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LIAPUNOV FUNCTIONS FOR NEURAL NETWORK MODELS

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Abstract. The dynamical behaviour of continuous time recurrent neural network models is studied with emphasis on global stability of a unique equilibrium. First we show in a unified context two Liapunov functions that were introduced in the nineties by Hopfield, Grossberg, Matsouka and Forti. Then we introduce a class of networks for which the model becomes a special cooperative system with a unique globally stable steady state. Finally, we show that periodic orbits may occur when the sufficient conditions for the existence of Liapunov functions are violated.

1. Introduction

Modelling neural networks has led to many mathematical challenges and research has been carried out in different branches of applied mathematics. The focus of this paper is global stability in continuous time recurrent neural networks with emphasis on the application of Liapunov functions. Consider a neural network with n nodes and with edge weights w_{ij} from node j to node i. We can use the adjacency matrix $W = (w_{ij})_{i,j=1,2,\dots,n}$ to represent the network. The activity rate of node i at time t is denoted by $x_i(t)$. In this paper we study the widely used Hopfield or Cowan-Wilson model

$$\dot{x} = -Ax + Wy + I, \qquad y_i = f_i(x_i), \tag{1.1}$$

where $I \in \mathbb{R}^n$ is the input, A is a diagonal matrix with positive entries, $f_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing ($f_i' > 0$), differentiable activation function having finite limits at both infinities. See [9, 10, 17] and equation (13.3) in Chapter 13 of [6]. The most frequently studied activation functions have the form

$$f(x) = \frac{1}{1+x^2}, \quad f(x) = \frac{1}{1+e^{-x}}, \quad f(x) = \tanh(x)$$

or are derived from these functions by simple linear transformations as $af(bx+c)$. The theorems presented in this paper are formulated for a general class of activation

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functions containing all these functions, i.e. both the symmetric and non-symmetric cases that are investigated in the literature [18].

The model can be formulated in a slightly different form, if a bias term Θ_i is introduced to each neuron. Then the i-th equation of the above system takes the form

$$\dot{x}_i = -a_i x_i + \sum_{j=1}^{n} w_{ij} f(x_j + \Theta_j) + I_i, \tag{1.2}$$

i.e. the term $f(x_j + \Theta_j)$ takes over the role of the term $f_j(x_j)$.

The research focused to synchronization of neurons originally, some recent review articles about this topic are [1, 15, 18]. The dynamical behaviour of (1.1) has also been studied since the pioneering works of Grossberg and Hopfield [9, 10]. The parameters I_i and Θ_i are used as bifurcation parameters in several papers considering small networks, e.g. [2, 3, 5], where the detailed bifurcation diagrams were determined analytically. Local bifurcations can be determined analytically also for larger networks when there are only four different weights in W [7]. In that paper the effect of inhibitory neurons is also studied in detail for a weight matrix with special structure and with parameters I_i and Θ_i. However, our focus in this paper is on the effect of the weights on the dynamical behaviour of the system.

It is known that a wide spectrum of dynamical behaviours can occur in this system depending on the values of the different parameters. Our goal here is to identify some important classes of networks when the trajectories tend to steady states or there is a globally stable steady state. In order to verify global stability, appropriately chosen Liapunov functions can be used. Before turning to the question of global stability, we note that simple calculation shows that the cube $[-K, K]^n$ is positively invariant for a suitable chosen value of K, and all trajectories enter this cube in finite time. Then a straightforward application of Brouwer's fixed point theorem shows that there is a steady state in the cube.

The structure of the paper is as follows. In the next two sections we show two Liapunov functions that were introduced in the nineties by Hopfield, Grossberg, Matsouka and Forti [8, 9, 10, 14]. In the fourth section we consider a class of networks for which (1.1) becomes a special cooperative system with a unique globally stable steady state. For this case the methods of monotone dynamical systems can be applied [16]. In the fifth section we will show that periodic orbits may occur when the conditions for the existence of Liapunov functions are violated. The periodic solutions are given analytically when the network is a directed cycle, the weights are nonnegative and the activation function is a step function. For sake of completeness, we present the proofs even for the previously known Liapunov functions. This also enables the Reader to see the extent to which the given Liapunov function can be generalised.

2. Cohen-Grossberg type Liapunov function

The strongest motivation for studying system (1.1) was the observation that all solutions tend to equilibria when the weight matrix is symmetric, paving the way for modelling addressable memory with a simple dynamical system. Hence in this section we assume that matrix W is symmetric and introduce a Liapunov function that enables us to prove that every trajectory tend to a steady state. This function was used in Hopfield's original paper [10] and was generalized to a wider class of dynamical systems by Cohen and Grossberg [9].

Let $F_i : \mathbb{R} \to \mathbb{R}$ be a differentiable function, for which $F_i'(z) = z f_i'(z)$ holds. Consider the following Liapunov function:

$$V(x) = \sum_{i=1}^{n} a_i F_i(x_i) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} f_i(x_i) f_j(x_j) - \sum_{i=1}^{n} I_i f_i(x_i).$$

Its partial derivative with respect to the k-th variable is

$$\partial_k V(x) = a_k F_k'(x_k) - \frac{1}{2} \sum_{j=1}^{n} w_{kj} f_k'(x_k) f_j(x_j) - \frac{1}{2} \sum_{i=1}^{n} w_{ik} f_i(x_i) f_k'(x_k) - I_k f_k'(x_k).$$

The symmetry of the weight matrix W implies that the last two terms are equal, hence

$$\partial_k V(x) = a_k x_k f'(x_k) - \sum_{i=1}^{n} w_{ik} f_i(x_i) f_k'(x_k) - I_k f_k'(x_k) = f_k'(x_k) \left(a_k x_k - \sum_{i=1}^{n} w_{ik} f_i(x_i) - I_k \right).$$

Observe that $V'(x)$ is zero if and only if x is a steady state, because $f' > 0$. Thus the Liapunov function can have a maximum or minimum only at the steady states. The calculation below shows that V is strictly increasing along other solutions, therefore every trajectory tends to an equilibrium.

Let $V^*(t) = V(x(t))$, where $t \mapsto x(t)$ is a non-constant solution of system (1.1). Then

$$\dot{V}^*(t) = \sum_{k=1}^{n} \partial_k V(x(t)) \dot{x}_k(t) = - \sum_{k=1}^{n} f'(x_k) \left(a_k x_k - \sum_{i=1}^{n} w_{ik} f_i(x_i) - I_k \right)^2 < 0.$$

Hence we have proved the following theorem.

THEOREM 1. *Assume that the weight matrix W is symmetric. Then every trajectory of system* (1.1) *tends to an equilibrium.*

Now we turn to another Liapunov function introduced in the nineties.

3. The Matsuoka–Forti Liapunov function

In this section we present a global stability result obtained by using a Liapunov function introduced independently by Matsuoka and Forti [14] [8]. Forti introduced the Liapunov function for a wider class of equations, hence his result is slightly weaker than Matsuoka's. Therefore, we follow Matsuoka's proof. Denote an equilibrium point of system (1.1) by $x^* \in \mathbb{R}^n$. Let us introduce the functions $g_i(z) = f_i(z + x_i^*) - f_i(x_i^*)$. These are strictly increasing differentiable functions with $g_i(0) = 0$. Let us denote their integral by $G_i(z) = \int_0^z g_i(u) \mathrm{d}u$. The relation $G_i' = g_i$ implies that the functions G_i have a strict minimum at 0.

Matsuoka introduced the Liapunov function

$$V(x) = G_1(x_1 - x_1^*) + G_2(x_2 - x_2^*) + \ldots + G_n(x_n - x_n^*).$$

It has a strict minimum at x^* because G_i have a strict minimum at 0 for all i. Let us calculate the derivative of V along solutions of system (1.1). Let x be a solution of (1.1) and let $V^*(t) = V(x(t))$. Then

$$\dot{V}^*(t) = \sum_{i=1}^{n} \partial_i V(x(t))\dot{x}_i(t) = \sum_{i=1}^{n} \partial_i V(x(t))\Big(-a_i x_i(t) + \sum_{j=1}^{n} w_{ij} f_i(x_j(t)) + I_i\Big).$$

Using the definition of V and that $G_i' = g_i$, we get (after omitting t)

$$\begin{aligned}\dot{V}^* &= \sum_{i=1}^{n} g_i(x_i - x_i^*)\Big(-a_i x_i + \sum_{j=1}^{n} w_{ij} f_j(x_j) + I_i\Big)\\ &= \sum_{i=1}^{n} (f_i(x_i) - f_i(x_i^*))\Big(-a_i x_i + \sum_{j=1}^{n} w_{ij} f_j(x_j) + I_i\Big).\end{aligned}$$

Using that

$$a_i x_i^* - \sum_{j=1}^{n} w_{ij} f_j(x_j^*) - I_i = 0$$

we can put the left hand side of this equation into the equation above, leading to

$$\dot{V}^* = \sum_{i=1}^{n} (f_i(x_i) - f_i(x_i^*))\Big(a_i(x_i^* - x_i) + \sum_{j=1}^{n} w_{ij} (f_j(x_j) - f_j(x_j^*))\Big).$$

For the sake of brevity, let us introduce the notations $u_i = x_i - x_i^*$ and $y_i = f_i(x_i) - f_i(x_i^*)$. Then

$$\begin{aligned}\dot{V}^* &= \sum_{i=1}^{n} y_i\Big(-a_i u_i + \sum_{j=1}^{n} w_{ij} y_j\Big) = y^\top(-a \circ u + Wy)\\ &= y^\top\Big(\frac{1}{M} y - a \circ u\Big) + y^\top\Big(-\frac{1}{M} y + Wy\Big).\end{aligned}$$

Observe that denoting the maximum of all f_i' by M, i.e. assuming $0 \leqslant f_i' \leqslant M$ for all i, if $a_i \geqslant 1$ $(i = 1, \ldots, n)$, we have $y^\top(1/M \cdot y - a \circ u) \leqslant 0$, since the mean value theorem yields $f_i(x_i) - f_i(x_i^*) = f_i'(c_i)(x_i - x_i^*)$, leading to

$$y_i\Big(\frac{1}{M} y_i - a_i u_i\Big) = d_i u_i\Big(\frac{1}{M} d_i u_i - a_i u_i\Big) = -d_i\Big(a_i - \frac{1}{M} d_i\Big) u_i^2 \leqslant 0,$$

where $d_i = f'(c_i) \leqslant M$.

On the other hand, the matrix in the quadratic form $y^\top(-1/M \cdot y + Wy)$, can be made symmetric $y^\top(-1/M \cdot y + Wy) = y^\top(-1/M \cdot I + 1/2(W + W^\top))y$. Summarising

$$\dot{V}^* \leqslant y^\top\Big(-\frac{1}{M} I + \frac{1}{2}(W + W^\top)\Big) y,$$

that is, V serves as a Liapunov function to system (1.1) if this matrix is negative definite. The above matrix is negative definite when the eigenvalues of $1/2(W+W^\top)$ are less than $1/M$. Since V has a strict minimum in x^*, then every trajectory tends to x^*, so the system has a unique steady state. Hence we proved the following.

THEOREM 2. *Assume that $0 \leqslant f_i' \leqslant M$ and $a_i \geqslant 1$ for all i. If the eigenvalues of matrix $1/2(W+W^\top)$ are less than $1/M$, then the equilibrium of system (1.1) is unique and asymptotically stable.*

This theorem is formulated for a general class of activation functions. This can be translated to the most widely studied special case when all the activation functions take the form $f(x) = a/(1+e^{-bx+c})$. Then we can formulate a condition for the parameters a,b,c for which global stability holds. In more detail, denoting by $\lambda_{max} > 0$ the greatest eigenvalue of $1/2(W+W^\top)$, a sufficient condition of global asymptotic stability is $a,b > 0$ and $ab < 4/\lambda_{max}$. We note that the factor $1/4$ is obtained as the maximum value of the derivative of $1/(1+e^{-x})$.

Since the eigenvalues are difficult to compute, it is useful to reformulate the theorem in terms of a matrix norm instead of the eigenvalues. The absolute values of the eigenvalues are less than any norm of the matrix, e.g. the maximum of the sums of rows, hence the following is a consequence of the theorem.

COROLLARY 1. *Assume that $0 \leqslant f_i' \leqslant M$, $a_i \geqslant 1$ and $1/2\sum_{j=1}^{n} |w_{ij}+w_{ji}| < 1/M$ for all i. Then the equilibrium of system (1.1) is unique and asymptotically stable.*

4. Lajmanovich-Yorke type Liapunov function

Studying an epidemic model, Lajmanovich and Yorke introduced a Liapunov function that enabled them to prove a global stability result [13]. Inspired by their idea, Bodó and Simon introduced a class of dynamical systems, for which the local stability of the trivial steady state determines the global stability of the system [4]. Here we present this general global stability result, relate it to the theory of monotone dynamical systems and then we generalize it to a wide class of activation functions. The result presented in [4] is valid for the activation function $1/(1+e^{-x})$, here we show that global stability holds when the activation function is strictly concave and positive in the positive half line.

Consider the dynamical system

$$\dot{x}(t) = g(x(t)), \tag{4.1}$$

in the cube $Q = \{x \in \mathbb{R}^n : 0 \leqslant x_i \leqslant K, i = 1,\dots,n\}$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable function satisfying the following assumptions.

(A1) The cube Q is strictly positively invariant, i.e. $x_k = 0$ implies $g_k(x) > 0$ and $x_k = K$ implies $g_k(x) < 0$.

(A2) If k is an index such that $x_k > y_k > 0$ and $x_k/y_k \geqslant x_j/y_j$ for all j, then $y_k g_k(x) < x_k g_k(y)$.

The following Proposition can be easily seen.

PROPOSITION 1. *The assumption (A2) implies the Kamke-Müller condition and strict R-concavity.*

(KM) *If* $u_i \leqslant v_i$ *for all* i *and* $u_k = v_k$, *then* $g_k(u) \leqslant g_k(v)$.

(SRC) *If* $x_i > 0$ *and* $g_i(x) = 0$ *for all* i *implies* $g_i(\lambda x) > 0$ *for all* i *when* $0 < \lambda < 1$.

Proof. We show that (A2) implies (SRC). Let $g_i(x) = 0$, $x_i > 0$ for all i and $y = \lambda x$ for $0 < \lambda < 1$. Then we can apply the property (A2) to every coordinate which yields $0 = y_i g_i(x) < x_i g_i(y)$ which is equivalent to $0 < g_i(\lambda x)$.

At last suppose that $u_i \leqslant v_i$ for all i and $u_k = v_k$. Now apply $(A2)$ to $x = u + \delta e_k$ and $y = v$. Then $(u_k + \delta)/v_k > 1 \geqslant u_i/v_i$ so $v_k g_k(u + \delta e_k) < (v_k + \delta) g_k(v)$ and taking the limit $\delta \searrow 0$ we get $g_k(u) \leqslant g_k(v)$. $\square$

REMARK 1. Strict R-concavity does not imply property (A2) even in one dimension.

To show this define $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \begin{cases} x^2 & \text{if } 0 \leqslant x < 1, \\ 2 - x & \text{if } x \geqslant 1 \end{cases}$$

and let $0 < y < x < 1$. Function g is (SRC), but $yx^2 > xy^2$.

Hence a system satisfying (A2) is a monotone dynamical system, see Section 3.1 in [16]. Moreover, (KM) and (SRC) together imply that there is at most one steady state ([12] Theorem 1.). The positive invariance of the cube implies the existence of the steady state, hence it has exactly one steady state. Cooperative systems with a unique equilibrium are globally stable when every forward semi-orbit has compact closure, [11], hence we have the following theorem.

THEOREM 3. *Assume that system* (4.1) *satisfies assumptions (A1) and (A2). Then there exists a (coordinate-wise) positive steady state that is globally asymptotically stable in* Q.

Now we apply Theorem 3 to system (1.1) when the inputs and the weights are nonnegative and each row of the weight matrix contains at least one non-zero entry. We note that this last assumption is not restrictive, since it means that there is now isolated neuron in the network. (Isolated neurons can be treated separately and the remaining part of the network can be considered instead of the original one.) It is easy to see that $x_i = 0$ implies $\dot{x}_i > 0$, and there exist a number K, such that $x_i = K$ implies $\dot{x}_i < 0$ for all indices i. Hence the cube $Q = [0, K]^n$ is strictly positively invariant. We show that assumption (A2) holds if the function f is strictly concave in the positive half of the real line and the inputs are non-negative. Assumption (A2) can be written as follows: assuming $x_k > y_k$ and $x_k/y_k \geqslant x_j/y_j$ for all j, we need to verify that

$$y_k g_k(x) = I_k y_k - a_k x_k y_k + \sum_{j=1}^{n} w_{jk} f_j(x_j) y_k < I_k x_k - a_k x_k y_k + \sum_{j=1}^{n} w_{jk} f_j(y_j) x_k = x_k g_k(y).$$

Hence it is enough to prove that $x_k > y_k$ and $x_k/y_k \geqslant x_j/y_j$ imply

$$f_j(x_j)y_k < f_j(y_j)x_k. \tag{4.2}$$

This will be proved by using the following proposition.

PROPOSITION 2. *If $f:[0,+\infty)\to\mathbb{R}$ is strictly concave and positive, then $f(cx) < cf(x)$ holds for all $c>1$ and for all $x>0$.*

Proof. Applying the definition of concavity in the interval $[0,cx]$ to the interior point $x = 1/c\cdot cx + (1-1/c)\cdot 0$, we obtain

$$f(x) = f\left(\frac{1}{c}cx + \left(1-\frac{1}{c}\right)\cdot 0\right) > \frac{1}{c}f(cx) + \left(1-\frac{1}{c}\right)f(0) > \frac{1}{c}f(cx),$$

that yields the a statement. □

Now we prove that inequality (4.2) holds. Introducing $c = x_k/y_k > 1$, the inequality $x_j/y_j \leqslant c$ holds for all j. Inequality (4.2) can be written in the form $f_j(x_j)y_k < f_j(y_j)cy_k$, hence we need to show that $f_j(x_j) < f_j(y_j)c$. The monotonicity of f_j implies $f_j(x_j) \leqslant f_j(cy_j)$, since $x_j \leqslant cy_j$. Therefore it is enough to prove that $f_j(cy_j) < cf_j(y_j)$, which follows directly from the concavity of f_j.

Thus we have proved the following theorem.

THEOREM 4. *Assume that the functions f_k are strictly concave in the half line $[0,+\infty)$, $I_k \geqslant 0$ for all k, all the entries of matrix W are nonnegative and each row of the weight matrix contains at least one non-zero entry. Then the equilibrium of system (1.1) is unique and asymptotically stable.*

We note that the result in [4] follows from this theorem when it is applied to the activation function having the form $f(x) = a/1+e^{-(bx+c)}$. Then for $a,b>0$ and $c\geqslant 0$ the conditions of the Theorem above hold, i.e. global stability follows. We note that here we have a restriction for the translation parameter c, while Theorem 2 holds for any value of c, instead it restricts the product ab.

5. Periodic solution

Theorems 2 and 4 yield sufficient conditions for the global stability of a unique steady state. Now, we investigate to what extent are these conditions necessary when the weights are non-negative. Theorem 4 requires the activation function to be concave on the positive half line, while in Theorem 2 the slope of the activation function is under a certain bound. We show an activation function that violates these conditions and enables the existence of a periodic solution, i.e. excludes global stability. (The system remains cooperative, hence the periodic orbit will not be stable [16].) The activation function will be the step function

$$f(x) = \begin{cases} 0 & \text{if } x < 1/2, \\ 1 & \text{if } x > 1/2. \end{cases} \tag{5.1}$$

because of two reasons. On one hand, analytical results are possible to achieve by manipulating with explicit formulas, on the other hand, these kind of functions can be thought of as the limits of sigmoid functions with increasing slope. This step function violates the conditions of the Theorems above.

In order to have an explicit formula for the periodic orbit, the network is chosen to be a directed cycle i.e. $w_{1n} = 1$ and $w_{i,i-1} = 1$ for $i = 2, \dots, n$, and all other entries of W are zero. In this case the differential equation takes the following form:

$$\dot{x}_1(t) = f(x_n(t)) - x_1(t), \tag{5.2}$$

$$\dot{x}_i(t) = f(x_{i-1}(t)) - x_i(t) \quad \text{for } i = 2, \dots, n \tag{5.3}$$

We will show that with a properly chosen T-periodic function p the periodic solution of this system takes the form $x_i(t) = p(t-(i-1)l)$ for $i = 1, 2, \dots, n$, where $T = nl$.

PROPOSITION 3. *Let $T, l \in \mathbb{R}$ be such that $nl = T$. Assume that there exists a continuous, T-periodic function p satisfying the inequalities*

$$p(t) < \frac{1}{2}, \text{ when } t \in (l, T/2 + l) \quad \text{and} \quad p(t) > \frac{1}{2}, \text{ when } t \in (T/2 + l, T + l) \tag{5.4}$$

and the differential equation

$$\dot{p}(t) = q(t) - p(t), \tag{5.5}$$

where q is a T-periodic function satisfying

$$q(t) = \begin{cases} 0 & \text{if } 0 < t < T/2, \\ 1 & \text{if } T/2 < t < T. \end{cases}$$

Then the functions x_i defined as $x_i(t) = p(t-(i-1)l)$ for $i = 1, 2, \dots, n$ are solutions of equations (5.2)-(5.3).

Proof. Observe first that $f(x_i(t)) = q(t - il)$. This follows from the fact that both sides can be only zero or one. The left hand side is zero when $x_i(t) < 1/2$, that is when $p(t-(i-1)l) < 1/2$. This inequality holds if $t-(i-1)l \in (l, T/2+l)$, i.e. when $t - il \in (0, T/2)$, which means that $q(t - il) = 0$, that is the right hand side is zero. In a similar way, one can see that if the left hand side is one then so is the right hand side. Then taking an index $i = 2, \dots, n$ we have

$$\dot{x}_i(t) = \dot{p}(t-(i-1)l) = q(t-(i-1)l) - p(t-(i-1)l) = f(x_{i-1}(t)) - x_i(t),$$

that is x_i satisfies (5.3). Similarly, it is easy to check that x_1 satisfies (5.2) as follows.

$$\dot{x}_1(t) = \dot{p}(t) = q(t) - p(t) = q(t - nl) - p(t) = f(x_n(t)) - x_1(t). \quad \square$$

Finally, we prove the existence of the periodic function p satisfying the Proposition above.

Solving the differential equation (5.5) in $(0, T/2)$ we get $p(t) = c_1 \mathrm{e}^{-t}$ and in $(T/2, T)$ we have $p(t) = 1 + c_2 \mathrm{e}^{-t}$. The continuity of p at $T/2$ and at T and its periodicity require

$$c_1 \mathrm{e}^{-T/2} = 1 + c_2 \mathrm{e}^{-T/2} \quad \text{and} \quad c_1 = 1 + c_2 \mathrm{e}^{-T}. \tag{5.6}$$

The inequalities in (5.4) hold if $p(l) = 1/2 = p(T/2 + l)$ and p is strictly decreasing (i.e. $c_1 > 0$) in $(0, T/2)$ and strictly increasing (i.e. $c_2 < 0$) in $(T/2, T)$. So we need

$$c_1 \mathrm{e}^{-l} = \frac{1}{2} = 1 + c_2 \mathrm{e}^{-l-T/2}. \tag{5.7}$$

Substituting c_1 and c_2 from (5.7) into the equations in (5.6), we can observe that the two equations are identical, both are equivalent to

$$\mathrm{e}^{l} + \mathrm{e}^{l-T/2} = 2.$$

Now we show that for any $n \geqslant 5$ there is a positive period T, for which this equation holds with $l = T/n$. Introducing $x = \mathrm{e}^{l}$ and using $T = nl$ this equation is equivalent to

$$h(x) := x + x^{1-n/2} = 2.$$

It is straightforward to check that $h(1) = 2$, $h'(1) < 0$ and $\lim_{x \to +\infty} h(x) = +\infty$. Hence the continuity of h implies that there is $x^* > 1$, for which $h(x^*) = 2$. Hence $l = \log(x^*)$ and $T = nl$ yield the required period.

Thus we have proved the following result about the periodic solution.

THEOREM 5. *If $n \geqslant 5$ is a natural number, W is a directed cycle with weights 1, the activation function f takes the form* (5.1)*, $I_k = 0$, $a_k = 1$, then* (1.1) *has a periodic solution.*

Conclusion

The main goal of this paper is to study the global stability of a widely used continuous time recurrent neural network model, given in (1.1). In the second and third sections we presented two previously known Liapunov functions, which can be used to guarantee stability under certain restrictions on the network. The Cohen-Grossberg type Liapunov function requires the network to be symmetric. The Matsuoka–Forti Liapunov function roughly requires that the entries of the matrix are relatively small compared to the steepness of the activation function.

In the fourth section we showed that global stability can also be guaranteed when the entries of the matrix are nonnegative and the activation function is positive and strictly concave in the positive half line.

In the fifth section we proved that if we relax the conditions on the activation function given in the previous two sections, then we can choose a network and an activation function such that a periodic solution exist. That is violating the conditions ensuring the existence of the Liapunov functions we loose global stability.

In the future, we plan to extend these results to the case when there are negative weights as well in certain columns, by using monotone systems.

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