau \\
 & \quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \tau^{\rho-1} (t_2^\rho - \tau^\rho)^{\alpha-1} |\varphi(\tau, x(\tau))| d\tau \\
 & \leq \Psi(t_1, t_2) + |\mathcal{I}(t_1) - \mathcal{I}(t_2)| \\
 & \quad + \frac{\Lambda}{\rho^\alpha \Gamma(\alpha + 1)} \left[2(t_2^\rho - t_1^\rho)^\alpha + (t_1^\rho - \mu^\rho)^\alpha - (t_2^\rho - \mu^\rho)^\alpha \right]. \tag{4.10}
 \end{aligned}$$

In the view of uniform continuity of $\mathcal{I}(t)$ on $[\mu, \mu + h]$ and inequality (4.10), we conclude that $\{(\mathcal{K}y)(t) : y \in S_h\}$ is equicontinuous. Thus, the operator \mathcal{K} is completely continuous. Therefore, by Lemma 2 (Schauder fixed point theorem) \mathcal{K} has a fixed point $\hat{x}(t) \in S_h$, *i.e.*

$$\begin{aligned}
 \hat{x}(t) &= x_1(t) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, \hat{x}(\tau)) d\tau \\
 &= \frac{x_0}{\Gamma(\gamma)} (t^\rho / \rho)^{1-\gamma} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, \hat{x}(\tau)) d\tau, \quad t \in [\mu, \mu + h]
 \end{aligned} \tag{4.11}$$

where

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if } t \in (0, \mu], \\ \hat{x}(t) & \text{if } t \in [\mu, \mu + h]. \end{cases}$$

From Lemma 1, it follows that $\tilde{x} \in C_{1-\gamma, \rho}[0, \mu + h]$ and

$$\tilde{x}(t) = \frac{x_0}{\Gamma(\gamma)} (t^\rho / \rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, \tilde{x}(\tau)) d\tau.$$

Hence, in the light of Lemma 3, $\tilde{x}(t)$ is a solution of the initial value problem (1.3) on $[0, \mu + h]$. This gives contradiction because $x(t)$ is non-continuable. \square

Now, we will give the second theorem of continuation, which is applied more conveniently.

THEOREM 4. *Suppose that (\mathcal{H}_1) is satisfied. Then, $x = x(t)$, $t \in (0, \mu)$ is non-continuable if and only if*

$$\limsup_{t \rightarrow \mu^-} |\Phi(t)| = +\infty, \tag{4.12}$$

where $\Phi(t) = (t, x(t))$ & $|\Phi(t)| = \sqrt{t^2 + x^2(t)}$.

Proof. Firstly, assume that $x = x(t)$ is continuable. Then, there exists solution $\hat{x}(t)$ of initial value problems (1.3) defined on $(0, \hat{\mu})$ such that, $x(t) = \hat{x}(t)$ for $t \in (0, \mu)$, which yields that $\lim_{t \rightarrow \mu^-} x(t) = \hat{x}(\mu)$. Therefore, $|\Phi(t)| \rightarrow |\Phi(\mu)|$ as $t \rightarrow \mu^-$, which gives a contradiction.

Secondly, assume that equation (4.12) is not true. Then, there exist a sequence $\{t_m\}$ and $M > 0$, where M is positive constant, such that

$$\begin{aligned} t_m < t_{m+1}, \quad m \in \mathbb{N}, \\ \lim_{m \rightarrow \infty} t_m = \mu, \quad |\Phi(t_m)| \leq M, \\ \text{i.e. } t_m^2 + x^2(t_m) \leq M^2. \end{aligned} \tag{4.13}$$

Since $x(t_m)$ be the bounded convergent subsequence, without loss of generality, we put

$$\lim_{m \rightarrow \infty} x(t_m) = x^*. \tag{4.14}$$

Now, for any given $\delta > 0$, there exists $T \in (0, \mu)$ such that, $|x(t) - x^*| < \delta$, $t \in (T, \mu)$, we show that

$$\lim_{t \rightarrow \mu^-} x(t) = x^*. \tag{4.15}$$

For sufficiently small $\zeta > 0$, let

$$S_1 = \left\{ (t, x) : t \in [\zeta, \mu], |x| \leq \sup_{t \in [\zeta, \mu]} |x(t)| \right\}. \tag{4.16}$$

In the light of continuity of φ on S_1 , we denote $\Phi = \max_{(t,y) \in S_1} |\varphi(t, y)|$. From equations (4.13) and (4.14), it follows that there exists m_0 such that $t_{m_0} > \zeta$ and for $m \geq m_0$, we have

$$|x(t_m) - x^*| \leq \frac{\delta}{2}.$$

If (4.14) is not true, then for $m \geq m_0$, there exists $\xi_m \in (t_m, \mu)$ such that, for $t \in (t_m, \xi_m)$, $|x(t) - x^*| < \delta$ and $|x(\xi_m) - x^*| \geq \delta$. Hence,

$$\begin{aligned} \delta &\leq |x(\xi_m) - x^*| \leq |x(t_m) - x^*| + |x(\xi_m) - x(t_m)| \\ &\leq \frac{\delta}{2} + \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_m} \tau^{\rho-1} (t_m^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right. \\ &\quad \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\xi_m} \tau^{\rho-1} (\xi_m^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\delta}{2} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left| \int_0^\zeta \tau^{\rho-1} \left[(t_m^\rho - \tau^\rho)^{\alpha-1} - (\xi_m^\rho - \tau^\rho)^{\alpha-1} \right] \varphi(\tau, x(\tau)) d\tau \right| \\
 &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left| \int_\zeta^{t_m} \tau^{\rho-1} \left[(t_m^\rho - \tau^\rho)^{\alpha-1} - (\xi_m^\rho - \tau^\rho)^{\alpha-1} \right] \varphi(\tau, x(\tau)) d\tau \right| \\
 &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left| \int_{t_m}^{\xi_m} \tau^{\rho-1} (\xi_m^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \\
 &\leq \frac{\delta}{2} + \frac{\|\mathcal{M}x\|_{[0,\zeta]} \rho^{1-\alpha}}{\Gamma(\alpha)} |\mathcal{J}(t_m) - \mathcal{J}(\xi_m)| \\
 &\quad + \frac{\Phi}{\rho^\alpha \Gamma(\alpha + 1)} \left[2(\xi_m^\rho - t_m^\rho)^\alpha + (t_m^\rho - \zeta^\rho)^\alpha - (\xi_m^\rho - \zeta^\rho)^\alpha \right], \tag{4.17}
 \end{aligned}$$

where $\mathcal{J}(t)$ is defined in the equation (4.2). By a continuity of $\mathcal{J}(t)$ on $[t_{m_0}, \mu]$, and for sufficiently large $m \geq m_0$, we have

$$\begin{aligned}
 &\frac{\|\mathcal{M}x\|_{[0,\zeta]} \rho^{1-\alpha}}{\Gamma(\alpha)} |\mathcal{J}(t_m) - \mathcal{J}(\xi_m)| \\
 &\quad + \frac{\Phi}{\rho^\alpha \Gamma(\alpha + 1)} \left[2(\xi_m^\rho - t_m^\rho)^\alpha + (t_m^\rho - \zeta^\rho)^\alpha - (\xi_m^\rho - \zeta^\rho)^\alpha \right] < \frac{\delta}{2}. \tag{4.18}
 \end{aligned}$$

From equations (4.17) and (4.18), we obtain

$$\delta \leq |x(\xi_m) - x^*| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

This contradiction gives that $\lim_{t \rightarrow \mu^-} x(t)$ exists.

By using same argument as in a proof of the previous theorem, easily we can prove the continuation of $x(t)$. \square

In the next part of this section, we discuss the global existence of solutions for the initial value problem (1.3), which is based on results obtained previously. Applying the second theorem of continuation (Theorem 4), we can immediately obtain the following conclusion about the global existence of solution for the initial value problem (1.3).

THEOREM 5. *Suppose that (\mathcal{H}_1) is satisfied. Let $x(t)$ is the solution of the initial value problem (1.3) on $(0, \mu)$. If $x(t)$ is bounded on $[\zeta, \mu)$ for some $\zeta > 0$, then $\mu = +\infty$.*

For illustrative the above theorem we give the following example

EXAMPLE 2. We consider the initial value problem as following:

$$\begin{cases}
 {}_2D_{0+}^{\frac{1}{2}, \frac{1}{4}} x(t) = \varphi(t, x(t)), & t \in (0, +\infty), \\
 {}_2I_{0+}^{\frac{3}{8}} x(t)|_{t=0} = 1,
 \end{cases} \tag{4.19}$$

here, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\gamma = \frac{5}{8}$, $\rho = 2$ and $\varphi(t, x(t)) = \frac{\exp(-t^2 x \sin t)}{\sqrt{t}(1-t)}$.

By applying Theorem 1, we know that the initial value problem (4.19) has at least one a local solution $x(t)$ on $(0, h]$ for some $h > 0$. In the view of the Lemma 4, $x(t)$ satisfies the following integral equation:

$$x(t) = \frac{1}{\Gamma(\frac{5}{8})} (t^2/2)^{-\frac{3}{8}} + \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} \int_0^t \tau \frac{\exp(-\tau^2 x \sin \tau)}{\sqrt{\tau(t^2 - \tau^2)}(1 - \tau)} d\tau.$$

Hence,

$$|x(t)| \leq \frac{1}{\Gamma(\frac{5}{8})} (t^2/2)^{-\frac{3}{8}} + \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} \int_0^t \frac{\tau}{\sqrt{\tau(t^2 - \tau^2)}} d\tau.$$

Assume that $[0, \mu)$ with $(\mu < +\infty)$ be a maximum existing interval of $x(t)$. Easily, we can see that for any $\zeta \in (0, \mu)$, $x(t)$ is bounded on $[\zeta, \mu)$. By using Theorem 5, we have $\mu = +\infty$, i.e. the maximum existing interval of $x(t)$ is $(0, +\infty)$.

In the light of (Lemma 7.14, in [11], Lemma 7.1.1, in [12] and Theorem 1, in [25]), we state a generalization of Gronwall’s lemma for singular kernels which is essential for our discussion.

LEMMA 7. Assume that $\phi : [0, \mu] \rightarrow [o, +\infty)$ is a real function and $\theta(\cdot)$ is a non-negative locally integrable function on $[0, \mu]$. Also, let there exists $\rho > 0$, $\omega > 0$, and $0 < \alpha < 1$, such that

$$\phi(t) \leq \theta(t) + \omega \int_0^t \tau^{\rho-1} \frac{\rho^\alpha}{(t^\rho - \tau^\rho)^\alpha} \phi(\tau) d\tau.$$

Then, there exists a constant $C = C(\alpha)$ such that for $t \in [0, \mu]$, we have

$$\phi(t) \leq \theta(t) + C \omega \int_0^t \tau^{\rho-1} \frac{\rho^\alpha}{(t^\rho - \tau^\rho)^\alpha} \theta(\tau) d\tau.$$

THEOREM 6. Suppose that (\mathcal{H}_1) is satisfied and there exist three non-negative continuous functions $f(t)$, $g(t)$, $\psi(t) : [0, +\infty) \rightarrow [0, +\infty)$ such that $|\varphi(t, x(t))| \leq g(t)f(|x(t)|) + \psi(t)$, where $g(\eta) \leq \eta$ for $\eta \geq 0$. Then, the initial value problem (1.3) has one solution in $C_{1-\gamma, \rho}[0, +\infty)$.

Proof. The local existence of a solution of the initial value problem (1.3) can be deduced from Theorem 1. By using Lemma 4, $x(t)$ satisfies the second kind Volterra fractional integral equation

$$x(t) = \frac{x_0}{\Gamma(\gamma)} (t^\rho/\rho)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \varphi(\tau, x(\tau))}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau.$$

Assume that $[0, \mu)$ with $(\mu < +\infty)$ be a maximum existing interval of $x(t)$. Then, we have

$$\begin{aligned}
 & |(t^\rho/\rho)^{1-\gamma}x(t)| \\
 &= \left| \frac{x_0}{\Gamma(\gamma)} + (t^\rho/\rho)^{1-\gamma} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau \right| \\
 &\leq \frac{x_0}{\Gamma(\gamma)} + \frac{(t^\rho/\rho)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} |\varphi(\tau, x(\tau))| d\tau \\
 &\leq \frac{x_0}{\Gamma(\gamma)} + \frac{\mu^{\rho(1-\gamma)}}{\rho^{\alpha-\gamma}\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} [g(\tau)f((\tau^\rho/\rho)^{1-\gamma}|x(\tau)|) + \psi(\tau)] d\tau \\
 &\leq \frac{x_0}{\Gamma(\gamma)} + \frac{\mu^{\rho(1-\gamma)}}{\rho^{\alpha-\gamma}\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} [g(\tau)f((\tau^\rho/\rho)^{1-\gamma}|x(\tau)|)] d\tau \\
 &\quad + \frac{\mu^{\rho(1-\gamma)}}{\rho^{\alpha-\gamma}\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \psi(\tau) d\tau \\
 &\leq \frac{x_0}{\Gamma(\gamma)} + \frac{\mu^{\rho(1-\gamma)}}{\rho^{\alpha-\gamma}\Gamma(\alpha)} \|g\|_{[0,\mu]} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} f((\tau^\rho/\rho)^{1-\gamma}|x(\tau)|) d\tau \\
 &\quad + \frac{\mu^{\rho(1-\gamma)}}{\rho^{\alpha-\gamma}\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \psi(\tau) d\tau.
 \end{aligned}$$

Now, we taking

$$\phi(t) = (t^\rho/\rho)^{1-\gamma}|x(t)|, \quad \theta(t) = \frac{x_0}{\Gamma(\gamma)} + \frac{\mu^{\rho(1-\gamma)}}{\rho^{\alpha-\gamma}\Gamma(\alpha)} \int_0^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \psi(\tau) d\tau$$

and

$$\omega = \frac{\mu^{\rho(1-\gamma)}}{\rho^{\alpha-\gamma}\Gamma(\alpha)} \|g\|_{[0,\mu]}.$$

By applying Lemma 7, we can see that $\phi(t) = (t^\rho/\rho)^{1-\gamma}|x(t)|$ is bounded on $[0, \mu)$. Hence, for any $\zeta \in (0, \mu)$, $x(t)$ is bounded on $[\zeta, \mu)$. By using Theorem 5, the initial value problem (1.3) has a solution $x(t)$ on $[0, +\infty)$. \square

The next theorem guarantees the existence and uniqueness of global solution for the initial value problem (1.3) on \mathbb{R}^+ .

THEOREM 7. *Suppose that (\mathcal{H}_1) is satisfied and there exists a nonnegative continuous function $g(t)$ defined on $[0, +\infty)$ such that $|\varphi(t, x(t)) - \varphi(t, y(t))| \leq g(t)|x(t) - y(t)|$. Then, the initial value problem (1.3) has a unique solution in $C_{1-\gamma, \rho}[0, +\infty)$.*

We can obtained the existence of a global solution by using similar arguments as above. By applying Lipschitz condition and Lemma 7, we can deduced the uniqueness of global solution. Here, we omitted the proof.

5. Concluding remarks

REMARK 1. If we take $\rho = 1$, in the initial value problems (1.3) and (1.4), then

1. the local existence Theorems 1 and 2, yield the local existence Theorems 1 and 2 [6], respectively, associated with Hilfer-type fractional differential equations with the initial value problems (5) and (6) [6], respectively.
2. For $\beta = 0$, the local existence Theorem 1 yields the local existence Theorem 3.1 [18], associated with Riemann-Liouville-type fractional differential equations with the initial value problem (1) [18].
3. For $\beta = 1$ the local existence Theorems 1 and 2, yield the local existence Theorems 3.1 and 3.2 [22], respectively, associated with Caputo-type fractional differential equations with the initial value problems (1.1) and (1.2) [22], respectively.

REMARK 2. If we take $\rho = 1$, in the initial value problems (1.3) and (1.4), then

1. the continuation Theorems 3 and 4, reduce to the continuation Theorems 3 and 4, for Hilfer-type fractional differential equations [6], respectively.
2. For $\beta = 0$, the continuation Theorems 3 and 4, reduce to the continuation Theorems 4.1 and 4.2, for Riemann-Liouville-type fractional differential equations [18], respectively.
3. For $\beta = 1$ the continuation Theorems 3 and 4, reduce to the continuation Theorems 4.2 and 4.4, for Caputo-type fractional differential equations [22], respectively.

REMARK 3. In this article, we proved new existence theorems of a local solutions for the generalized fractional differential equations, which is Hilfer-Katugampola-type with the certain singularity functions. Also, we obtained two continuation theorems and we established global existence theorems for Hilfer-Katugampola-type fractional differential equations, which had been not investigated before. Our discussion in this article generalizes the existing results in the literature.

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