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THREE DIMENSIONAL SYSTEM OF GLOBALLY MODIFIED MAGNETOHYDRODYNAMICS EQUATIONS WITH INFINITE DELAYS

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Abstract. Existence and uniqueness of strong solutions for three dimensional system of globally modified magnetohydrodynamics equations containing infinite delays terms are established together with some qualitative properties of the solution in this work. The existence is proved by making use of Galerkin's method, Cauchy-Lipshitz's theorem, a priori estimates, the Aubin-Lions compactness theorem. Moreover, we study the asymptotic behavior of the solution.

1. Introduction and statement of the problem

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary $\Gamma = \partial\Omega$, and $N > 0$ be fixed. We define $F_N : (0,+\infty) \to (0,1]$ by

$$F_N(r) = \min\left\{1, \frac{N}{r}\right\},\ r \in \mathbb{R}^+$$

and consider the following system of globally modified magnetohydrodynamics equations (GMMHDE)

$$\begin{cases} \dfrac{\partial u}{\partial t} + F_N(\|u\|_{V_1})[(u\cdot\nabla)u] - \dfrac{1}{R_e}\Delta u - SF_N(\|(u,B)\|_V)[(B\cdot\nabla)B] \\ +\nabla\left(p + S\dfrac{|B|^2}{2}\right) = f_1(t) \text{ in } (\tau,T)\times\Omega, \\ \dfrac{\partial B}{\partial t} + F_N(\|(u,B)\|_V)[(u\cdot\nabla)B - (B\cdot\nabla)u] + \dfrac{1}{R_m}\text{curl}(\text{curl}B) = f_2(t) \text{ in } (\tau,T)\times\Omega, \\ \text{div } u = 0,\ \text{div } B = 0 \text{ in}(\tau,T)\times\Omega, \\ u(\tau,x) = u_0(x),\ B(\tau,x) = B_0(x) \text{ for all } x\in\Omega, \\ u = 0,\ \ B\cdot n = 0 \text{ and curl}B\times n = 0 \text{ on } \Gamma, \end{cases} \tag{1.1}$$

where u, B and p represent respectively the fluid velocity, the magnetic field and the pressure. f_1 and f_2 are given external forces fields. R_e and R_m are the so-called

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Paper DEA-13-21

Reynolds and magnetic Reynolds numbers, respectively and $S = \frac{M^2}{R_e R_m}$ is a positive constant, where M is the Hartman number. $|B|^2 = B \cdot B$ and represents the length of the magnetic field, n is the unit outward normal on Γ and τ the initial time. It is manifest that the system of equations in (1.1) does not represent the MHD model due to un-physical terms introduced such as $F_N(\|u\|_{V_1}), F_N(\|(u,B)\|_V)$. These terms can find their existence in the original model of globally modified Navier Stokes introduced in [5]. As clearly demonstrated in [5], $F_N(\|u\|_{V_1})$ prevent the rapid grow of velocity gradient and help to obtain uniqueness of weak solution in 3d, property which is lacking for Navier Stokes in 3d. Hence Mathematically, there is a merit of studying this system. Recently globally modified Navier Stokes coupled with the magnetic field or the heat equation have been proposed and analysed in [13, 14, 15]. It is clearly observed in those later works that the “perturbation terms” added play a crucial role in describing the unique solvability of the system. Just like the MHD model (cf. [9]), the expressions describing the coupling between the velocity and magnetic fields are represented. The question we would like to investigate in this work is simple and summarizes as follows: The system of equations (1.1) has a unique strong solution and stable, what happens if there is a delay?

This question has been answered in [14] where finite delays were considered.

Many problems in applied science, physics, and engineering are modeled mathematically by delay differential equations. The reason of introducing the time delay in (1.1) followed the work of [28], but we also note the contribution in [2, 3, 4, 29, 32, 38]. It is observed that delays terms may appear when we want to control the system by applying a force which takes into account not only the present state but the complete history of the system. In this paper, we introduce the following system of 3d globally modified magnetohydrodynamics equations with infinite delays terms (GMMHDED)

$$\begin{cases} \dfrac{\partial u}{\partial t} + F_N(\|u\|_{V_1})[(u\cdot\nabla)u] - \dfrac{1}{R_e}\Delta u - SF_N(\|(u,B)\|_V)[(B\cdot\nabla)B] \\ +\nabla\left(p + S\frac{|B|^2}{2}\right) = f_1(t) + g_1(t,(u_t,B_t)) \text{ in } (\tau,T)\times\Omega, \\ \dfrac{\partial B}{\partial t} + F_N(\|(u,B)\|_V)[(u\cdot\nabla)B - (B\cdot\nabla)u] + \dfrac{1}{R_m}\text{curl}(\text{curl}B) \\ = f_2(t) + g_2(t,(u_t,B_t)) \text{ in } (\tau,T)\times\Omega, \\ \text{div } u = 0, \text{ div } B = 0 \text{ in}(\tau,T)\times\Omega, \\ u = 0,\ B\cdot n = 0 \text{ and curl}B\times n = 0 \text{ on } \Gamma, \\ u(\tau+s,x) = \phi_1(s,x),\ B(\tau+s,x) = \phi_2(s,x),\ s\in(-\infty,0],\ x\in\Omega, \end{cases} \tag{1.2}$$

where $g_1(t,(u_t,B_t))$ and $g_2(t,(u_t,B_t))$ are another external forces containing some hereditary characteristic (delays terms), where u_t and B_t are functions defined on $(-\infty,0]$ by the relations $u_t(s) = u(t+s)$ and $B_t(s) = B(t+s)$ respectively. ϕ_1 and ϕ_2 are given functions defined in the interval $(-\infty,0]$. Since the initial time is τ, we deduce from the last line of (1.2) that $(u(\tau),B(\tau)) = (\phi_1(0),\phi_2(0))$. This system as we see is a modification of the magnetohydrodynamics (MHD) equations with delays, for an incompressible resistive viscous fluid subjected to a Lorentz force due to the presence of a magnetic field. The GMMHDED (1.2) is inspired from the globally modified

Navier-Stokes equations (GMNSE) with infinite delays studied in [28]. Such models (with delays) have been intensively investigated for many years (see [2, 3, 4, 29, 32, 38], just to cite some); but globally modified MHD with delays remain to be explored. This work follow our initial works [13, 14, 15] where the focus is on dynamics of globally modified Navier-Stokes coupled with magnetic field or the heat. We should also mentioned that the inspiration from this work comes from the work of J. Real and the coauthors in [28]. It is worth mentioning that our work differ from the one of J. Real and co-authors because we are dealing here with more equations, and there are more nonlinearities in our context, implying that the investigations are more involved even though some of the Proofs presented here are inspired from the works in [2, 3, 4, 28, 29, 32, 38]. This work is mainly concerned about the existence and uniqueness of solution of system (1.2) and its long term behavior when the forcing terms are independent of time.

The rest of the paper is structured as follows: in section 2, we recall some spaces useful for the variational formulation of problem (1.2). We also present some mathematical properties and estimates related to the operators involved in the model. In section 3 we establish the existence and the uniqueness of the solutions of the model. Section 4 (the last one) is devoted to the asymptotic behavior of that solution.

2. Preliminaries

In order to write down in mathematical terms (1.2), some notations and preliminaries need to be introduced. The material is borrowed mainly from [1, 43]. We recall the abstract spaces for model (1.2) and its abstract formulation. Bold notations will denote a vector or a tensor. We consider the well known Hilbert spaces $L^2(\Omega), H^m(\Omega), H_0^m(\Omega)$ and we set

$$\mathbb{L}^2(\Omega) := (L^2(\Omega))^3,\ \mathbb{H}^m(\Omega) := (H^m(\Omega))^3,\ \mathbb{H}_0^m(\Omega) := (H_0^m(\Omega))^3, \mathbb{L}_0^2(\Omega) := (L_0^2(\Omega))^3 \tag{2.1}$$

where $L_0^2(\Omega) = \left\{ q \in L^2(\Omega);\ \int_\Omega q(x)dx = 0 \right\}$. It is noted that for a vector w we set

$$\|w\|_{\mathbb{L}^r(\Omega)}^r = \int_\Omega |w(x)|^r dx\,,$$

where $|\cdot|$ denotes the Euclidean norm $|w|^2 = w \cdot w$. We shall frequently use Sobolev imbedding: for a real number $p \in \mathbb{R}$, $1 \leqslant p \leqslant 6$, the space $\mathbb{H}^1(\Omega)$ is imbedded into $\mathbb{L}^p(\Omega)$. In particular, there exists a constant c_p (that depends only on p, Ω and $d = 3$) such that

$$\text{for all} \quad v \in \mathbb{H}_0^1,\ \ \|v\|_{\mathbb{L}^p(\Omega)} \leqslant c_p \|\nabla v\|\,. \tag{2.2}$$

When $p = 2$, this is Poincare's inequality and c_2 is Poincare's constant. In the case of the maximum norm, the following imbedding holds

$$\text{for all } r > d = 3,\ \ \mathbb{W}^{1,r}(\Omega) \subset \mathbb{L}^\infty(\Omega)$$

in particular, for each $r > d = 3$, there exists $c_{\infty,r}$ such that

$$\text{for all} \quad v \in \mathbb{H}_0^1(\Omega) \cap \mathbb{W}^{1,r},\ \ \|v\|_{\mathbb{L}^\infty(\Omega)} \leqslant c_{\infty,r} \|\nabla v\|_{\mathbb{L}^r(\Omega)}\,. \tag{2.3}$$

Owing to Poincare's inequality, the semi-norm $|\cdot|$ is a norm on $\mathbb{H}_0^1(\Omega)$, equivalent to the full norm. As it is directly related gradient operator, we take this semi-norm as norm on $\mathbb{H}_0^1(\Omega)$, and we use it to define the dual norm on its dual space $\mathbb{H}^{-1}(\Omega)$:

$$\text{for all } \; f \in \mathbb{H}^{-1}(\Omega), \quad \|f\|_{\mathbb{H}^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v\rangle}{|v|},$$

where $\langle\cdot\rangle$ is the duality pairing between $\mathbb{H}^{-1}(\Omega)$ and $\mathbb{H}_0^1(\Omega)$. As usual for handling time dependent problems, it is convenient to consider functions defined on a time interval (a,b) with values in a functional space, say Y. More precisely, we denote by $\|\cdot\|_Y$ the norm on Y and for any number r with $1 \leqslant r \leqslant \infty$, we define

$$L^r(a,b;Y) = \{w \text{ measurable in } (a,b)\,;\ \int_a^b \|w(t)\|_Y^r\, dt < \infty\}$$

equipped with the norm

$$\|w\|_{L^r(a,b;Y)}^r = \int_a^b \|w(t)\|_Y^r\, dt$$

with the usual modification if $r = \infty$. It is a Banach space if Y is a Banach space, and when $r = 2$, it is a Hilbert space if Y is also a Hilbert space.

We also introduce the following spaces

$$\begin{aligned}
&\mathscr{V}_1 = \left\{u \in (\mathscr{C}_c^\infty(\Omega))^3 : \operatorname{div} u = 0\right\},\\
&V_1 = \text{the closure of } \mathscr{V}_1 \text{ in } \mathbb{H}_0^1(\Omega),\\
&H_1 = \left\{u \in \mathbb{L}^2(\Omega) : \operatorname{div} u = 0 \text{ and } u\cdot n = 0 \text{ on } \Gamma\right\},\\
&\mathscr{V}_2 = \left\{B \in (\mathscr{C}^\infty(\overline{\Omega}))^3 : \operatorname{div} B = 0,\ B\cdot n = 0 \text{ on } \Gamma\right\},\\
&V_2 = \left\{B \in \mathbb{H}^1(\Omega) : \operatorname{div} B = 0;\ B\cdot n = 0 \text{ on } \Gamma\right\},\\
&H_2 = \text{the closure of } \mathscr{V}_2 \text{ in } \mathbb{L}^2(\Omega).
\end{aligned} \tag{2.4}$$

Thus $H_2 = H_1$. We endow H_i, $i = 1,2$ with the inner product of $\mathbb{L}^2(\Omega)$ and the norm of $\mathbb{L}^2(\Omega)$ denote respectively by $(.,.)_{L^2}$ and $|.|_{L^2}$.

We equip V_1 with the following inner product

$$((u,v))_1 = \sum_{i=1}^3 \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{L^2}. \tag{2.5}$$

We equip V_2 with the scalar product

$$((u,v))_2 = (\operatorname{curl} u, \operatorname{curl} v)_{L^2}. \tag{2.6}$$

Where $\operatorname{curl} u = \nabla \wedge u$. We note that by Poincaré's inequality, the scalar product $((.,.))_1$ defined in (2.5) coincides with the well known inner product in $\mathbb{H}_0^1(\Omega)$. The norm generated by $((.,.))_2$ is equivalent to the norm induced by $\mathbb{H}^1(\Omega)$ on V_2 (see [16, Chapter VII]). Hereafter, we set

$$H = H_1 \times H_2,\ V = V_1 \times V_2. \tag{2.7}$$

The dual space of V is denoted by V'. We endow H with the inner products defined as: for all $\varphi = (u, B)$, $\psi = (v, C) \in H$.

$$\begin{aligned}(\varphi,\psi) &= (u,v)_{L^2} + (B,C)_{L^2},\\ [\varphi,\psi] &= (u,v)_{L^2} + S(B,C)_{L^2}.\end{aligned}$$

They generate equivalent norms (for $0 < S < \infty$)

$$|\varphi|_H^2 = (\varphi,\varphi) = |u|_{L^2}^2 + |B|_{L^2}^2,\ [\varphi]_H^2 = [\varphi,\varphi] = |u|_{L^2}^2 + S|B|_{L^2}^2. \tag{2.8}$$

We also endow V with the inner products

$$((\varphi,\psi)) = \frac{1}{R_e}((u,v))_1 + \frac{1}{R_m}((B,C))_2,\ [[\varphi,\psi]] = \frac{1}{R_e}((u,v))_1 + \frac{S}{R_m}((B,C))_2, \tag{2.9}$$

which in turn generate the equivalent norms on V

$$\|\varphi\|_V^2 = ((\varphi,\varphi)),\ [[\varphi]]_V^2 = [[\varphi,\varphi]]. \tag{2.10}$$

In order to give an abstract formulation of problem (1.2), we introduce the operators $\mathscr{A}_1 \in \mathscr{L}(V_1, V_1')$, $\mathscr{A}_2 \in \mathscr{L}(V_2, V_2')$, and $\mathscr{A} \in \mathscr{L}(V, V')$ defined by

$$\begin{aligned}\langle \mathscr{A}_1 u, v\rangle &= ((u,v))_1, \text{ for all } u, v \in V_1,\\ \langle \mathscr{A}_2 B, C\rangle &= ((B,C))_2, \text{ for all } B, C \in V_2,\\ \langle \mathscr{A}\varphi, \psi\rangle &= ((\varphi,\psi)), \text{ for all } \varphi, \psi \in V.\end{aligned} \tag{2.11}$$

with domains

$$\begin{aligned}D(\mathscr{A}_1) &= \{u \in V_1 : \mathscr{A}_1 u \in H_1\},\\ D(\mathscr{A}_2) &= \{u \in V_2 : \mathscr{A}_2 u \in H_2\},\\ D(\mathscr{A}) &= D(\mathscr{A}_1) \times D(\mathscr{A}_2).\end{aligned}$$

By the regularity of Γ, $D(\mathscr{A}) = \mathbb{H}^2 \cap V$. From the continuity of the embedding of V_i into H_i, $i = 1, 2$, there exists constant κ_i, $i = 1, 2$ such that

$$|u|_{L^2} \leqslant \kappa_1 \|u\|_{V_1} \text{ for all } u \in V_1,\ |B|_{L^2} \leqslant \kappa_2 \|B\|_{V_2} \text{ for all } B \in V_2. \tag{2.12}$$

The best constant κ_i is equal to $\frac{1}{\sqrt{\lambda_1^i}}$, where λ_1^i is the first eigenvalue of the compact operator $\mathscr{A}_i^{-1}$ from H_i into itself. As in [36], we introduce the trilinear form $\mathscr{B}_0$ on $V \times V \times V$ by

$$\mathscr{B}_0(\varphi_1,\varphi_2,\varphi_3) = b(u_1,u_2,u_3) - Sb(B_1,B_2,u_3) + b(u_1,B_2,B_3) - b(B_1,u_2,B_3), \tag{2.13}$$

for all $\varphi_i = (u_i, B_i) \in V (i = 1, 2, 3)$, where $b(\cdot,\cdot,\cdot)$ is a continuous trilinear form defined on $\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

and satisfies the following standard relations,

$$
\begin{aligned}
&b(u,v,v)=0,\ \forall u\in V_1,\ v\in \mathbb{H}^1(\Omega),\\
&b(u,v,w)=-b(u,w,v),\ \forall u\in V_1,\ v,w\in \mathbb{H}^1(\Omega),\\
&|b(u,v,w)|\leqslant c\|u\|_{V_1}^{1/2}|\mathscr{A}_1u|_{L^2}^{1/2}\|v\|_{V_1}|w|_{L^2},\ \forall u\in D(\mathscr{A}_1),v\in V_1,w\in H_1\\
&|b(b_1,b_2,u)|\leqslant c|b_1|_{L^2}^{1/4}\|b_1\|_{V_2}^{3/4}\|u\|_{V_1}\|b_2\|_{V_2},\ \forall b_1,b_2\in V_2,u\in V_1,\\
&|b(b_1,b_2,u)|\leqslant c\|b_1\|_{V_2}|\mathscr{A}_2b_2|_{L^2}|u|_{L^2},\ \forall b_1\in V_2,b_2\in D(\mathscr{A}_2),u\in H_1,\\
&|b(b_1,u_1,b_2)|\leqslant c\|b_1\|_{V_2}|\mathscr{A}_1u_1|_{L^2}|b_2|_{L^2},\ \forall b_1\in V_2,u_1\in D(\mathscr{A}_1),b_2\in H_2.\\
&|b(u,v,w)|\leqslant |u|_{L^6}|\nabla v|_{L^2}|w|_{L^2}^{1/2}|w|_{L^6}^{1/2},\ \forall u,v,w\in \mathbb{H}^1(\Omega).
\end{aligned}
\tag{2.14}
$$

REMARK 1. Using the inclusion of $\mathbb{H}^1(\Omega)$ in $\mathbb{L}^p(\Omega)$ for $1\leqslant p\leqslant 6$, we infer that trilinear form $b(\cdot,\cdot,\cdot)$ also satisfies

$$
|b(u,v,w)|\leqslant \|u\|_{V_1}\|v\|_{V_1}|w|_{L^2}^{1/2}\|w\|_{V_1}^{1/2},\ \forall u,v,w\in V_1. \tag{2.15}
$$

From (2.14), we infer that

$$
\begin{aligned}
&\mathscr{B}_0(\varphi_1,\varphi_2,\varphi_2)=0,\ \forall \varphi_1,\varphi_2\in V,\\
&\mathscr{B}_0(\varphi_1,\varphi_2,\varphi_3)=-\mathscr{B}_0(\varphi_1,\varphi_3,\varphi_2),\ \forall \varphi_i\in V,\ i=1,2,3.
\end{aligned}
\tag{2.16}
$$

Now we introduce the continuous bilinear form $\mathscr{B}:V\times V\to V'$ by

$$
\langle \mathscr{B}(\varphi_1,\varphi_2),\varphi_3\rangle=\mathscr{B}_0(\varphi_1,\varphi_2,\varphi_3). \tag{2.17}
$$

We also introduce a diagonal matrix $\mathbf{M}=(m_{ij})_{1\leqslant i,j\leqslant 6}\in M_6(\mathbb{R})$ defined by:

$$
\begin{cases}
m_{ii}=1 \text{ if } 1\leqslant i\leqslant 3,\\
m_{ii}=S \text{ if } 4\leqslant i\leqslant 6,\\
m_{ij}=0 \text{ if } i\neq j.
\end{cases}
\tag{2.18}
$$

Note that

$$
\mathscr{B}_0(\varphi_1,\varphi_2,\mathbf{M}\varphi_2)=b(u_1,u_2,u_2)+Sb(u_1,B_2,B_2)-S[b(B_1,B_2,u_2)+b(B_1,u_2,B_2)]. \tag{2.19}
$$

It follows from (2.14) and (2.19) that

$$
\begin{aligned}
&\mathscr{B}_0(\varphi_1,\varphi_2,\mathbf{M}\varphi_2)=0\ \forall \varphi_1,\varphi_2\in V,\\
&\mathscr{B}_0(\varphi_1,\varphi_2,\mathbf{M}\varphi_3)=-\mathscr{B}_0(\varphi_1,\varphi_3,\mathbf{M}\varphi_2),\ \forall \varphi_i\in V,\ i=1,2,3.
\end{aligned}
\tag{2.20}
$$

We recall that (see [36]) $\mathscr{B}_0$ and $\mathscr{B}$ satisfy the following estimates

$$
\begin{aligned}
&|\mathscr{B}_0(\varphi_1,\varphi_2,\varphi_3)|\leqslant c\|\varphi_1\|_V\|\varphi_2\|_V^{1/2}|\mathscr{A}\varphi_2|_H^{1/2}|\varphi_3|_H,\ \forall \varphi_1\in V,\varphi_2\in D(\mathscr{A}),\varphi_3\in H,\\
&\|\mathscr{B}(\varphi,\varphi)\|_{V'}\leqslant c|\varphi|_H^{1/2}\|\varphi\|_V^{3/2}.
\end{aligned}
\tag{2.21}
$$

Hereafter we set

$$\begin{aligned}\mathscr{B}_0^N(\varphi_1,\varphi_2,\varphi_3) = & F_N(\|u_2\|_{V_1})b(u_1,u_2,u_3) - SF_N(\|(u_2,B_2)\|_V)b(B_1,B_2,u_3)\\ & + F_N(\|(u_2,B_2)\|_V)b(u_1,B_2,B_3) - F_N(\|(u_2,B_2)\|_V)b(B_1,u_2,B_3)\,,\end{aligned}$$

$$\left\langle \mathscr{B}^N(\varphi_1,\varphi_2),\varphi_3\right\rangle = \mathscr{B}_0^N(\varphi_1,\varphi_2,\varphi_3),\ \forall \varphi_i = (u_i,B_i)\in V,\ i=1,2,3. \tag{2.22}$$

Arguing similarly as in the proof of (2.21), we can check that the following inequalities hold

$$\begin{aligned}|\mathscr{B}_0^N(\varphi_1,\varphi_2,\varphi_3)| \leqslant & cN\|\varphi_1\|_V^{1/2}|\mathscr{A}\varphi_1|_H^{1/2}|\varphi_3|_H\\ & + cSN\|\varphi_1\|_V^{1/2}|\mathscr{A}\varphi_1|_H^{1/2}|\varphi_3|_H,\ \forall \varphi_1\in V, \varphi_2\in D(\mathscr{A}), \varphi_3\in H\,.\end{aligned} \tag{2.23}$$

Secondly

$$\begin{aligned}|\mathscr{B}_0^N(\varphi_1,\varphi_1,\varphi_2)| \leqslant & cN|\varphi_1|_H^{1/4}|\mathscr{A}\varphi_1|_H^{3/4}|\varphi_2|_H\\ & + cSN|\varphi_1|_H^{1/4}|\mathscr{A}\varphi_1|_H^{3/4}|\varphi_2|_H,\ \forall \varphi_1\in D(\mathscr{A}),\ \varphi_2\in H\,,\end{aligned} \tag{2.24}$$

thirdly

$$\begin{aligned}\|\mathscr{B}^N(\varphi_1,\varphi_2)\|_{V'} \leqslant & c|\varphi_1|_H^{1/4}\|\varphi_1\|_V^{3/4}|\varphi_2|_H^{1/4}\|\varphi_2\|_V^{3/4}\\ & + cS|\varphi_1|_H^{1/4}\|\varphi_1\|_V^{3/4}|\varphi_2|_H^{1/4}\|\varphi_2\|_V^{3/4},\ \forall \varphi_i = (u_i,B_i)\in V\,,\end{aligned} \tag{2.25}$$

next

$$\|\mathscr{B}^N(\varphi_1,\varphi_2)\|_{V'} \leqslant cN\|\varphi_1\|_V + cNS\|\varphi_1\|_V\,, \tag{2.26}$$

and finally

$$\begin{aligned}|\mathscr{B}_0^N(\varphi_1,\varphi_1,\varphi_2)| \leqslant & cN\|\varphi_1\|_V^{1/2}|\mathscr{A}\varphi_1|_H^{1/2}|\varphi_2|_H\\ & + c\|\varphi_1\|_V^{3/2}|\mathscr{A}\varphi_1|_H^{1/2}|\varphi_2|_H,\ \forall \varphi_1\in D(\mathscr{A}),\ \varphi_2\in H\,.\end{aligned} \tag{2.27}$$

The analysis of (1.2) will also require the following version of Gronwall's lemma (see [35])

LEMMA 1. *Let $T>0$ and let κ be a non-negative function in $L^1(0,T)$. Let $c>0$ be a constant and $\psi\in\mathscr{C}^0(0,T)$ a function that satisfies*

$$\textit{for all } t\in[0,T],\ \ 0\leqslant \psi(t)\leqslant c+\int_0^t \kappa(s)\psi(s)ds,$$

then ψ satisfies the bound

$$\psi(t)\leqslant ce^{\displaystyle\int_0^t \kappa(s)ds}.$$

Here, $\mathscr{C}^0(0,T)$ denotes the set of continuous functions on $[0,T]$.

Let X a Banach space, we define $B_X(a,r)$ as an open ball of center a and the radius r in the space X.

One possibility to deal with infinite delays is to follow [28, 29, 30]), which entails to consider, for any $\gamma > 0$, the space

$$\mathscr{C}_\gamma(H) = \Big\{\varphi \in \mathscr{C}((-\infty,0];H) : \text{ such that } \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \text{ is well defined,} \\ \text{and an element of } H\Big\}.$$

This is a Banach space with the norm

$$\|\varphi\|_\gamma := \sup_{s\in(-\infty,0]} e^{\gamma s}|\varphi(s)|_H \,.$$

Following [28], more assumptions are required. For that purpose, we assume for $i = 1,\ 2$ and for some fixed $\gamma > 0$ that $g_i : (\tau,T)\times\mathscr{C}_\gamma(H) \to \mathbb{L}^2(\Omega)$ satisfies

(h_1) For any $\xi = (\xi_1,\xi_2) \in \mathscr{C}_\gamma(H)$, the mapping

$$\begin{aligned} g_i(.,\xi) : (\tau,T) &\to \mathbb{L}^2(\Omega) \\ t \quad &\mapsto g_i(t,\xi) \quad \text{is measurable}. \end{aligned}$$

(h_2) $g_i(t,0) = 0$ for all $t\in(\tau,T)$.

(h_3) there exists a constant $L_{g_i} > 0$ such that for any $t\in(\tau,T)$ and for all $\xi,\eta \in \mathscr{C}_\gamma(H)$,

$$|g_i(t,\xi) - g_i(t,\eta)|_{L^2} \leqslant L_{g_i}\|\xi-\eta\|_\gamma \,.$$

REMARK 2. (h_2) and (h_3) imply that for all $\xi\in\mathscr{C}_\gamma(H)$ $|g_i(t,\xi)|_{L^2} \leqslant L_{g_i}\|\xi\|_\gamma$ so that $|g_i(.,\xi)| \in L^\infty(\tau,T)$.

If we set $g = (g_1, g_2)$, then from (h_3), $g(t,.)$ is Lipschitz-continuous on $\mathscr{C}_\gamma(H)$.

Using the notations above, we can rewrite (1.2) in the form

$$\begin{cases} \dfrac{dy}{dt} + \mathscr{A}y + \mathscr{B}^N(y,y) = F + G_t \;\text{ on }\; \mathscr{D}'(\tau,T;V'), \\ y(\tau+s,x) = \phi(s,x),\ s\in(-\infty,0],\ x\in\Omega \end{cases} \tag{2.28}$$

where $y = (u, B)$, $F = (f_1, f_2)$, $\phi = (\phi_1,\phi_2)$ and $G_t = (g_1(t,(y_t)), g_2(t,(y_t)))$ with $y_t = (u_t, B_t)$. We can now define a concept of solution associated to (2.28).

DEFINITION 1. We suppose $(u(\tau), B(\tau)) \in H$, $f_i \in L^2(\tau,T;V_i')$ and $g_i : (\tau,T)\times \mathscr{C}_\gamma(H)) \to \mathbb{L}^2(\Omega)$ satisfies (h_1)–(h_3) for some fixed $\gamma > 0$, $= 1,\ 2$.

A weak solution of (2.28) is any pair $y = (u, B) \in L^2(\tau,T;V)$ such that

$$\begin{cases} \dfrac{dy}{dt} + \mathscr{A}y + \mathscr{B}^N(y,y) = F + G_t \;\text{ on }\; \mathscr{D}'(\tau,T;V') \\ y(\tau+s,x) = \phi(s,x),\ s\in(-\infty,0],\ x\in\Omega \end{cases} \tag{2.29}$$

or equivalently for all $\varphi=(v,c)\in V$

$$\begin{cases}\left(\dfrac{dy}{dt},\varphi\right)+((y,\varphi))+\mathscr{B}_0^N(y,y,\varphi)=\langle f_1,v\rangle+\langle f_2,c\rangle+\langle g_1(t,y_t),v\rangle+\langle g_2(t,y_t),c\rangle\,,\\ y(\tau+s,x)=\phi(s,x),\ s\in(-\infty,0],\ x\in\Omega\,.\end{cases} \tag{2.30}$$

REMARK 3. Definition 1 provides also the variational formulation of problem (1.2).

If $y=(u,B)\in L^2(\tau,T;V')$ satisfies $(2.29)_1$, it follows from (2.26),(2.27) and (h_1) that $\dfrac{dy}{dt}\in L^2(\tau,T;V')$, and consequently (see [41]), $y\in\mathscr{C}([\tau,T);H)$ so that $y(\tau)$ exists.

In addition, by taking $\varphi=My$ in $(2.30)_1$ and using $(2.20)_1$ we infer that y satisfies the following energy equality

$$\begin{aligned}&|u(t)|_{L^2}^2+S|B(t)|_{L^2}^2+\tfrac{2}{R_e}\int_\tau^t\|u(\xi)\|_{V_1}^2d\xi+\frac{2S}{R_m}\int_\tau^t\|B(\xi)\|_{V_2}^2d\xi\\ &\quad=|u_0|_{L^2}^2+S|B_0|_{L^2}^2+2\int_\tau^t(f_1(\xi),u(\xi))d\xi+2S\int_\tau^t(f_2(\xi),B(\xi))d\xi\\ &\qquad+2\int_\tau^t\big(g_1(\xi,(u_\xi,B_\xi)),u(\xi)\big)\,d\xi+2S\int_\tau^t\big(g_2(\xi,(u_\xi,B_\xi)),B(\xi)\big)\,d\xi\,.\end{aligned} \tag{2.31}$$

3. Existence and uniqueness result

In this section, we prove that problem (2.29) has a unique weak solution which is, under some conditions a strong solution. Before doing this, we recall from [5, 34, 37] the following properties of F_N, where the proof can be found in [5, 34]. These properties are the main tools in the proof of the uniqueness result. We first recall that;

$$\begin{aligned}&|F_N(p)-F_N(r)|\leqslant\tfrac{|p-r|}{r},\ \forall p,r\in\mathbb{R}^+,\ r\neq0,\\ &|F_N(\|u\|_{V_1})-F_N(\|v\|_{V_1})|\leqslant\tfrac{\|u-v\|_{V_1}}{\|v\|_{V_1}},\ u,v\in V_1,\ v\neq0,\\ &|F_M(p)-F_N(r)|\leqslant\tfrac{|M-N|}{r}+\tfrac{|p-r|}{r},\ \forall p,r,M,N\in\mathbb{R}^+,r\neq0\\ &|F_N(\|u\|_{V_1})-F_N(\|v\|_{V_1})|\leqslant\tfrac{1}{N}F_N(\|u\|_{V_1})F_N(\|v\|_{V_1})\|u-v\|_{V_1},\ u,v\in V_1\,.\end{aligned} \tag{3.1}$$

In the rest of this paper we will denote by c, a generic positive constant (possibly depending on $S,R_e,R_m,\kappa_1,\kappa_2,\Omega,L_{g_1},L_{g_2}$), which can vary even within the same line. However, this constant is always independent of time and initial data. We start by proving the uniqueness result; for this purpose, we have the following.

THEOREM 1. *There exists at most one weak solution (u,B) of (2.29) in the sense of definition 1.*

Proof. Let $y_i=(u_i,B_i)$, $i=1,2$ be weak solutions to (2.29) that belong to $L^2(0,T;V)$. We set $\delta y=(\delta u,\delta B)=y_1-y_2$, $u_{it}(s)=u_i(t+s)$, $B_{it}(s)=B_i(t+s)$, $s\in$

$(-\infty,0]$. Then $(\delta u, \delta B)$ satisfies

$$\begin{cases} \dfrac{d\delta y}{dt} + \mathscr{A}\delta y = -\left(\mathscr{B}^N(y_1,y_1) - \mathscr{B}^N(y_2,y_2)\right) + \left(G(t,(u_{1t},B_{1t})) - G(t,(u_{2t},B_{2t}))\right), \\ \delta y(\tau) = 0. \end{cases} \tag{3.2}$$

Taking the scalar product in H of (3.2) with $M\delta y$, we obtain

$$\begin{aligned} \frac{d\mathscr{Y}}{dt} + \frac{2}{R_e}\|\delta u\|_{V_1}^2 + \frac{2S}{R_m}\|\delta B\|_{V_2}^2 &= -2(\mathscr{B}^N(y_1,y_1) - \mathscr{B}^N(y_2,y_2), M\delta y) \\ &\quad + 2\left(G(t,(u_{1t},B_{1t})) - G(t,(u_{2t},B_{2t})), M\delta y\right) \end{aligned} \tag{3.3}$$

with $\mathscr{Y} = |\delta u|_{L^2}^2 + S|\delta B|_{L^2}^2$ and $2(-\mathscr{B}^N(y_1,y_1) + \mathscr{B}^N(y_2,y_2), M\delta y)$ satisfies the following (see [13] for the details)

$$2\left(-\mathscr{B}^N(y_1,y_1) + \mathscr{B}^N(y_2,y_2), My\right) \leqslant \left(cN^4 + cN^8\right)\mathscr{Y}. \tag{3.4}$$

Using (3.4) and hypothesis (h_3) in (3.3), we obtain

$$\frac{d\mathscr{Y}}{dt} + \frac{2}{R_e}\|\delta u\|_{V_1}^2 + \frac{2S}{R_m}\|\delta B\|_{V_2}^2 \leqslant \left(cN^4 + cN^8\right)\mathscr{Y} + 2\left\{L_{g_1} + SL_{g_2}\right\}\|\delta y_t\|_\gamma |\delta y|_H \tag{3.5}$$

Observe that $\delta y(s) = (0,0)$ if $s \leqslant \tau$,

$$\|\delta y_t\|_\gamma = \sup_{s\in(-\infty,0]} e^{\gamma s}|\delta y(t+s)|_H \leqslant \sup_{s\in[\tau-t,0]} |\delta y(t+s)|_H. \tag{3.6}$$

Dropping momentarily the term $\frac{2}{R_e}\|\delta u\|_{V_1}^2 + \frac{2S}{R_m}\|\delta B\|_{V_2}^2$ in (3.5), we have

$$\begin{aligned} \mathscr{Y}(t) &\leqslant \left(cN^4 + cN^8\right)\int_\tau^t \mathscr{Y}(\xi)d\xi + 2\eta\int_\tau^t \sup_{s\in[\tau-\xi,0]} |\delta y(\xi+s)|_H |\delta y|_H d\xi \\ &\leqslant \left(cN^4 + cN^8\right)\int_\tau^t \mathscr{Y}(\xi)d\xi + 2\eta\int_\tau^t \sup_{s\in[\tau,\xi]} |\delta y(s)|_H^2 d\xi \\ &\leqslant \left((cN^4 + cN^8)\max\{1,S\} + 2\eta\right)\int_\tau^t \sup_{s\in[\tau,\xi]} |\delta y(s)|_H^2 d\xi, \end{aligned} \tag{3.7}$$

where $\eta = L_{g_1} + SL_{g_2}$.

From (3.7), we have for any $t \in [\tau, T]$

$$\min\{1,S\}\sup_{s\in[\tau,t]} |\delta y(s)|_H^2 \leqslant \left((cN^4 + cN^8)\max\{1,S\} + 2\eta\right)\int_\tau^t \sup_{s\in[\tau,\xi]} |\delta y(s)|_H^2 d\xi. \tag{3.8}$$

The use of Lemma 1 leads to $\sup_{s\in[\tau,t]} |\delta y(s)|_H^2 \leqslant 0$ from which we infer that $u_1 = u_2$ and $B_1 = B_2$. $\square$

REMARK 4. It is worth mentioning that the uniqueness of solution is one of the important property of this model because precisely we do not have that property for the corresponding 3d magnetohydrodynamics version. We can now thing about a complete study of attractor in classical way [42]. This by the way is the object of our next investigation.

Now, we state the existence result.

THEOREM 2. *We suppose* $(u(\tau), B(\tau)) = (\phi_1(0), \phi_2(0)) \in H$, $f_i \in L^2(\tau, T; H_i)$ *and* $g_i : (\tau, T) \times \mathscr{C}_\gamma(H)) \to \mathbb{L}^2(\Omega)$ *satisfies* (h_1)–(h_3) *for some fixed* $\gamma > 0$, $i = 1, 2$. *Let* $\phi = (\phi_1, \phi_2) \in \mathscr{C}_\gamma(H)$ *be given, with* $R := \|\phi\|_\gamma$. *Then there exists a unique weak solution* (u, B) *of (2.29), which is in fact a strong solution in the sense that it belongs to*

$$\mathscr{C}(\tau, T; V) \cap L^2(\tau + \varepsilon, T; D(\mathscr{A}_1) \times D(\mathscr{A}_2)) \ \textit{for all}\ 0 < \varepsilon < T - \tau. \tag{3.9}$$

Moreover, if $(\phi_1(0), \phi_2(0)) \in V$, *then* (u, B) *satisfies*

$$(u, B) \in \mathscr{C}(\tau, T; V) \cap L^2(\tau, T; D(\mathscr{A}_1) \times D(\mathscr{A}_2)). \tag{3.10}$$

Proof. We split it in several steps.

Step 1: A Galerkin scheme. Since the injection $V \subset H$ is compact, let $\{(w_i, \psi_i), i = 1, 2, \ldots\} \subset V$ be an orthonormal basis of H, where $\{w_i, i = 1, 2, \ldots\}$, $\{\psi_i, i = 1, 2, \ldots\}$ are eigenfunctions of $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively. We set $V_n = H_n = \text{span}\{(w_1, \psi_1), \ldots, (w_n, \psi_n)\}$ and denote by $P_n = (P_n^1, P_n^2)$, the orthogonal projector from H onto V_n for the scalar product $(.,.)$ defined by $(2.8)_1$. Note that P_n is also the orthogonal projector from $D(\mathscr{A}), V, V'$ onto V_n. We look for $y_n = P_n(u, B) = (u_n, B_n) \in H_n$ solution to the ordinary differential equations with delay

$$\begin{cases} \dfrac{dy_n}{dt} + \mathscr{A} y_n + P_n \mathscr{B}^N(y_n, y_n) = P_n F + P_n G_t \\ y_n(\tau + s) = P_n(\phi_1(s), \phi_2(s)) = (P_n^1 \phi_1(s), P_n^2 \phi_2(s)),\ s \in (-\infty, 0]. \end{cases} \tag{3.11}$$

According to (h_1)–(h_3), the above system of the ordinary differential equations with infinite delay satisfies the conditions for existence and uniqueness of solution y_n on an interval $[\tau, T_n]$, $T_n \leqslant T$ (see Theorem 1.1 of [17]). It will follow from a priori estimates that y_n exists on the interval $[\tau, T]$.

Step 2: A priori estimates. As in remark 3, y_n satisfies the following energy inequality:

$$\begin{aligned} &\frac{d}{dt}|u_n(t)|_{L^2}^2 + S\frac{d}{dt}|B_n(t)|_{L^2}^2 + \frac{2}{R_e}\|u_n(t)\|_{V_1}^2 + \frac{2S}{R_m}\|B_n(t)\|_{V_2}^2 \\ &\leqslant 2(P_n^1 f_1(t), u_n(t)) + 2S(P_n^1 f_2(t), B_n(t)) + 2(P_n^1 g_1(t, (u_{n,t}, B_{n,t})), u_n(t)) \\ &\quad + 2S(P_n^2 g_2(t, (u_{n,t}, B_{n,t})), B_n(t)). \end{aligned} \tag{3.12}$$

We need to estimate the terms on the right hand side of (3.12). First by Young's and Cauchy-Schwartz's inequalities, we have

$$2|(P_n^1 f_1(t), u_n(t))| \leqslant 2\|f_1(t)\|_{V_1'}\|u_n(t)\|_{V_1} \leqslant \tfrac{1}{2R_e}\|u_n(t)\|_{V_1}^2 + c\|f_1(t)\|_{V_1'}^2, \tag{3.13}$$

$$2S|(P_n^2 f_2(t), B_n(t))| \leqslant 2S\|f_2(t)\|_{V_2'}\|B_n(t)\|_{V_2} \leqslant \frac{S}{2R_m}\|B_n(t)\|_{V_2}^2 + c\|f_2(t)\|_{V_2'}^2, \tag{3.14}$$

$$\begin{aligned}2|(P_n^1 g_1(t,(u_{n,t},B_{n,t})),u_n(t))| &\leqslant 2\|g_1(t,(u_{n,t},B_{n,t}))\|_{V_1'}\|u_n(t)\|_{V_1}\\ &\leqslant 2c\|g_1(t,(u_{n,t},B_{n,t}))\|_{H_1}\|u_n(t)\|_{V_1}\\ &\leqslant 2c\|(u_{n,t},B_{n,t})\|_\gamma\|u_n(t)\|_{V_1}\\ &\leqslant \frac{1}{2R_e}\|u_n(t)\|_{V_1}^2 + c\|(u_{n,t},B_{n,t})\|_\gamma^2,\end{aligned} \tag{3.15}$$

$$\begin{aligned}2S|(P_n^2 g_2(t,(u_{n,t},B_{n,t})),B_n(t))| &\leqslant 2S\|g_2(t,(u_{n,t},B_{n,t}))\|_{V_2'}\|B_n(t)\|_{V_2}\\ &\leqslant \frac{S}{2R_m}\|B_n\|_{V_2}^2 + c\|(u_{n,t},B_{n,t})\|_\gamma^2.\end{aligned} \tag{3.16}$$

where (h_2)–(h_3) have been used to derive (3.15) and (3.16). Inserting the estimates (3.13)–(3.16) in (3.11) and integrating from τ to some $\tau \leqslant t \leqslant T$, we obtain

$$\begin{aligned}&|u_n(t)|_{L^2}^2 + S|B_n(t)|_{L^2}^2 + \frac{1}{R_e}\int_\tau^t \|u_n(\xi)\|_{V_1}^2 d\xi + \frac{S}{R_m}\int_\tau^t \|B_n(\xi)\|_{V_2}^2 d\xi\\ &\leqslant |\phi_1(0)|_{L^2}^2 + S|\phi_2(0)|_{L^2}^2 + c\int_\tau^t \|f_1(\xi)\|_{V_1'}^2 d\xi\\ &\quad + c\int_\tau^t \|f_2(\xi)\|_{V_2'}^2 d\xi + c\int_\tau^t \|(u_{n,\xi},B_{n,\xi})\|_\gamma^2 d\xi.\end{aligned} \tag{3.17}$$

Furthermore,

$$\begin{aligned}&\|(u_{n,t},B_{n,t})\|_\gamma^2\\ &= \sup_{\theta\in(-\infty,0]} e^{2\gamma\theta}|(u_n(t+\theta),B_n(t+\theta))|_H^2\\ &= \sup_{\theta\in(-\infty,0]} e^{2\gamma\theta}\left\{|u_n(t+\theta)|_{L^2}^2 + |B_n(t+\theta))|_{L^2}^2\right\}\\ &\leqslant \sup_{\theta\in(-\infty,0]} e^{2\gamma\theta}\left\{|\phi_1(0)|_{L^2}^2 + S|\phi_2(0)|_{L^2}^2 + c\int_\tau^{t+\theta}\|f_1(\xi)\|_{V_1'}^2 d\xi\right.\\ &\quad\left. + c\int_\tau^{t+\theta}\|f_2(\xi)\|_{V_2'}^2 d\xi + c\int_\tau^{t+\theta}\|(u_{n,\xi},B_{n,\xi})\|_\gamma^2 d\xi\right\}\\ &\leqslant \max\left\{\sup_{\theta\in(-\infty,\tau-t]} e^{2\gamma\theta}[\phi(\theta+t-\tau)]^2,\right.\\ &\quad \sup_{\theta\in[\tau-t,0]} e^{2\theta\gamma}\left(|\phi_1(0)|_{L^2}^2 + S|\phi_2(0)|_{L^2}^2 + c\int_\tau^{t+\theta}\|f_1(\xi)\|_{V_1'}^2 d\xi\right.\\ &\quad\left.\left. + c\int_\tau^{t+\theta}\|f_2(\xi)\|_{V_2'}^2 d\xi + c\int_\tau^{t+\theta}\|(u_{n,\xi},B_{n,\xi})\|_\gamma^2 d\xi\right)\right\}\\ &\leqslant \max\left\{\sup_{\theta\in(-\infty,\tau-t]}\left(e^{2\gamma\theta}[\phi(\theta+t-\tau)]^2\right),\right.\\ &\quad\left.[\phi(0)]_H^2 + c\int_\tau^t\|f_1(\xi)\|_{V_1'}^2 d\xi + c\int_\tau^t\|f_2(\xi)\|_{V_2'}^2 d\xi + c\int_\tau^t\|(u_{n,\xi},B_{n,\xi})\|_\gamma^2 d\xi\right\}.\end{aligned} \tag{3.18}$$

Observing that

$$\begin{aligned}\sup_{\theta\in(-\infty,\tau-t]} e^{\gamma\theta}\left[\phi(\theta+t-\tau)\right] &= \sup_{s\in(-\infty,0]} e^{\gamma(s-(t-\tau))}\left[\phi(s)\right]\\ &= \sup_{s\in(-\infty,0]} e^{\gamma s}\left[\phi(s)\right]e^{-(t-\tau)}\\ &\leqslant \|\phi\|_{\gamma}\,.\end{aligned}$$

and $[(u(\tau),B(\tau))]=[\phi(0)]\leqslant\|\phi\|_{\gamma}$, we deduce from (3.18)

$$\|(u_{n,t},B_{n,t})\|_{\gamma}^{2}\leqslant R^2+c\int_{\tau}^{t}\|f_1(\xi)\|_{V_1'}^{2}d\xi+c\int_{\tau}^{t}\|f_2(\xi)\|_{V_2'}^{2}d\xi+c\int_{\tau}^{t}\|(u_{n,\xi},B_{n,\xi})\|_{\gamma}^{2}d\xi\,. \tag{3.19}$$

By the Lemma 1, we have

$$\|(u_{n,t},B_{n,t})\|_{\gamma}^{2}\leqslant R^2e^{c(t-\tau)}+c\int_{\tau}^{t}\left(\|f_1(\xi)\|_{V_1'}^{2}d\xi+\|f_2(\xi)\|_{V_2'}^{2}\right)d\xi\, e^{c(t-\tau)}. \tag{3.20}$$

Thus, there exists a constant $\mathscr{K}_1=\mathscr{K}_1(R,\tau,Lg_1,Lg_2,T,f_1,f_2)>0$ such that

$$\|(u_{n,t},B_{n,t})\|_{\gamma}^{2}\leqslant\mathscr{K}_1, \tag{3.21}$$

which together with (3.17) gives

$$\begin{aligned}&|\left(u_n(t)|_{L^2}^2+S|B_n(t)|_{L^2}^2\right)+\left(\frac{1}{R_e}\int_{\tau}^{T}\|u_n(\xi)\|_{V_1}^{2}d\xi+\frac{S}{R_m}\int_{\tau}^{T}\|B_n(\xi)\|_{V_2}^{2}d\xi\right)\\ &\leqslant R^2+c\int_{\tau}^{T}\|f_1(\xi)\|_{V_1'}^{2}d\xi+c\int_{\tau}^{T}\|f_2(\xi)\|_{V_2'}^{2}d\xi+c\mathscr{K}_1(T-\tau)\,.\end{aligned} \tag{3.22}$$

(3.22) proves that the sequence $y_n=(u_n,B_n)$ remains in a bounded set of $L^\infty(\tau,T;H)\cap L^2(\tau,T;V)\cap\mathscr{C}_\gamma(H)$. Hence, we can use a compactness argument (see [42]) to extract a subsequence from $y_n=(u_n,B_n)$ still denoted by $y_n=(u_n,B_n)$ satisfying

$$y_n\to y\begin{cases}\text{weak-star in } L^\infty(\tau,T;H)\,,\\ \text{weakly in } L^2(\tau,T;V)\,,\\ \text{strongly in } L^2(\tau,T;H)\,,\\ \text{a.e., in } (\tau,T)\times\Omega\,,\end{cases} \tag{3.23}$$

with $y=(u,B)\in L^\infty(\tau,T;H)\cap L^2(\tau,T;V)\cap\mathscr{C}_\gamma(H)$.

But the estimates (3.22) are not enough to pass to the limit in (2.29) and deduce the solution of (1.2). More precisely, we have two main difficulties, firstly, we need to pass to the limit on the $G(t,(u_{n,t},B_{n,t}))$, this will be done on Step 3; secondly, we need to prove that

$$\begin{aligned}&F_N(\|u_n\|_{V_1})\to F_N(\|u\|_{V_1})\text{ as } n\to\infty\,,\\ &F_N(\|(u_n,B_n)\|_V)\to F_N(\|(u,B)\|_V)\text{ as } n\to\infty\,,\end{aligned} \tag{3.24}$$

To overcome the second difficulty, we need to find a stronger estimate and it is the aim of the lines below.

Taking the inner product in H between the first equation of (3.11) with $\mathscr{A}y_n$, we obtain

$$\begin{aligned}&\frac{d}{dt}\|y_n\|_V^2+2|\mathscr{A}y_n|_H^2=2(f_1,\mathscr{A}_1u_n)+2(f_2,\mathscr{A}_2B_n)-2\mathscr{B}_0^N(y_n,y_n,\mathscr{A}y_n)\\&+2(g_1(t,(u_{n,t},B_{n,t}),\mathscr{A}_1u_n))+2(g_2(t,(u_{n,t},B_{n,t}),\mathscr{A}_2B_n)).\end{aligned}\tag{3.25}$$

Now using (2.23) and Young's inequality with the exponents $(4,4/3)$, we have

$$2|\mathscr{B}_0^N(y_n,y_n,\mathscr{A}y_n)|\leqslant cN\|y_n\|_V^{1/2}|\mathscr{A}y_n|_H^{3/2}\leqslant\tfrac14|\mathscr{A}y_n|_H^2+cN^4\|y_n\|_V^2.\tag{3.26}$$

In addition, by Young's inequality, (h_2)–(h_3), (3.21) one obtains

$$\begin{aligned}2|(f_1,\mathscr{A}_1u_n)|+2|(f_2,\mathscr{A}_2B_n)|&\leqslant\tfrac14|\mathscr{A}_1u_n|_{L^2}^2+\tfrac14|\mathscr{A}_2B_n|_{L^2}^2+c|f_1|_{L^2}^2+c|f_2|_{L^2}^2\\&=\tfrac14|\mathscr{A}y_n|_H^2+c|f_1|_{L^2}^2+c|f_2|_{L^2}^2\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}&2|(g_1(t,(u_{n,t},B_{n,t}),\mathscr{A}_1u_n))+(g_2(t,(u_{n,t},B_{n,t}),\mathscr{A}_2B_n))|\\&\leqslant\frac12|\mathscr{A}_1u_n|_{L^2}^2+\frac12|\mathscr{A}_2B_n|_{L^2}^2+cK_1^2=\frac12|\mathscr{A}y_n|_H^2+c\mathscr{K}_1^2.\end{aligned}\tag{3.28}$$

It follows from (3.26)–(3.28) that

$$\frac{d}{dt}\|y_n\|_V^2+|\mathscr{A}y_n|_H^2\leqslant c|f_1|_{L^2}^2+c|f_2|_{L^2}^2+c\mathscr{K}_1^2+cN^4\|y_n\|_V^2.\tag{3.29}$$

Now we distinguish two cases:
Case 1: $y(\tau)=(u(\tau),B(\tau))\in H$.
Integrating (3.29) between s and t for $\tau<s\leqslant t\leqslant T$, we obtain

$$\begin{aligned}&\|y_n(t)\|_V^2+\int_s^t|\mathscr{A}y_n(\xi)|_H^2d\xi\\&\leqslant\|y_n(s)\|_V^2+c\int_s^t\left(|f_1(\xi)|_{L^2}^2+|f_2(\xi)|_{L^2}^2+\mathscr{K}_1^2\right)d\xi+cN^4\int_s^t\|y_n(\xi)\|_V^2d\xi.\\&\leqslant\|y_n(s)\|_V^2+c\int_\tau^t\left(|f_1(\xi)\|_{L^2}^2+|f_2(\xi)|_{L^2}^2+\mathscr{K}_1^2\right)d\xi+cN^4\int_\tau^t\|y_n(\xi)\|_V^2d\xi.\end{aligned}\tag{3.30}$$

Momentarily dropping the term $\int_s^t|\mathscr{A}y_n(\xi)|_H^2d\xi$ in (3.30) and integrating once more between τ and $\tau+\varepsilon$ for some $\varepsilon\in(0,T-\tau)$, we have

$$\begin{aligned}\int_\tau^{\tau+\varepsilon}\|y_n(t)\|_V^2ds\leqslant&\int_\tau^{\tau+\varepsilon}\|y_n(s)\|_V^2ds\\&+\int_\tau^{\tau+\varepsilon}\left(c\int_\tau^t\left(|f_1(\xi)\|_{L^2}^2+|f_2(\xi)|_{L^2}^2+\mathscr{K}_1^2\right)d\xi\right)ds\\&+cN^4\int_\tau^{\tau+\varepsilon}\left(\int_\tau^t\|y_n(\xi)\|_V^2d\xi\right)ds.\end{aligned}\tag{3.31}$$

Since $\tau+\varepsilon \leqslant T$, it follows from (3.31) that

$$\varepsilon\|y_n(t)\|_V^2 \leqslant \int_\tau^T \|y_n(s)\|_V^2 ds + c(T-\tau)\int_\tau^T \left(|f_1(\xi)|_{L^2}^2+|f_2(\xi)|_{L^2}^2+\mathscr{K}_1^2\right) d\xi$$
$$+c(T-\tau)N^4\int_\tau^T \|y_n(\xi)\|_V^2 d\xi\,. \tag{3.32}$$

From the estimate (3.22), we infer that the right hand side of (3.32) is bounded independently of n. Coming back to (3.30) and dropping the term $\|y_n(t)\|_V^2$, we get for some $\varepsilon\in[0,T-\tau]$

$$\int_{\tau+\varepsilon}^T |\mathscr{A} y_n(\xi)|_H^2 d\xi$$
$$\leqslant \|y_n(s)\|_V^2 + c\int_\tau^T \left(|f_1(\xi)|_{L^2}^2+|f_2(\xi)|_{L^2}^2+\mathscr{K}_1^2\right) d\xi + cN^4\int_\tau^T \|y_n(\xi)\|_V^2 d\xi\,. \tag{3.33}$$

We then deduce that $y_n\in L^\infty(\tau+\varepsilon,T;V)$. Therefore

$$y_n\in L^\infty(\tau+\varepsilon,T;V)\cap L^2(\tau+\varepsilon,T;D(\mathscr{A}_1)\times D(\mathscr{A}_2)) \text{ for all } 0<\varepsilon<T-\tau. \tag{3.34}$$

Note from (3.11) that

$$\frac{dy_n}{dt} = -\mathscr{A} y_n - P_n\mathscr{B}^N(y_n,y_n)+P_nF+P_nG.$$

Then using (2.23) we deduce that

$$\text{the sequence } \left\{P_n\mathscr{B}^N(y_n,y_n)\right\} \text{ is bounded in } L^2(\tau+\varepsilon,T;H).$$

Therefore, from (3.21) and (3.34), we infer that the sequence

$$\frac{d}{dt}(u_n,B_n) \text{ is also bounded in } L^2(\tau+\varepsilon,T;H). \tag{3.35}$$

Since $D(\mathscr{A})=D(\mathscr{A}_1)\times D(\mathscr{A}_2)\subset V\subset H$ with compact injection, we derive from [25, Theorem 5.1, Chapter 1] that there exists an element $(u,B)\in L^\infty(\tau+\varepsilon,T;V)\cap L^2(\tau+\varepsilon,T;D(\mathscr{A}))$, and a subsequence of (u_n,B_n) (still) denoted (u_n,B_n) such that for all $T>\tau+\varepsilon$, we have

$$(u_n,B_n)\to(u,B)\begin{cases}\text{weak-star in } L^\infty(\tau+\varepsilon,T;V),\\ \text{weakly in } L^2(\tau+\varepsilon,T;D(\mathscr{A})),\\ \text{strongly in } L^2(\tau+\varepsilon,T;V),\\ \text{a.e., in } (\tau+\varepsilon,T)\times\Omega,\end{cases} \tag{3.36}$$

and

$$\frac{d}{dt}(u_n,B_n)\to\frac{d}{dt}(u,B) \text{ weakly in } L^2(\tau+\varepsilon,T;H). \tag{3.37}$$

From (3.36), we can assume, eventually extracting a subsequence of y_n still denoted y_n such that

$$\begin{aligned} &\|u_n\|_{V_1} \to \|u\|_{V_1} \text{ a.e. in } (\tau+\varepsilon, T),\\ &\|(u_n, B_n)\|_V \to \|(u,B)\|_V \text{ a.e. in } (\tau+\varepsilon, T), \end{aligned} \tag{3.38}$$

and therefore

$$\begin{aligned} &F_N(\|u_n\|_{V_1}) \to F_N(\|u\|_{V_1}) \text{ a.e. in } (\tau+\varepsilon, T),\\ &F_N(\|(u_n, B_n)\|_V) \to F_N(\|(u,B)\|_V) \text{ a.e. in } (\tau+\varepsilon, T). \end{aligned} \tag{3.39}$$

Case 2: $(\phi_1(0), \phi_2(0)) \in V$.
We mention that

$$\|(\phi_{1n}(0), \phi_{2n}(0))\|_V = \|P_n(\phi_1(0), \phi_2(0))\|_V \leqslant \|y(\tau)\|_V.$$

Now, dropping the term $|\mathscr{A} y_n|_H^2$ in (3.29), we have the following differential inequality

$$\frac{d}{dt}\|y_n\|_V^2 \leqslant c|f_1|_{L^2}^2 + c|f_2|_{L^2}^2 + c\mathscr{K}_1^2 + cN^4\|y_n\|_V^2, \tag{3.40}$$

from which we obtain by using Lemma 1

$$\begin{aligned} \|y_n(t)\|_V^2 \leqslant{}& \|y(\tau)\|_V^2 \exp\left[cN^4(t-\tau)\right]\\ &+ c\exp\{cN^4(t-\tau)\}\int_\tau^t \left(|f_1(\xi)|_{L^2}^2 + |f_2(\xi)|_{L^2}^2 + c\mathscr{K}_1^2\right) d\xi. \end{aligned} \tag{3.41}$$

Hence, we derive from (3.29) and (3.32) that $(y_n) = (u_n, B_n)$ satisfies

$$\|(u_n, B_n)(t)\|_V^2 \leqslant \mathscr{K}_2, \quad \int_\tau^T \left(|\mathscr{A}_1 u_n(\xi)|_{L^2}^2 + |\mathscr{A}_2 B_n(\xi)|_{L^2}^2\right) d\xi \leqslant \mathscr{K}_3, \tag{3.42}$$

which proves that (u_n, B_n) is bounded in $L^\infty(\tau, T; V) \cap L^2(\tau, T; D(\mathscr{A}_1) \times D(\mathscr{A}_2))$.

Note that in (3.42), $\mathscr{K}_2$ and $\mathscr{K}_3$ are positive constants independent of n and depending only on data $\Omega, R_e, R_m, S, f_1, f_2, T, u_0, B_0, Lg_1$ and Lg_2.

Note that from (3.11) that

$$\frac{dy_n}{dt} = -\mathscr{A} y_n - P_n \mathscr{B}^N(y_n, y_n) + P_n F + P_n G.$$

Then using (2.23) we deduce that the sequence

$$\left\{P_n \mathscr{B}^N(y_n, y_n)\right\}_n \text{ is bounded in } L^2(\tau, T; H).$$

Therefore, from (3.33) and (3.21), we infer that the sequence

$$\frac{d}{dt}(u_n, B_n) \text{ is also bounded in } L^2(\tau, T; H). \tag{3.43}$$

Since $D(\mathscr{A}) = D(\mathscr{A}_1) \times D(\mathscr{A}_2) \subset V \subset H$ with compact injection, we derive from [25, Theorem 5.1, Chapter 1] that there exists an element $(u, B) \in L^\infty(\tau, T; V) \cap$

$L^2(\tau,T;D(\mathscr{A}))$, and a subsequence of (u_n, B_n) (still) denoted (u_n, B_n) such that for all $T>\tau$, we have

$$(u_n,B_n) \to (u,B) \begin{cases} \text{weak-star in } L^\infty(\tau,T;V), \\ \text{weakly in } L^2(\tau,T;D(\mathscr{A})), \\ \text{strongly in } L^2(\tau,T;V), \\ \text{a.e., in } (\tau,T)\times\Omega, \end{cases} \tag{3.44}$$

and

$$\frac{d}{dt}(u_n, B_n) \to \frac{d}{dt}(u, B) \text{ weakly in } L^2(\tau,T;H). \tag{3.45}$$

From (3.44), we infer that

$$\begin{aligned} &\|u_n\|_{V_1} \to \|u\|_{V_1} \text{ a.e. in } (\tau,T), \\ &\|(u_n,B_n)\|_V \to \|(u,B)\|_V \text{ a.e. in } (\tau,T), \end{aligned} \tag{3.46}$$

and therefore

$$\begin{aligned} &F_N(\|u_n\|_{V_1}) \to F_N(\|u\|_{V_1}) \text{ a.e. in } (\tau,T), \\ &F_N(\|(u_n,B_n)\|_V) \to F_N(\|(u,B)\|_V) \text{ a.e. in } (\tau,T). \end{aligned} \tag{3.47}$$

Step 3: Passage to the limit. We want to take the limit in (3.11) when n goes to $+\infty$. We focus our attention on the term $G(t,(u_{n,t}, B_{n,t}))$ we refer the reader to [5, 13] for the other terms. More precisely, we want to prove that

$$G(t,(u_{n,t}, B_{n,t})) \to G(t,(u_t, B_t)) \text{ when } n \to +\infty. \tag{3.48}$$

We proceed like in [28] where the globally modified Navier-Stokes with infinite delays is investigated. We start by proving that

$$(u_{n,t}, B_{n,t}) \to (u_t, B_t) \text{ in } \mathscr{C}_\gamma(H),\ \forall t \in (-\infty, T]. \tag{3.49}$$

Since

$$\begin{aligned} &\sup_{\theta\leqslant 0} e^{\gamma\theta}\,|y_n(t+\theta)-y(t+\theta)|_H \\ &= \max\Bigg\{ \sup_{\theta\in(-\infty,\tau-t]} e^{\gamma\theta}\,|P_n\phi(\theta+t-\tau)-\phi(\theta+t-\tau)|_H; \\ &\qquad \sup_{\theta\in[\tau-t,0]} e^{\gamma\theta}\,|y_n(t+\theta)-y(t+\theta)|_H \Bigg\} \\ &\leqslant \max\left\{ \sup_{s\leqslant 0} e^{\gamma s}e^{\gamma(\tau-t)}\,|P_n\phi(s)-\phi(s)|_H;\ \sup_{r\in[\tau,t]} e^{\gamma(r-t)}\,|y_n(r)-y(r)|_H \right\} \\ &= \max\left\{ e^{s(\tau-t)}\,\|P_n\phi-\phi\|_\gamma;\ \sup_{r\in[\tau,t]} |y_n(r)-y(r)|_H \right\}. \end{aligned}$$

Hence, the relation (3.49) will hold if we prove that

$$P_n\phi \to \phi \text{ in } \mathscr{C}_\gamma(H) \tag{3.50}$$

and

$$(u_n, B_n) \to (u, B) \text{ in } \mathscr{C}([\tau,T];H). \tag{3.51}$$

We start by proving (3.50). Indeed, if we assume that it is not held, then there exists $\varepsilon > 0$ and a subsequence $(\theta_n)_n$ such that

$$e^{\gamma\theta_n}|P_n\phi(\theta_n) - \phi(\theta_n)|_H > \varepsilon. \tag{3.52}$$

Hence following [28], we assume that $\theta_n \to -\infty$, otherwise if $\theta_n \to \theta \neq -\infty$, then $P_n\phi(\theta_n) \to \phi(\theta)$. In fact,

$$|P_n\phi(\theta_n) - \phi(\theta)|_H \leqslant |P_n\phi(\theta_n) - P_n\phi(\theta)|_H + |P_n\phi(\theta) - \phi(\theta)|_H \to 0 \text{ as } n \to +\infty.$$

Assume that $\theta_n \to -\infty$ as $n \to +\infty$, if we set $x := \lim\limits_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)$, we obtain

$$\begin{aligned} e^{\gamma\theta_n}|P_n\phi(\theta_n) - \phi(\theta_n)|_H &= \left|P_n(e^{\gamma\theta_n}\phi(\theta_n)) - e^{\gamma\theta_n}\phi(\theta_n)\right|_H \\ &\leqslant \left|P_n(e^{\gamma\theta_n}\phi(\theta_n)) - P_nx\right|_H + |P_nx - x|_H + \left|x - e^{\gamma\theta_n}\phi(\theta_n)\right|_H \\ &\to 0 \text{ as } n \to +\infty \end{aligned}$$

which is a contradiction with (3.52), hence we have (3.50).

Next, we will prove (3.51).

From the convergence of y_n to y in $L^2(\tau,T;H)$ given by (3.23), we deduce that $y_n(t) \to y(t)$ in H *a.e.* $t \in (\tau,T]$.

Since

$$y_n(t) - y_n(s) = \int_s^t y'(\xi)d\xi \ \forall s,t \in [\tau,T]$$

we have

$$\|y_n(t) - y_n(s)\|_{V'} \leqslant \int_\tau^T \|y'(\xi)\|_{V'} d\xi.$$

Then from remark 3, we deduce that $\int_\tau^T \|y'(\xi)\|_{V'} d\xi$ is bounded. Thus y_n is equicontinuous on $[\tau,T]$ with values in V'. In addition, $y_n(t)$ is bounded in V'. In fact, due to the convergence of $y_n(t)$ to $y(t)$ in H for $a.e. t \in [\tau,T]$, we infer that $y_n(t)$ is bounded in H and by the compactness injection of H in to V', we claim that $y_n(t)$ is bounded in V'. By the Ascoli-Arzela theorem, we have

$$y_n \to y \text{ in } \mathscr{C}((\tau,T];V'). \tag{3.53}$$

Also by the convergence of y_n to y in $L^\infty(\tau,T;H)$, we obtain that for any sequence $\{t_n\} \subset [\tau,T]$ with $t_n \to t$, one has

$$y_n(t_n) \to y(t) \text{ weakly in } H. \tag{3.54}$$

Now we prove (3.51) by a contradiction argument.

If (3.51) does not hold, then using the fact that $y_n \in \mathscr{C}([\tau,T];H)$, there would exist $\varepsilon > 0$, a value $t_0 \in [\tau,T]$ and subsequences labelled the same y_n and $t_n \subset [\tau,T]$ with $t_0 = \lim_{n\to+\infty} t_n$ such that

$$|y_n(t_n) \to y(t_0)|_H \geqslant \varepsilon. \tag{3.55}$$

To prove that this is absurd, we will use an energy method. Observe that the sequence y_n also satisfies the following energy inequality

$$\begin{aligned}&|y_n(t)|_H^2+\frac{1}{R_e}\int_s^t\|u_n(\xi)\|_{V_1}^2\,d\xi+\frac{S}{R_e}\int_s^t\|B_n(\xi)\|_{V_2}^2\,d\xi\\&\leqslant|y_n(s)|_H^2+c\int_s^t\left(\|f_1(\xi)\|_{V_1'}^2+\|f_2(\xi)\|_{V_2'}^2\right)d\xi+\mathscr{K}_1(t-s).\end{aligned}\tag{3.56}$$

On the other hand, since the functions y_n are bounded in $L^\infty(\tau,T;H)$, we deduce the existence of $\eta_G=(\eta_{g_1},\eta_{g_2})\in L^2(\tau,T;H)$ such that $(g_1(t,y_{n,t}),g_2(t,y_{n,t}))$ converges weakly to η_G in $L^2(\tau,T;H)$. Then passing to the limit in (2.30), we deduce that y is a solution of

$$\begin{aligned}&\frac{d}{dt}(y(t),(v,C))_H+\frac{1}{R_e}((u(t),v))_{V_1}+\frac{S}{R_e}((B(t),C))_{V_2}+\mathscr{B}^N(y(t),y(t),(v,C))\\&=(f_1(t),v)+(f_2(t),C)+(\eta_{g_1}(t,(u_t,B_t)),v)+(\eta_{g_2}(t,(u_tB_t)),C).\end{aligned}\tag{3.57}$$

Thus we deduce that, y satisfies the energy inequality

$$\begin{aligned}&|y(t)|_H^2+\frac{1}{R_e}\int_s^t\|u(\xi)\|_{V_1}^2\,d\xi+\frac{S}{R_e}\int_s^t\|B(\xi)\|_{V_2}^2\,d\xi\\&\leqslant|y(s)|_H^2+c\int_s^t\left(\|f_1(\xi)\|_{V_1'}^2+\|f_2(\xi)\|_{V_2'}^2\right)d\xi+\int_s^t|\eta_G(\xi)|_H^2\,d\xi.\end{aligned}\tag{3.58}$$

Since

$$\int_s^t|\eta_G(\xi)|_H^2\,d\xi\leqslant\liminf_{n\to\infty}\int_s^t\left|(g_1(\xi,y_{n,\xi}),g_2(\xi,y_{n,\xi}))\right|_H^2d\xi\leqslant\mathscr{K}_1(t-s).$$

It is noted that y satisfies also the inequality (3.56) with the same constant $\mathscr{K}_1$.

Now we consider the functions $J_n,\ J:[\tau,T]\to\mathbb{R}$ defined by

$$J_n(t)=\frac{1}{2}|y_n(t)|_H^2-c\int_\tau^t\left(\|f_1(\xi)\|_{V_1'}^2+\|f_2(\xi)\|_{V_2'}^2\right)d\xi-\mathscr{K}_1t,$$

and

$$J(t)=\frac{1}{2}|y(t)|_H^2-c\int_\tau^t\left(\|f_1(\xi)\|_{V_1'}^2+\|f_2(\xi)\|_{V_2'}^2\right)d\xi-\mathscr{K}_1t.$$

J_n and J are continuous and non-increasing functions.

Next, we show that for $t\geqslant s,\ J_n(t)-J_n(s)\leqslant0$. From a direct definition of J_n, we have

$$\begin{aligned}J_n(t)-J_n(s)&=\frac{1}{2}|y_n(t)|_H^2-\frac{1}{2}|y_n(s)|_H^2-c\int_\tau^t\left(\|f_1(\xi)\|_{V_1'}^2+\|f_2(\xi)\|_{V_2'}^2\right)d\xi-\mathscr{K}_1t+\mathscr{K}_1s\\&=\frac{1}{2}|y_n(t)|_H^2-\frac{1}{2}|y_n(s)|_H^2-c\int_s^t\left(\|f_1(\xi)\|_{V_1'}^2+\|f_2(\xi)\|_{V_2'}^2\right)d\xi+\mathscr{K}_1(s-t)\\&\leqslant0.\end{aligned}$$

Similarly, we can prove that J is a non-increasing function. Moreover by the convergence of y_n to y $a.e.$ in time with value on H, it holds that

$$J_n(t) \to J(t) \ a.e.\text{in } H. \tag{3.59}$$

Now, we want to prove that

$$y_n(t_n) \to y(t_0) \ \text{ in } \ H, \tag{3.60}$$

which contradicts (3.55).

Firstly, from (3.54) we recall that

$$y_n(t_n) \to y(t_0) \text{ weakly in } H, \tag{3.61}$$

then

$$|y(t_0)|_H \leqslant \liminf_{n\to\infty} |y_n(t_n)|_H \,. \tag{3.62}$$

Therefore if we prove that

$$\limsup_{n\to\infty} |y_n(t_n)|_H \leqslant |y(t_0)|_H \tag{3.63}$$

we obtain $\lim\limits_{n\to\infty} |y_n(t_n)|_H \to |y(t_0)|$ which jointly with (3.61) imply (3.60).

If $t_0 = \tau$, it follows from (3.50) and (3.58) that $\limsup\limits_{n\to\infty} |y_n(t_n)|_H \leqslant |y(\tau)|$. So we may assume that $t_0 > \tau$; this is important since we will approach this value t_0 by a sequence $\{\tilde{t}_k\}$, this means that $\lim_{k\to\infty} \tilde{t}_k \to t_0$, with $\tilde{t}_k$ being taken only when (3.59) is valid. Since $y(\cdot)$ is continuous at t_0 and $\tilde{t}_k \to t_0$, for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$|J(\tilde{t}_k) - J(t_0)| < \frac{\varepsilon}{2}.$$

On the other hand, taking $n > n(k_\varepsilon)$ such that $t_n > \tilde{t}_{k_\varepsilon}$, as J_n is non-increasing and for all $\tilde{t}_k$, the convergence (3.59) holds, one has that

$$J_n(t_n) - J(t_0) \leqslant |J_n(\tilde{t}_{k_\varepsilon}) - J(\tilde{t}_{k_\varepsilon})| + |J(\tilde{t}_{k_\varepsilon}) - J(t_0)|$$

and obviously taking $n > n'(k_\varepsilon)$, it is possible due to (3.59) to obtain

$$|J_n(\tilde{t}_{k_\varepsilon}) - J(\tilde{t}_{k_\varepsilon})| \leqslant \frac{\varepsilon}{2}.$$

Moreover, we deduce from (3.23)

$$\int_\tau^{t_n} (F(\xi), y_n(\xi))d\xi \to \int_\tau^{t_0} (F(\xi), y(\xi))d\xi,$$

so we conclude that (3.63) holds. Thus (3.60) and finally (3.51) are also true as we wanted to prove.

Now we are ready to pass to the limit in (3.26). Assume initially that $y(\tau) = \phi(0) \in H$, the first consequence of the convergence proved above since g_i satisfies (h_3) is that

$$(g_1(.,(u_{n,.}, B_{n,.})).g_2(.,(u_{n,.}, B_{n,.}))) \to (g_1(.,(u_{.}, B_{.})).g_2(.,(u_{.}, B_{.}))) \text{ in } L^2(\tau, T; H).$$

Hence, we can identify $(\eta g_1, \eta g_2) = (g_1, g_2)$ in (3.57) so that y is a solution of (2.29). $\square$

In the next lines, we prove that the solution of (2.29) given by Theorem 2 is continuous in respect to the initial data as well as in the parameter N. More precisely, we prove the following result.

THEOREM 3. *Assume that $f_i \in L^2(\tau,T;H_i)$ and $g_i : (\tau,T) \times \mathscr{C}_\gamma(H)) \to \mathbb{L}^2(\Omega)$ satisfies (h_1)–(h_3) for some fixed $\gamma > 0$, $i = 1, 2$. Let $\phi_i = (\phi_{i,1}, \phi_{i,2}) \in \mathscr{C}_\gamma(H)$ be given, with $R_i := \|\phi_i\|_\gamma$ and $N_i > 0$, $y_i(\tau) = (u_i(\tau), B_i(\tau)) \in V$, $i = 1;2$ be given. Let $y_i = (u_i, B_i)$ be the solutions of (2.29) corresponding to the parameter N_i and the initial values $y_i(\tau) = (u_i(\tau), B_i(\tau))$, $i = 1;2$. Then*

$$(u_1, B_1) \to (u_2, B_2) \text{ in } C(\tau,T;V) \cap \mathscr{D}(\tau,T;\mathscr{D}(\mathscr{A}_1) \times \mathscr{D}(\mathscr{A}_2))$$

when $N_1 \to N_2$, $(u_1(\tau), B_1(\tau)) \to (u_2(\tau), B_2(\tau))$ and $\phi_1 \to \phi_2$. More precisely, let $y = y_1 - y_2$ and $\phi = \phi_1 - \phi_2$, the following estimates hold true.

$$\sup_{\theta\in[\tau,t]} \|y(\theta)\|_V^2 \leqslant \left[\|y(\tau)\|_V^2 + (t-\tau)\|\phi\|_\gamma^2 + c(N_1-N_2)^2 \int_\tau^t \mathscr{Z}_1(\xi)d\xi\right] \times \exp\left[c(\eta+N_1^4)(t-\tau) + c\int_\tau^t \mathscr{Z}_1(\xi)d\xi\right], \tag{3.64}$$

and

$$\int_\tau^t |\mathscr{A}y(\xi)|_H^2 d\xi \leqslant \left[\|y(\tau)\|_V^2 + (t-\tau)\|\phi\|_\gamma^2 + c(N_1-N_2)^2 \int_\tau^t \mathscr{Z}_1(\xi)d\xi\right] \times \left[1 + \left(c(N_1^4+\eta)(t-\tau) + c\int_\tau^t \mathscr{Z}_1(\xi)d\xi\right) \times \exp\left[c(N_1^4+\eta)(t-\tau) + c\int_\tau^t \mathscr{Z}_1(\xi)d\xi\right]\right], \tag{3.65}$$

for all $t \in [\tau,T]$ with $\mathscr{Z}_1 = |\mathscr{A}_1 u_2|_{L^2}^2 + |\mathscr{A}_2 B_2|_{L^2}^2$.

Proof. Since $y = y_1 - y_2 = (u_1, B_1) - (u_2, B_2) = (\delta u, \delta B)$ and $\phi = \phi_1 - \phi_2 = (\phi_{1,1} - \phi_{2,1}, \phi_{1,2} - \phi_{2,2})$, then $y = (\delta u, \delta B)$ satisfies

$$\frac{dy}{dt} + \mathscr{A}y + \mathscr{B}^{N_1}(y_1,y_2) - \mathscr{B}^{N_2}(y_2,y_2) = G(t,(u_{1t}, B_{1t})) - G(t,(u_{2t}, B_{2t})). \tag{3.66}$$

From [5, 13], we have

$$\begin{aligned} R_1 &\equiv F_{N_1}(\|u_1\|_{V_1})b(u_1,u_1,\mathscr{A}_1\delta u) - F_{N_2}(\|u_2\|_{V_1})b(u_2,u_2,\mathscr{A}_1\delta u) \\ &= F_{N_1}(\|u_1\|_{V_1})b(\delta u,u_1,\mathscr{A}_1\delta u) + F_{N_2}(\|u_2\|_{V_1})b(u_2,\delta u,\mathscr{A}_1\delta u) \\ &\quad + [F_{N_1}(\|u_1\|_{V_1}) - F_{N_2}(\|u_2\|_{V_1})]\, b(u_2,u_1,\mathscr{A}_1\delta u). \end{aligned} \tag{3.67}$$

Making similar reasoning as in (3.67), we can also check that:

$$\begin{aligned} R_2 &\equiv F_{N_1}(\|(u_1,B_1)\|_V)b(B_1,B_1,\mathscr{A}_1\delta u) - F_{N_2}(\|(u_2,B_2)\|_V)b(B_2,B_2,\mathscr{A}_1\delta u) \\ &= F_{N_1}(\|(u_1,B_1)\|_V)b(\delta B,B_1,\mathscr{A}_1\delta u) + F_{N_2}(\|(u_2,B_2)\|_V)b(B_2,\delta B,\mathscr{A}_1\delta u) \\ &\quad + [F_{N_1}(\|(u_1,B_1)\|_V) - F_{N_2}(\|(u_2,B_2)\|_V)]\, b(B_2,B_1,\mathscr{A}_1\delta u) \\ &\equiv R_2^1 + R_2^2 + R_2^3, \end{aligned} \tag{3.68}$$

$$\begin{aligned} R_3 &\equiv F_{N_1}(\|(u_1,B_1)\|_V)b(u_1,B_1,\mathscr{A}_2\delta B) - F_{N_2}(\|(u_2,B_2)\|_V)b(u_2,B_2,\mathscr{A}_2\delta B) \\ &= F_{N_1}(\|(u_1,B_1)\|_V)b(\delta u,B_1,\mathscr{A}_2\delta B) + F_{N_2}(\|(u_2,B_2)\|_V)b(u_2,\delta B,\mathscr{A}_2\delta B) \\ &\quad + [F_{N_1}(\|(u_1,B_1)\|_V) - F_{N_2}(\|(u_2,B_2)\|_V)]\, b(u_2,B_1,\mathscr{A}_2\delta B) \\ &\equiv R_3^1 + R_3^2 + R_3^3, \end{aligned} \tag{3.69}$$

$$\begin{aligned} R_4 &\equiv F_{N_1}(\|(u_1,B_1)\|_V)b(B_1,u_1,\mathscr{A}_2\delta B) - F_{N_2}(\|(u_2,B_2)\|_V)b(B_2,u_2,\mathscr{A}_2\delta B) \\ &= F_{N_1}(\|(u_1,B_1)\|_V)b(\delta B,u_1,\mathscr{A}_2\delta B) + F_{N_2}(\|(u_2,B_2)\|_V)b(B_2,\delta u,\mathscr{A}_2\delta B) \\ &\quad + [F_{N_1}(\|(u_1,B_1)\|_V) - F_{N_2}(\|(B_2,u_1)\|_V)]\, b(B_2,u_1,\mathscr{A}_2\delta B) \\ &\equiv R_4^1 + R_4^2 + R_4^3. \end{aligned} \tag{3.70}$$

Also, we can check that

$$\left(\mathscr{B}^{N_1}(y_1,y_1) - \mathscr{B}^{N_2}(y_2,y_2), \mathscr{A} y\right) = R_1 - SR_2 + R_3 - R_4. \tag{3.71}$$

Hence, taking the scalar product in H of (3.66) with $\mathscr{A} y$, we obtain

$$\frac{d}{dt}\|y\|_V^2 + 2|\mathscr{A} y|_H^2 = -2R_1 + 2SR_2 - 2R_3 + 2R_4 + 2(G(t,(u_{1t},B_{1t})) - G(t,(u_{2t},B_{2t})),\mathscr{A} y). \tag{3.72}$$

We can check that (see [5, 13])

$$2|R_1| \leqslant \frac{2}{12}|\mathscr{A} y|_H^2 + cN_1^4\|y\|_V^2 + c|\mathscr{A}_1 u_2|_{L^2}^2\|y\|_V^2 + c|\mathscr{A}_1 u_2|_{L^2}^2\left(\|y\|_V^2 + (N_1 - N_1)^2\right). \tag{3.73}$$

SR_2^i, R_3^i and R_4^i, $i = 1,2,3$ satisfy the following estimates (see [13] for more details):

$$2S|R_2^1| = 2SF_{N_1}(\|(u_1,B_1)\|_V)|b(\delta B,B_1,\mathscr{A}_1\delta u)| \leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + cN_1^4\|y\|_V^2, \tag{3.74}$$

$$2S|R_2^2| = 2SF_{N_2}(\|(u_2,B_2)\|_V)|b(B_2,\delta B,\mathscr{A}_1\delta u)| \leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + c|\mathscr{A}_2 B_2|_{L^2}^2\|y\|_V^2, \tag{3.75}$$

$$\begin{aligned} 2S|R_2^3| &= 2S\,|[F_{N_1}(\|(u_1,B_1)\|_V) - F_{N_2}(\|(u_2,B_2)\|_V)]|\,|b(B_2,B_1,\mathscr{A}_1\delta u)| \\ &\leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + c|\mathscr{A}_2 B_2|_{L^2}^2\left[\|y\|_V^2 + (N_1 - N_2)^2\right], \end{aligned} \tag{3.76}$$

$$2|R_3^1| = 2F_{N_1}(\|(u_1,B_1)\|_V)|b(\delta u,B_1,\mathscr{A}_2\delta B)| \leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + cN_1^4\|y\|_V^2, \tag{3.77}$$

$$2|R_3^2| = 2F_{N_2}(\|(u_2,B_2)\|_V)|b(u_2,\delta B,\mathscr{A}_2\delta B)| \leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + c|\mathscr{A}_1 u_2|_{L^2}^2\|y\|_V^2, \tag{3.78}$$

$$\begin{aligned} 2|R_3^3| &= 2|\,[F_{N_1}(\|(u_1,B_1)\|_V) - F_{N_2}(\|(u_2,B_2)\|_V)]\,||b(u_2,B_1,\mathscr{A}_2\delta B)| \\ &\leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + c|\mathscr{A}_1 u_2|_{L^2}^2\left[\|y\|_V^2 + (N_1 - N_2)^2\right], \end{aligned} \tag{3.79}$$

$$2|R_4^1| = 2F_{N_1}(\|(u_1,B_1)\|_V)|b(\delta B,u_1,\mathscr{A}_2\delta B)| \leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + cN_1^4\|y\|_V^2, \tag{3.80}$$

$$2|R_4^2| = 2F_{N_2}(\|(u_2,B_2)\|_V)|b(B_2,\delta u,\mathscr{A}_2\delta B)| \leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + c|\mathscr{A}_2 B_2|_{L^2}^2\|y\|_V^2, \tag{3.81}$$

$$\begin{aligned} 2|R_4^3| &= 2\,|[F_{N_1}(\|(u_1,B_1)\|_V) - F_{N_2}(\|(B_2,u_1)\|_V)]|\,|b(B_2,u_1,\mathscr{A}_2\delta B)| \\ &\leqslant \tfrac{1}{12}|\mathscr{A} y|_H^2 + c|\mathscr{A}_2 B_2|_{L^2}^2\left[\|y\|_V^2 + (N_1 - N_2)^2\right]. \end{aligned} \tag{3.82}$$

$$\begin{aligned} &|2\,(G(t,(u_{1t},B_{1t})) - G(t,(u_{2t},B_{2t})),\mathscr{A} y)| \\ &\leqslant 2\max\left\{L_{g_1},L_{g_2}\right\}\|y_t\|_\gamma|\mathscr{A} y|_H + \tfrac{1}{12}\,|\mathscr{A}\,\delta y|_H^2 + c\,\|\delta y_t\|_\gamma^2. \end{aligned}$$

Just like proving (3.21) we can prove that

$$\|(\delta u_t,\delta_{B_t})\|_\gamma^2 \leqslant \|\phi\|_\gamma^2 + \sup_{\theta\in[\tau,t]} e^{2\gamma(\theta-t)}\,|y(\theta)|_H^2 \leqslant \|\phi\|_\gamma^2 + c\eta \sup_{\theta\in[\tau,t]}\|y(\theta)\|_V^2 \tag{3.83}$$

where we have used also the embedding of V into H and set $\eta = \max\{L_{g_1}, L_{g_2}\}$. Then

$$|2\,(G(t,(u_{1t},B_{1t})) - G(t,(u_{2t},B_{2t})),\mathscr{A}y)| \leqslant \frac{1}{12}\,|\mathscr{A}y|_H^2 + \|\phi\|_\gamma^2 + c\eta \sup_{\theta\in[\tau,t]}\|y(\theta)\|_V^2\,. \tag{3.84}$$

Now inserting these estimates (3.73)–(3.84) in (3.72), we obtain

$$\begin{aligned}\frac{d}{dt}\|y\|_V^2 + |\mathscr{A}y|_H^2 &\leqslant c(N_1^4 + |\mathscr{A}_1u_2|_{L^2}^2 + |\mathscr{A}_2B_2|_{L^2}^2 + \eta)\sup_{\theta\in[\tau,t]}\|y(\theta)\|_V^2\\ &\quad + c(|\mathscr{A}_1u_2|_{L^2}^2 + |\mathscr{A}_2B_2|_{L^2}^2)(N_1-N_2)^2 + \|\phi\|_\gamma^2\\ &\equiv c(N_1^4 + \eta + \mathscr{Z}_1)\sup_{\theta\in[\tau,t]}\|y(\theta)\|_V^2 + c\mathscr{Z}_1(N_1-N_2)^2 + \|\phi\|_\gamma^2\,.\end{aligned} \tag{3.85}$$

Integrating (3.85) from τ to t, we have

$$\begin{aligned}\|y(t)\|_V^2 + \int_\tau^t |\mathscr{A}y(\xi)|_H^2 d\xi &\leqslant \|y(\tau)\|_V^2 + (t-\tau)\,\|\phi\|_\gamma^2 + c(N_1-N_2)^2\int_\tau^t \mathscr{Z}_1(\xi)d\xi\\ &\quad + \int_\tau^t c(\eta + N_1^4 + \mathscr{Z}_1(\xi))\sup_{\theta\in[\tau,\xi]}\|y(\theta)\|_V^2\,d\xi\,,\end{aligned} \tag{3.86}$$

which leads to

$$\begin{aligned}\sup_{\theta\in[\tau,t]}\|y(\theta)\|_V^2 + \int_\tau^t |\mathscr{A}y(\xi)|_H^2 d\xi &\leqslant \|y(\tau)\|_V^2 + (t-\tau)\,\|\phi\|_\gamma^2 + c(N_1-N_2)^2\int_\tau^t \mathscr{Z}_1(\xi)d\xi\\ &\quad + \int_\tau^t c(\eta + N_1^4 + \mathscr{Z}_1(\xi))\sup_{\theta\in[\tau,\xi]}\|y(\theta)\|_V^2\,d\xi\,.\end{aligned} \tag{3.87}$$

It follows from Lemma 1 and (3.87) when dropping the term $\int_\tau^t |\mathscr{A}y(\xi)|_H^2 d\xi$ that

$$\begin{aligned}\sup_{\theta\in[\tau,t]}\|y(\theta)\|_V^2 &\leqslant \left[\|y(\tau)\|_V^2 + (t-\tau)\,\|\phi\|_\gamma^2 + c(N_1-N_2)^2\int_\tau^t \mathscr{Z}_1(\xi)d\xi\right]\\ &\quad\times\exp\left[c(\eta + N_1^4)(t-\tau) + c\int_\tau^t \mathscr{Z}_1(\xi)d\xi\right],\end{aligned} \tag{3.88}$$

which proves (3.64).

Now using (3.87) and (3.88), we get

$$\int_\tau^t |\mathscr{A} y(\xi)|_H^2 d\xi \leqslant \left[\|y(\tau)\|_V^2 + (t-\tau)\,\|\phi\|_\gamma^2 + c(N_1-N_2)^2 \int_\tau^t \mathscr{Z}_1(\xi)d\xi \right] \times \left[1 + \left(c(N_1^4+\eta)(t-\tau) + c\int_\tau^t \mathscr{Z}_1(\xi)d\xi \right) \times \exp\left[c(N_1^4+\eta)(t-\tau) + c\int_\tau^t \mathscr{Z}_1(\xi)d\xi \right] \right]. \quad \square \tag{3.89}$$

REMARK 5. Theorem 3 also provides the uniqueness of the strong solution of problem (1.2).

4. Stationary solution

In this section, we are interested in proving that problem (1.2) with some restrictions, admits stationary solutions. Also, we prove under additional assumptions that the stationary solution is unique and is globally asymptotically exponentially stable.

The restrictions we impose to give sense to a stationary solution are that f_1, f_2, g_1 and g_2 are independent on time. One of the worries is that how (g_1, g_2) acts over a fixed element of H. Inspired by what was done in [28], we consider $(g_1(w), g_2(w))$ as $(g_1(\widetilde{w}), g_2(\widetilde{w}))$, where $\widetilde{w} \in H$ is the element that has the only value w for all time $t \leqslant 0$. $\widetilde{w}$ is an element of $\mathscr{C}_\gamma(H)$ and $\|\widetilde{w}\|_\gamma = |w|_H$ for some fixed $\gamma > 0$; so we will continue denoting w instead of $\widetilde{w}$ since no confusion arises. The hypothesis (h_3) is reformulated as follows: There exists a constant $L_{g_i} > 0$ such that for all $\xi, \eta \in H$,

$$|g_i(\xi) - g_i(\eta)|_{L^2(\Omega)} \leqslant L_{g_i} |\xi - \eta|_H . \tag{4.1}$$

We consider the following system

$$\begin{cases} \dfrac{dy}{dt} + \mathscr{A}y + \mathscr{B}^N(y,y) = F + G \ \text{ on } \ \mathscr{D}'(\tau, T; V') \\ y(\tau+s) = (\phi_1(s), \phi_2(s)),\ s \in (-\infty, 0] \end{cases} \tag{4.2}$$

where $\mathscr{A}$ and $\mathscr{B}^N$ are defined in section 3.

By a stationary solution to (4.2) we mean an element $(u^*, B^*) \in V$ such that for all $\varphi = (v, C) \in V$,

$$((y,\varphi)) + \mathscr{B}_0^N(y,y,\varphi) = \langle f_1, v\rangle + \langle f_2, C\rangle + \langle g_1(u^*, B^*), v\rangle + \langle g_2(u^*, B^*), C\rangle . \tag{4.3}$$

We will prove the existence result by Galerkin's method before studiying its asymptotic behavior.

4.1. Existence result

THEOREM 4. *We assume that f_1, f_2, g_1, g_2 are independent on time, and (4.1) holds. If $1 > \left\{cLg_1(\frac{R_e}{\lambda_1^1})^{1/2} + cLg_2(\frac{R_m}{\lambda_1^2})^{1/2}\right\}$, Problem (4.3) has at least one solution (u^*, B^*) which belongs to $\mathscr{D}(\mathscr{A})$.*

In addition, if

$$\min\{1,S\} \geqslant \max\left\{R_e(\lambda_1^1)^{-1}, SR_m(\lambda_1^2)^{-1}\right\}\left(cN^4+cN^8\right) \\ +cL_{g_1}\left(\frac{R_e}{\lambda_1^1}\right)^{1/2} + ScL_{g_2}\left(\frac{R_m}{\lambda_1^2}\right)^{1/2}, \tag{4.4}$$

this solution is unique.

Proof. Like dealing with the evolutionary case, the existence of a solution $y^* = (u^*, B^*)$ of problem (4.3) is proved by the Galerkin's method as follows.

Since the injection $V \subset H$ is compact, let $\{(w_i, \psi_i), i = 1,2,\ldots\} \subset V$ be an orthonormal basis of H, where $\{w_i, i = 1,2,\ldots\}$, $\{\psi_i, i = 1,2,\ldots\}$ are eigenfunctions of $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. We set $V_m = H_m = \text{span}\{(w_1,\psi_1),\ldots,(w_m,\psi_m)\}$ and denote by $P_m = (P_m^1, P_m^2)$, the orthogonal projector from H onto V_m for the scalar product $(.,.)$ defined by $(2.8)_1$. Note that P_m is also the orthogonal projector from $D(\mathscr{A}), V, V'$ onto V_m. We look for $y_m^* = (u_m^*, B_m^*)$, verifying for $\varphi = (v, C) \in V_m$

$$\langle \mathscr{A} y_m^*, \varphi\rangle + \left\langle P_n\mathscr{B}^N(y_m^*, y_m^*), \varphi\right\rangle = \langle f_1, v\rangle + \langle f_2, C\rangle + \langle g_1(u_m^*, B_m^*), v\rangle \\ + \langle g_2(u_m^*, B_m^*), C\rangle\ . \tag{4.5}$$

Since we will apply a consequence of Brouwer's fixed point theorem, see ([46], Lemma 41, page 23), we define the operators $\ell_m : V_m \to V_m$ by, for all $y = (u, B)$, $\varphi = (v, C) \in V_m$.

$$((\ell_m y, \varphi)) = \langle \mathscr{A} y, \varphi\rangle + \left\langle P_m\mathscr{B}^N(y,y), \varphi\right\rangle - \langle f_1, v\rangle - \langle f_2, C\rangle - \langle g_1(u,B), v\rangle \\ - \langle g_2(u,B), C\rangle\ . \tag{4.6}$$

Since the right hand side of (4.6) is a linear continuous map from V_m to $\mathbb{R}$, by the Riez theorem, each $\ell_m y \in V_m$ is well defined. We now prove that ℓ_m is continuous.

Let $(y_n) \subset V_m$ be a sequence which converges to y in V_m, since $\mathscr{A}$ and $\mathscr{B}^N$ are continuous from V_m to V_m' then $\mathscr{A} y_n \to \mathscr{A} y$ and $\mathscr{B}^N(y_n, y_n) \to \mathscr{B}^N(y,y)$. In addition,

$$\|g_i(y_n) - g_i(y)\|_{V_{i,m}'} \leqslant |g_i(y_n) - g_i(y)|_{L^2} \leqslant L_{g_i}|y_n - y|_H \leqslant cL_{g_i}|y_n - y|_V\ .$$

By the compactness of the embedding $V_m \hookrightarrow H_m$ and (4.6), we infer that $R_m y_n \to R_m y$ in V_m as $n \to \infty$. On the other hand, for all $y = (u, B) \in V_m$,

$$\begin{aligned}((\ell_m y, y)) &= \langle \mathscr{A} y, y\rangle - \langle f_1, u\rangle - \langle f_2, B\rangle - \langle g_1(u,B), u\rangle - \langle g_2(u,B), B\rangle \\ &\geqslant \|y\|_V^2 - (\lambda_1^1)^{-1/2}|f_1|_{L^2}\|u\|_{V_1} - (\lambda_1^2)^{-1/2}|f_2|_{L^2}\|B\|_{V_2} \\ &\quad -c(\lambda_1^1)^{-1/2}L_{g_1}\|y\|_V\|v\|_{V_1} - c(\lambda_1^2)^{-1/2}L_{g_2}\|y\|_V\|C\|_{V_2} \\ &\geqslant \|y\|_V\left[\|y\|_V\left(1-\left\{cL_{g_1}(\tfrac{R_e}{\lambda_1^1})^{1/2} + cL_{g_2}(\tfrac{R_m}{\lambda_1^2})^{1/2}\right\}\right)\right. \\ &\quad \left. -(\tfrac{R_e}{\lambda_1^1})^{1/2}|f_1|_{L^2} - (\tfrac{R_m}{\lambda_1^2})^{1/2}|f_2|_{L^2}\right]\end{aligned} \tag{4.7}$$

where $c = \left(\max\left\{\lambda_1^1 R_e, \lambda_1^2 R_m\right\}\right)^{1/2}$.

Then, $((\ell_m y, y))$ is non negative on the sphere of V_m with radius

$$\beta \geqslant \frac{(\frac{R_e}{\lambda_1^1})^{1/2}\,|f_1|_{L^2} + (\frac{R_m}{\lambda_1^2})^{1/2}\,|f_2|_{L^2}}{1 - \left\{cL_{g_1}(\frac{R_e}{\lambda_1^1})^{1/2} + cL_{g_2}(\frac{R_m}{\lambda_1^2})^{1/2}\right\}}.$$

So by a consequence of Brouwer fixed point theorem, for each $m \geqslant 1$, there exists $y_m^* = (u_m^*, B_m^*) \in V_m$ solution of (4.5). Moreover, replacing ϕ in the first equation of (4.5) by $\mathscr{A} y_m^*$, we obtain

$$\begin{aligned} |\mathscr{A} y_m^*|_H^2 = &(f_1, \mathscr{A}_1 u_m^*) + (f_2, \mathscr{A}_2 B_m^*) - \mathscr{B}_0^N(y_m^*, y_m^*, \mathscr{A} y_m^*) \\ &+ (g_1(u_m^*, B_m^*), \mathscr{A}_1 u_m^*)) + (g_2(u_m^*, B_m^*), \mathscr{A}_2 B_m^*)). \end{aligned} \tag{4.8}$$

Now using (2.23) and Young's inequality with the exponents $(4, 4/3)$, we have

$$|\mathscr{B}_0^N(y_m^*, y_m^*, \mathscr{A} y_m^*)| \leqslant cN\|y_m^*\|_V^{1/2}|\mathscr{A} y_m^*|_H^{3/2} \leqslant \tfrac{1}{8}|\mathscr{A} y_m^*|_H^2 + cN^4\|y_m\|_V^2. \tag{4.9}$$

In addition, by Young's inequality and (4.1) one obtains

$$\begin{aligned} |(f_1, \mathscr{A}_1 u_m^*)| + |(f_2, \mathscr{A}_2 B_m^*)| &\leqslant \tfrac{1}{8}|\mathscr{A}_1 u_m^*|_{L^2}^2 + \tfrac{1}{8}|\mathscr{A}_2 B_m^*|_{L^2}^2 + c|f_1|_{L^2}^2 + c|f_2|_{L^2}^2 \\ &= \tfrac{1}{8}|\mathscr{A} y_m^*|_H^2 + c|f_1|_{L^2}^2 + c|f_2|_{L^2}^2 \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &|(g_1(u_m^*, B_m^*), \mathscr{A}_1 u_m^*)) + (g_2(t, (u_m^*, B_m^*), \mathscr{A}_2 B_m^*))| \\ &\leqslant \tfrac{1}{4}|\mathscr{A}_1 u_m^*|_{L^2}^2 + \tfrac{1}{4}|\mathscr{A}_2 B_m^*|_{L^2}^2 + \left\{(cL_{g_1})^2 + (cL_{g_2})^2\right\}\|y_m^*\|_V^2 \\ &= \tfrac{1}{4}|\mathscr{A} y_m^*|_H^2 + \left\{(cL_{g_1})^2 + (cL_{g_2})^2\right\}\|y_m^*\|_V^2. \end{aligned} \tag{4.11}$$

It follows from (4.9)–(4.11) that

$$|\mathscr{A} y_m^*|_H^2 \leqslant c|f_1|_{L^2}^2 + c|f_2|_{L^2}^2 + \left((cL_{g_1})^2 + (cL_{g_2})^2 + cN^4\right)\|y_m^*\|_V^2. \tag{4.12}$$

From (4.12), we infer that the sequence y_m^* is bounded in $\mathscr{D}(\mathscr{A})$; consequently, using the compact injection of $\mathscr{D}(\mathscr{A})$ in V, we can extract a subsequence of y_m^* still denoted by y_m^* which converges weakly in $\mathscr{D}(\mathscr{A})$ and strongly in V to an element $(u^*, B^*) \in \mathscr{D}(\mathscr{A})$. Finally taking the limit in (4.5), we prove that (u^*, B^*) is a solution of a stationary problem (4.3).

For the uniqueness, let $y^* = (u^*, B^*)$ and $\widetilde{y}^* = (\widetilde{u}^*, \widetilde{B}^*)$ two solutions of (4.3), we set $y = y^* - \widetilde{y}^*$; $u = u^* - \widetilde{u}^*$ and $B = u^* - \widetilde{u}^*$ then

$$\mathscr{A} y = -\left(\mathscr{B}^N(y^*, y^*) - \mathscr{B}^N(\tilde{y}^*, \tilde{y}^*)\right) + \left(G(u^*, B^*) - G(\tilde{u}^*, \widetilde{B}^*))\right). \tag{4.13}$$

Taking the inner product in H of (4.13) with My and proceed like proving the uniqueness result in the non-stationary case and taking into account (4.1), we obtain

$$\begin{aligned} \frac{1}{R_e}\|u\|_{V_1}^2 + \frac{S}{R_m}\|B\|_{V_2}^2 = &-(\mathscr{B}^N(y^*, y^*) - \mathscr{B}^N(\widetilde{y}^*, \widetilde{y}^*), My) + \left(g_1(u^*, B^*) - g_1(\widetilde{u}^*, \widetilde{B}^*), u\right) \\ &+ \left(g_2(u^*, B^*) - g_2(\widetilde{u}^*, \widetilde{B}^*), B\right). \end{aligned} \tag{4.14}$$

But

$$\begin{aligned}
&\left|\left(g_1(u^*,B^*)-g_1(\widetilde{u}^*,\widetilde{B}^*),u\right)+\left(g_2(u^*,B^*)-g_2(\widetilde{u}^*,\widetilde{B}^*),SB\right)\right|\\
&\leqslant cL_{g_1}(\lambda_1^1)^{-1/2}\|(u,B)\|_V\|u\|_{V_1}+ScL_{g_2}(\lambda_1^2)^{-1/2}\|(u,B)\|_V\|B\|_{V_2}\\
&\leqslant\left\{cL_{g_1}(\tfrac{R_e}{\lambda_1^1})^{1/2}+ScL_{g_2}(\tfrac{R_m}{\lambda_1^2})^{1/2}\right\}\|(u,B)\|_V^2 .
\end{aligned}\tag{4.15}$$

On the other hand,

$$\begin{aligned}
&\left|\left(-\mathscr{B}^N(y^*,y^*)+\mathscr{B}^N(\tilde{y}^*,\tilde{y}^*),My\right)\right|\\
&\leqslant\left(cN^4+cN^8\right)\left\{\frac{R_e(\lambda_1^1)^{-1}}{R_e}\|u\|_{V_1}^2+\frac{SR_m(\lambda_1^2)^{-1}}{R_m}\|B\|_{V_2}^2\right\}\\
&\leqslant\left(cN^4+cN^8\right)\max\left\{R_e(\lambda_1^1)^{-1},SR_m(\lambda_1^2)^{-1}\right\}\|(u,B)\|_V^2 .
\end{aligned}\tag{4.16}$$

Including (4.15) and (4.16) in (4.14), we obtain

$$\begin{aligned}
&\min\{1,S\}\|(u,B)\|_V^2\\
&\leqslant\left[\max\left\{R_e(\lambda_1^1)^{-1},SR_m(\lambda_1^2)^{-1}\right\}\left(cN^4+cN^8\right)+cL_{g_1}(\tfrac{R_e}{\lambda_1^1})^{1/2}+ScL_{g_2}(\tfrac{R_m}{\lambda_1^2})^{1/2}\right]\\
&\times\|(u,B)\|_V^2 .
\end{aligned}\tag{4.17}$$

Consequently, if (4.4) holds, $\|(u,B)\|_V^2\leqslant 0$ then, $u^*=\widetilde{u}^*$ and $B^*=\widetilde{B}^*$. □

4.2. Stability of the stationary solution

As announced before, we prove here that the unique solution of (4.3) given by Theorem 4 is globally asymptotically exponentially stable. More precisely, we prove the following result.

THEOREM 5. *Assume that f_1,f_2,g_1,g_2 are independent on time and (4.4) is valid. Assume that the assumptions in Theorem 2 are valid.*

Then for some fixed $\gamma>0$, there exists a value $0<\beta<2\gamma$ such that for the solution $y(.,0,\phi)=(u(.,0,\phi),B(.,0,\phi))$ of problem (1.2) with $\tau=0$ and $\phi\in\mathscr{C}_\gamma(H)$, the following estimates hold for all $t\geqslant 0$:

$$[y(t,0,\phi)-y^*]^2\leqslant e^{-\beta t}\left([\phi(0)-y^*]^2+\left\{L_{g_1}^2\frac{R_e}{\lambda_1^1}+L_{g_2}^2\frac{R_m}{S\lambda_1^2}\right\}\frac{\|\phi-y^*\|_\gamma^2}{2\gamma-\beta}\right),\tag{4.18}$$

and

$$\begin{aligned}
&\|y_t(t,0,\phi)-y^*\|_\gamma^2\\
&\leqslant\max\left\{e^{-2\gamma t}\|\phi-y^*\|_\gamma^2;e^{-\beta t}\left\{[\phi(0)-y^*]^2+\left\{L_{g_1}^2\frac{R_e}{\lambda_1^1}+L_{g_2}^2\frac{R_m}{S\lambda_1^2}\right\}\frac{\|\phi-y^*\|_\gamma^2}{2\gamma-\beta}\right\}\right\}
\end{aligned}\tag{4.19}$$

where y^* is the unique solution of (4.5) given by Theorem 4.

Proof. Just for simplification, we denote $y(t)=y(.,0,\phi)$. We also denote $w(t)=y(t)-y^*$. From equations (4.2) and (4.3), one has for any $z=(v,C)$

$$\begin{aligned}\frac{d}{dt}(w(t),z)_H+((w(t),z))_V=&-(\mathscr{B}^N(y(t),y(t))-\mathscr{B}^N(y^*,y^*),z)\\&+(g_1(u_t,B_t)-g_1(u^*,B^*),v)\\&+(g_2(u_t,B_t)-g_2(u^*,B^*),C)\,.\end{aligned}\tag{4.20}$$

Seeing how (3.17) is obtained, we have

$$\begin{aligned}&\frac{d}{dt}\left(|u_n(t)|^2_{L^2}+S|B_n(t)|^2_{L^2}\right)+\frac{2}{R_e}\|u_n(t)\|^2_{V_1}+\frac{2S}{R_m}\|B_n(t)\|^2_{V_2}\\&\leqslant L^2_{g_1}\frac{R_e}{\lambda^1_1}\|w_t\|^2_\gamma+\frac{1}{R_e}\|u\|^2_{V_1}+L^2_{g_2}\frac{R_m}{S\lambda^2_1}\|w_t\|^2_\gamma+\frac{S}{R_m}\|B\|^2_{V_2}\\&\quad+2\left(cN^4+cN^8\right)\max\left\{\frac{1}{\lambda^1_1},\frac{S}{\lambda^2_1}\right\}\|(u,B)\|^2_V\,,\end{aligned}\tag{4.21}$$

which leads to

$$\begin{aligned}&\frac{d}{dt}[w(t)]^2+\left[\min\{1,S\}-2\left(cN^4+cN^8\right)\max\left\{\frac{1}{\lambda^1_1},\frac{S}{\lambda^2_1}\right\}\right]\|w(t)\|^2_V\,.\\&\qquad\leqslant\left(L^2_{g_2}\frac{R_m}{S\lambda^2_1}+L^2_{g_1}\frac{R_e}{\lambda^1_1}\right)\|w_t\|^2_\gamma\,.\end{aligned}\tag{4.22}$$

Introducing in (4.22) an exponential term $e^{\beta t}$ with a positive value $0<\beta<2\gamma$ and integrating from 0 to t, we obtain

$$\begin{aligned}&e^{\beta t}[w(t)]^2+\left[\min\{1,S\}-2\left(cN^4+cN^8\right)\max\left\{\frac{1}{\lambda^1_1},\frac{S}{\lambda^2_1}\right\}\right.\\&\quad\left.-\beta\max\left\{\frac{1}{\lambda^1_1},\frac{1}{\lambda^2_1}\right\}\right]\int_0^t e^{\beta\xi}\|w(\xi)\|^2_V\,d\xi\,.\\&\leqslant[w(0)]^2+\left(L^2_{g_2}\frac{R_m}{S\lambda^2_1}+L^2_{g_1}\frac{R_e}{\lambda^1_1}\right)\int_0^t e^{\beta\xi}\left\|w_\xi\right\|^2_\gamma d\xi\,.\end{aligned}\tag{4.23}$$

But

$$\begin{aligned}&\int_0^t e^{\beta\xi}\left\|w_\xi\right\|^2_\gamma d\xi\\&=\int_0^t e^{\beta\xi}\sup_{\theta\in(-\infty,0]}e^{2\gamma\theta}\,|(w(\xi+\theta)|^2_H\,d\xi\\&=\int_0^t e^{\beta\xi}\max\left\{\sup_{\theta\leqslant-\xi}e^{2\gamma\theta}\,|(w(\xi+\theta)|^2_H;\ \sup_{\theta\in[-\xi,0]}e^{2\gamma\theta}\,|(w(\xi+\theta)|^2_H\right\}d\xi\\&=\int_0^t\max\left\{e^{-(2\gamma-\beta)\xi}\|\phi-y^*\|^2_\gamma;\ \sup_{\theta\in[-s,0]}e^{(2\gamma-\beta)\theta}e^{\beta(\xi+\theta)}\,|(w(\xi+\theta)|^2_H\right\}d\xi\\&\leqslant\int_0^t\left\{e^{-(2\gamma-\beta)\xi}\|\phi-y^*\|^2_\gamma+\sup_{r\in[0,\xi]}e^{r\beta}\,|(w(r)|^2_H\right\}d\xi\end{aligned}\tag{4.24}$$

where we have used $\beta \leqslant 2\gamma$ to deduce the last inequality. Using this in (4.23), we obtain

$$\begin{aligned}
e^{\beta t}[w(t)]^2 \leqslant [w(0)]^2 &+ \left(L_{g_2}^2 \frac{R_m}{S\lambda_1^2} + L_{g_1}^2 \frac{R_e}{\lambda_1^1}\right) \|\phi - y^*\|_\gamma^2 \int_0^t e^{-(2\gamma-\beta)\xi} d\xi \\
&+ \left[2\left(cN^4 + cN^8\right) \max\left\{\frac{1}{\lambda_1^1}, \frac{S}{\lambda_1^2}\right\}\right. \\
&\left. + \beta \max\left\{\frac{1}{\lambda_1^1}, \frac{1}{\lambda_1^2}\right\} + c\left(L_{g_2}^2 \frac{R_m}{S\lambda_1^2} + L_{g_1}^2 \frac{R_e}{\lambda_1^1}\right) - \min\{1, S\}\right] \\
&\times \int_0^t \sup_{r\in[0,\xi]} e^{\beta r} \|w(r)\|_V^2 \, d\xi\,.
\end{aligned} \tag{4.25}$$

We can choose β such that the coefficient of the last integral in (4.25) is negative. Doing that, we deduce (4.18).

On the other hand,

$$\begin{aligned}
\|w(t)\|_\gamma^2 &= \sup_{\theta\leqslant 0} e^{2\gamma\theta} |w(t+\theta)|_H^2 \\
&= \max\left\{\sup_{\theta\leqslant -t} e^{2\gamma\theta} |(w(t+\theta)|_H^2;\ \sup_{\theta\in[-t,0]} e^{2\gamma\theta} |(w(t+\theta)|_H^2\right\} \\
&\leqslant \max\left\{e^{-2\gamma t} \|\phi - y^*\|_\gamma^2,\ \sup_{\theta\in[-t,0]} e^{2\gamma\theta} |(w(t+\theta)|_H^2\right\}.
\end{aligned} \tag{4.26}$$

Using (4.18), we have

$$\begin{aligned}
&\sup_{\theta\in[-t,0]} e^{2\gamma\theta} |(w(t+\theta)|_H^2 \\
&\leqslant \sup_{\theta\in[-t,0]} e^{2\gamma\theta} e^{-\beta(t+\theta)} \left([\phi(0) - y^*]^2 + \left\{L_{g_1}^2 \frac{R_e}{(\lambda_1^1)} + L_{g_2}^2 \frac{R_m}{(S\lambda_1^2)}\right\} \frac{\|\phi - y^*\|_\gamma^2}{2\gamma - \beta}\right) \\
&= \sup_{\theta\in[-t,0]} e^{(2\gamma-\beta)\theta} e^{-\beta t} \left([\phi(0) - y^*]^2 + \left\{L_{g_1}^2 \frac{R_e}{(\lambda_1^1)} + L_{g_2}^2 \frac{R_m}{(S\lambda_1^2)}\right\} \frac{\|\phi - y^*\|_\gamma^2}{2\gamma - \beta}\right) \\
&\leqslant e^{-\beta t} \left([\phi(0) - y^*]^2 + \left\{L_{g_1}^2 \frac{R_e}{(\lambda_1^1)} + L_{g_2}^2 \frac{R_m}{(S\lambda_1^2)}\right\} \frac{\|\phi - y^*\|_\gamma^2}{2\gamma - \beta}\right)
\end{aligned}$$

since $e^{(2\gamma-\beta)\theta} \leqslant 1$ for $\theta \in [-t, 0]$. Hence, we deduce (4.19). □

Conflict of interest. The authors declare that there is no conflict of interest.

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