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A BOUND FOR ZEROS OF SOLUTIONS TO A HIGHER ORDER NON–HOMOGENEOUS ODE WITH POLYNOMIAL COEFFICIENTS

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Abstract. Let $P_k(z)$ $(k=1,2,\ldots,n)$ and $G(z)$ be polynomials with complex in general coefficients. The paper deals with the higher order differential equation

$$v^{(n)}(z)+P_1(z)v^{(n-1)}(z)+\ldots+P_n(z)v(z)=G(z).$$

We derive estimates for the sums of the zeros of solutions to this equation. These estimates give us bounds for the function counting the zeros of solutions and information about the zero-free domain. Some other applications are also discussed.

1. Introduction and statement of the main result

Let

$$P_k(z)=\sum_{j=0}^{\nu_k}c_{kj}z^j \ \ (\nu_k<\infty,\ k=1,\ldots,n) \text{ and } G(z)=\sum_{j=0}^{\nu_G}\psi_j z^j \ \ (z\in\mathbb{C},\nu_G<\infty)$$

be polynomials with complex, in general, coefficients. The paper deals with the zeros of solutions to the equation

$$\frac{d^n v}{dz^n}+\sum_{k=1}^{n}P_k(z)\frac{d^{n-k}v}{dz^{n-k}}=G(z)\ \ (v=v(z)). \tag{1.1}$$

For arbitrary initial conditions the Cauchy problem to (1.1) has solutions which are entire functions, cf. [14, Proposition 8.1]. The literature devoted to the zeros of the solutions of ordinary differential equations (ODEs) is very rich. Besides, the main tool is the Nevanlinna theory. The excellent exposition of the Nevanlinna theory and its applications to differential equations is given in the book [14]. In that book the results of many mathematicians are reflected.

The recent results on complex zeros of solutions to ODEs can be found in the papers [3, 6, 15, 16, 17, 18], and references given therein. In particular, the paper [18] studies the convergence of the zeros of a non-trivial (entire) solution to the linear differential equation

$$f''+\{Q_1(z)e^{P_1(z)}+Q_2(z)e^{P_2(z)}+Q_3(z)e^{P_3(z)}\}f=0,$$

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where P_j are polynomials of degree $n \geqslant 1$ and $Q_j(\not\equiv 0)$ are entire functions of order less than n $(j = 1,2,3)$. The real zeros of solutions to equations with polynomial coefficients were investigated in the papers by Eremenko and Merenkov [5], and by C. Z. Huang [13]. The remarkable results on the zeros of a wide class of ordinary differential equations with polynomial coefficients, whose solutions are classical orthogonal polynomials, have been established by N. Anghel [1]. In addition, in the paper [2] N. Anghel investigated the following question: when is an entire function of finite order, the solution to a complex second order homogeneous linear differential equation with polynomial coefficients ? He gives two (equivalent) answers to this question, one of which involves certain Stieltjes-like relations for the zeros of solutions, the second one requires the vanishing of all but finitely many suitable expressions constructed via the relations of the sums of the zeros of the function derived in [7].

Certainly, we could not survey the whole subject here and refer the reader to the listed publications.

It should be noted that in the above cited works mainly the asymptotic distributions of zeros are investigated. At the same time, bounds for the zeros of solutions are very important in various applications, but to the best of our knowledge, they have been investigated considerably less than the asymptotic distributions. In the paper [8] the author has established bounds for the sums of the zeros of solutions for the second order homogeneous equations with polynomial coefficients. In the interesting paper [4], some results from [8] have been extended to the equation $u^{(m)} = P(z)u$, where P is a polynomial and $m > 2$. In the papers [11] and [10] the main result from [8] have been extended to the second order ODEs having singular points and to non-homogeneous second order ODEs, respectively. In addition, in the paper [9] the author has derived a bound for the products of the zeros of solutions to second order ODEs with polynomial coefficients.

In this paper we generalize the main results from the papers [8] and [10] to equation (1.1). That generalization requires a considerably new approach. Our main tool is a combined usage of the new solution estimates for the higher order differential equations and recent estimates for the roots of entire functions. Besides, we generalize the well known [14, Theorem 8.3] on the order of solutions of non-homogeneous ODEs.

To formulate our main result note that any initial condition for (1.1) can be reduced to the initial condition

$$v(0) = 1, v^{(k)}(0) = 0 \ \ (k = 1,\dots,n-1). \tag{1.2}$$

For a continuous function f and a positive number r put $\hat{f}(r) = \sup_{|z|\leqslant r} |f(z)|$. Clearly, $r^k \hat{P}_k(r) \leqslant p_k(r)$ $(k = 1,\dots,n)$ and $\hat{G}(z) \leqslant g(r)$, where

$$p_k(r) := \sum_{j=0}^{\nu_k} |c_{k,j}| r^{k+j} \text{ and } g(r) := \sum_{j=0}^{\nu_G} |\psi_j| r^j.$$

Below we check that there are constants $\alpha_k \leqslant (p_k(1))^{1/k}$ and $\beta_k \leqslant (p_k(1))^{1/k}$, such that

$$(p_k(r))^{1/k} \leqslant \beta_k + \alpha_k r^{\rho_0} \ \ (r > 0;\ k = 1,\dots,n),$$

where $\rho_0 = 1 + \max_{k=1,\ldots,n}(\nu_k/k)$. Put

$$a_0 = \max_{k=1,\ldots,n} 2^{(n-1)/k}\alpha_k, \ \tilde{r}_j := \left(\frac{j}{a_0\rho_0}\right)^{1/\rho_0} \text{ and } \chi_0 = 2(ea_0\rho_0)^{1/\rho_0}.$$

We will show that

$$\theta_0 := b_0 \sum_{j=1}^{\infty} \frac{\tilde{r}_j^n[g(\tilde{r}_j) + p_n(\tilde{r}_j)]}{2^j} < \infty, \tag{1.3}$$

where

$$b_0 = \frac{1}{n!2^{n-1}} \sum_{k=1}^{n} \exp\,[2^{(n-1)/k}\beta_k],$$

Enumerate the zeros $z_k(v)$ of the solution $v(z)$ to problem (1.1), (1.2), with the multiplicities taken into account, in the non-decreasing order of their absolute values: $|z_k(v)| \leqslant |z_{k+1}(v)| \ \ (k = 1,2,\ldots)$. Now we are in a position to formulate the main result of the paper.

THEOREM 1. *The zeros* $z_k(v) \ (k = 1,2,\ldots)$ *of the solution* $v(z)$ *to problem* (1.1), (1.2) *satisfy the inequalities*

$$\sum_{k=1}^{j} \frac{1}{|z_k(v)|} \leqslant \chi_0 \left[\theta_0 + \sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho_0}}\right] \ (j = 1,2,\ldots). \tag{1.4}$$

This theorem is proved in the next three sections.

Below we show that Theorem 1 gives us a bound for the function counting the zeros of solutions and information about the zero-free domain. Some other applications are also discussed.

2. Solution estimates

Consider the equation

$$\frac{d^n u}{dz^n} + \sum_{k=1}^{n} Q_k(z)\frac{d^{n-k}u}{dz^{n-k}} = F(z) \ \ (u = u(z)), \tag{2.1}$$

where $Q_k(z) \ \ (k = 1,\ldots,n)$ and $F(z)$ are continuous functions.

A solution of (2.1) is an n-times continuously differentiable function $u(z)$ defined for all $z \in \mathbb{C}$ and satisfying (2.1), and the given initial conditions. Since the equation is linear, the existence and uniqueness of solutions is well-known, cf. [14].

To estimate solutions of (2.1) we need the following

LEMMA 1. *Let* $x_k \ \ (k = 1,\ldots,m < \infty)$ *be positive numbers. Then*

$$(\sum_{k=1}^{m} x_k)^j \leqslant 2^{(m-1)(j-1)} \sum_{k=1}^{m} x_k^j \ \ (j = 1,2,\ldots).$$

Proof. Let

$$f(x)=\frac{(1+x)^j}{1+x^j}\ \ (x>0).$$

Simple calculations show that $f'(1)=0$ and $f(1)=2^{j-1}$. So $(1+x)^j\leqslant 2^{j-1}(1+x^j)$ and

$$(x_1+x_2)^j\leqslant 2^{j-1}(x_1^j+x_2^j).$$

Hence,

$$(x_1+x_2+x_3)^j\leqslant 2^{j-1}(x_1^j+(x_2+x_3)^j)\leqslant 2^{j-1}(x_1^j+2^{j-1}(x_2^j+x_3^j)\leqslant 2^{2j}(x_1^j+x_2^j+x_3^j)..$$

If

$$(\sum_{k=1}^{m-1}x_k)^j\leqslant 2^{(m-2)(j-1)}\sum_{k=1}^{m-1}x_k^j,$$

then we have

$$(\sum_{k=1}^{m}x_k)^j\leqslant 2^{(m-2)(j-1)}[\sum_{k=1}^{m-2}x_k^j+(x_{m-1}+x_m)^j]$$

$$\leqslant 2^{(j-1)(m-2)}(\sum_{k=1}^{m-2}x_k^j+2^{j-1}(x_{m-1}^j+x_m^j))\leqslant 2^{(j-1)(m-1)}\sum_{k=1}^{m}x_k^j.$$

This induction proves the lemma. □

Put

$$\gamma_m(x)=\sum_{k=0}^{\infty}\frac{x^k}{(mk)!}\ \ (x\geqslant 0)$$

for an integer $m\geqslant 1$. Since

$$\sum_{k=0}^{\infty}\frac{x^k}{(mk)!}\leqslant\sum_{k=0}^{\infty}\frac{(\sqrt[m]{x})^k}{k!},$$

we have

$$e^{\sqrt[m]{x}}\geqslant\gamma_m(x)\ \ (x\geqslant 0;\ m=1,2,\dots). \tag{2.2}$$

LEMMA 2. *Let $u(z)$ be a solution of equation (2.1) with the zero initial condition*

$$u^{(k)}(0)=0\ \ (k=0,\dots,n-1). \tag{2.3}$$

Then

$$\hat u^{(n)}(r)\leqslant\frac{1}{2^{n-1}}\hat F(r)\sum_{k=1}^{n}\gamma_k(2^{n-1}\hat Q_k(r)r^k)\ \ (r\geqslant 0).$$

Proof. Put $w(z)=\frac{d^nu(z)}{dz^n}$. Then

$$\frac{d^{n-k}u(z)}{dz^{n-k}}=(J^kw)(z),$$

where

$$(Jw)(z) = \int_0^z w(s)ds \text{ and } (J^k w)(z) = \frac{1}{(k-1)!}\int_0^z (z-s)^{k-1} w(s)ds.$$

From (2.1) we have

$$w(z) + \sum_{k=1}^{n} Q_k(z)(J^k w)(z) = F(z). \tag{2.4}$$

For a fixed $t \in [0, 2\pi)$ and $z = re^{it}$ we get

$$\sup_{|z|\leqslant r} |(Jw)(z)| = \sup_{|z|\leqslant r} \Big|\int_0^{re^{it}} w(s)ds\Big| \leqslant \int_0^r \hat{w}(r_1)dr_1 = (J\hat{w})(r).$$

Similarly,

$$\sup_{|z|\leqslant r} |(J^k w)(z)| \leqslant \frac{1}{(k-1)!}\int_0^r (r-r_1)^{k-1}\hat{w}(r_1)dr_1 = (J^k\hat{w})(r).$$

Hence, making use of (2.4), we get

$$\hat{w}(r) \leqslant \sum_{k=1}^{n} \hat{Q}_k(r)(J^k\hat{w})(r) + \hat{F}(r).$$

Put

$$(Vf)(r) = \sum_{k=1}^{n} \hat{Q}_k(r)(J^k f)(r) \quad (r > 0)$$

for a continuous function f. Clearly, V is a Volterra operator and

$$\hat{w}(r) \leqslant (V\hat{w})(r) + \hat{F}(r). \tag{2.5}$$

Let C_+ be the cone of all continuous positive functions defined on $[0,\infty)$. For operators A and B defined in C_+ we write $A \geqslant 0$ and $A \geqslant B$ if $(Af)(t) \geqslant 0$ $(t \geqslant 0, f \in C_+)$ and $A - B \geqslant 0$. Since V is a Volterra operator and $V \geqslant 0$ we have

$$(I-V)^{-1} = \sum_{j=0}^{\infty} V^j \geqslant 0.$$

Now (2.5) implies

$$\hat{w}(r) \leqslant ((I-V)^{-1}\hat{F})(r) = \sum_{j=0}^{\infty} (V^j\hat{F})(r) \ \ (r > 0). \tag{2.6}$$

We can write

$$V = \sum_{k=1}^{n} V_k,$$

where $(V_k f)(r) = \hat{Q}_k(r)(J^k f)(r) \ (f \in C_+)$. Applying the inequality

$$(J^k \hat{w})(r) = \frac{1}{(k-1)!} \int_0^r (r-s)^{k-1} \hat{w}(s) ds \leqslant \hat{w}(r) \frac{r^k}{(k)!}, \tag{2.7}$$

we get

$$(V_k^j \hat{w})(r) \leqslant \hat{Q}_k^j(r)(J^{kj} \hat{w})(r) \leqslant \hat{w}(r) \hat{Q}_k^j(r) \frac{r^{kj}}{(kj)!}.$$

By Lemma 1,

$$V^j = (\sum_{k=1}^n V_k)^j \leqslant 2^{(n-1)(j-1)} \sum_{k=1}^n V_k^j.$$

Consequently,

$$\begin{aligned}(I-V)^{-1} f(r) &\leqslant \sum_{k=0}^{\infty} (V^j f)(r) \leqslant \hat{f}(r) \sum_{k=1}^n \sum_{j=0}^{\infty} 2^{(n-1)(j-1)} \hat{Q}_k^j(r) \frac{r^{kj}}{(kj)!} \\ &= \frac{1}{2^{n-1}} \hat{f}(r) \sum_{k=1}^n \gamma_k(2^{n-1} \hat{Q}_k(r) r^k) \ (f \in C_+).\end{aligned}$$

Hence, due to (2.5),

$$\hat{w}(r) \leqslant \frac{1}{2^{n-1}} \hat{F}(r) \sum_{k=1}^n \gamma_k(\hat{Q}_k(r) 2^{n-1} r^k),$$

as claimed. □

Taking into account that $u(z) = (J^n u^{(n)})(z)$, due to Lemma 2 and (2.7), we arrive at

COROLLARY 1. *Let $u(z)$ be a solution of problem (2.1), (2.3). Then*

$$\hat{u}(r) \leqslant \frac{r^n}{n! 2^{n-1}} \hat{F}(r) \sum_{k=1}^n \gamma_k(2^{n-1} \hat{Q}_k(r) r^k) \ (r \geqslant 0).$$

Moreover, according to (2.2),

$$\hat{u}(r) \leqslant \frac{r^n}{n! 2^{n-1}} \hat{F}(r) \sum_{k=1}^n \exp\left[r(2^{n-1} \hat{Q}_k(r))^{1/k}\right] \ (r \geqslant 0).$$

3. Zeros of entire functions

Consider the entire function

$$h(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{(k!)^\alpha} \ (0 < \alpha \leqslant 1,\ z \in \mathbb{C},\ a_0 = 1,\ a_k \in \mathbb{C},\ k \geqslant 1). \tag{3.1}$$

Enumerate the zeros $z_k(h)$ of h with the multiplicities in non-decreasing order of their absolute values: $|z_k(h)| \leqslant |z_{k+1}(h)|$ $(k = 1,2,\dots)$ and assume that

$$\theta(h) := [\sum_{k=1}^{\infty} |a_k|^2]^{1/2} < \infty. \tag{3.2}$$

We need Theorem 5.3.1 from [7].

THEOREM 2. *Let h be defined by* (3.1) *and condition* (3.2) *hold. Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(h)|} \leqslant \theta(h) + \sum_{k=1}^{j} \frac{1}{(k+1)^{\alpha}} \quad (j = 1,2,\dots).$$

We are going to apply this theorem to an entire function $f(z)$ satisfying the inequality

$$\hat{f}(r) \leqslant q(r)\exp[Br^{\rho}] \quad (r = |z| > 0,\ B = \text{const} > 0;\ \rho \geqslant 1), \tag{3.3}$$

where $q(r)$ is a polynomial in r with nonnegative coefficients. To this end we prove the following lemma.

LEMMA 3. *Let an entire function $f(z)$ satisfy inequality* (3.3)*. Then the Taylor coefficients f_j $(j = 1,2,\dots)$ of $f(z)$ satisfy the inequality*

$$|f_j| \leqslant q(r_j)\left(\frac{eB\rho}{j}\right)^{j/\rho} \quad (j = 1,2,\dots),$$

where

$$r_j := \left(\frac{j}{B\rho}\right)^{1/\rho}.$$

Proof. By the well-known inequality for the coefficients of a power series, for any $r > 0$ we have

$$|f_j| \leqslant \frac{\hat{f}(r)}{r^j} \leqslant \frac{q(r)e^{Br^{\rho}}}{r^j}. \tag{3.4}$$

First, let $q(r) \equiv 1$: $|\hat{f}(r)| \leqslant \exp[Br^{\rho}]$. Employing the usual method for finding extrema it is easy to see that the function in the right-hand side of this inequality takes its smallest value in the range $r > 0$ for $r = r_j$, and therefore

$$|f_j| \leqslant \frac{\exp[Br_j^{\rho}]}{r_j^j} = \left(\frac{eB\rho}{j}\right)^{j/\rho}.$$

If $q(r)$ is an arbitrary polynomial with nonnegative coefficients, then due to (3.4)

$$|f_j| \leqslant q(r_j)\frac{\exp[Br_j^{\rho}]}{r_j^j} \leqslant q(r_j)\left(\frac{eB\rho}{j}\right)^{j/\rho},$$

as claimed. $\square$

Put $\chi := 2(eB\rho)^{1/\rho}$. Then Lemma 3 yields

$$|f_j| \leqslant q(r_j)\frac{\chi^j}{2^j j^{j/\rho}}.$$

Now consider the function $f_\chi(z) := f(z/\chi)$. Then

$$f_\chi(z) = \sum_{k=0}^{\infty} \tilde{f}_j z^j$$

with

$$|\tilde{f}_j| = |f_j|/\chi^j \leqslant \frac{q(r_j)}{2^j j^{j/\rho}} \leqslant \frac{q(r_j)}{2^j (j!)^{1/\rho}}.$$

Hence, with $d_j = \tilde{f}_j (j!)^{1/\rho}$ we can write

$$f_\chi(z) = \sum_{k=0}^{\infty} \frac{d_k z^k}{(k!)^{1/\rho}}.$$

Since $q(r)$ is a polynomial, simple calculations show that $q(r) \leqslant \text{const}\, r^{\deg(q)}$ $(r \geqslant 1)$. Hence it follows that

$$\theta(f,\chi) := [\sum_{j=1}^{\infty} |d_j|^2]^{1/2} = \Big[\sum_{j=1}^{\infty} \Big((j!)^{1/\rho}\frac{|f_j|}{\chi^j}\Big)^2\Big]^{1/2} \leqslant \Big[\sum_{j=1}^{\infty} \Big(\frac{q(r_j)}{2^j}\Big)^2\Big]^{1/2} < \infty. \quad (3.5)$$

Making use of Theorem 2, we get

$$\sum_{k=1}^{j} \frac{1}{|z_k(f_\chi)|} \leqslant \theta(f,\chi) + \sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho}} \ (j = 1,2,\ldots).$$

But $z_k(f) = \chi z_k(f_\chi)$. We thus arrive at

COROLLARY 2. *Let an entire function f satisfy the inequality* (3.3) *and* $f(0) = 1$. *Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \leqslant \chi\Big[\theta(f,\chi) + \sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho}}\Big] \ (j = 1,2,\ldots),$$

where $\theta(f,\chi)$ is defined by (3.5) *and* $\chi = 2(eB\rho)^{1/\rho}$.

4. Proof of Theorem 1

Substitute $u(z) = v(z) - 1$ into (2.1). Then problem (2.1), (2.3) takes the form (1.1), (1.2) with $P_k(z) = Q_k(z), G(z) = F(z) - P_n(z)$. Since $r^k \hat{P}_k(r) \leqslant p_k(r)$ $(r \geqslant 0)$, $\hat{G}(r) \leqslant g(r)$, Corollary 1 yields the inequality

$$\hat{v}(r) \leqslant 1 + \frac{r^n}{n! 2^{n-1}}(g(r) + p_n(r)) \sum_{k=1}^{n} \exp\,[(2^{n-1} p_k(r))^{1/k}] \ (r \geqslant 0). \quad (4.1)$$

Clearly, $p_k(r) \leqslant p_k(1)\ \ (r \leqslant 1)$, and

$$p_k(r) \leqslant p_k(1) r^{k+\nu_k}\ \ (r > 1).$$

Thus

$$(p_k(r))^{1/k} \leqslant (p_k(1))^{1/k}(1+r^{\rho_0})\ \ (r > 0).$$

Therefore, there are constants $\alpha_k \leqslant (p_k(1))^{1/k}$ and $\beta_k \leqslant (p_k(1))^{1/k}$, such that

$$(p_k(r))^{1/k} \leqslant \beta_k + \alpha_k r^{\rho_0}\ \ (r > 0).$$

Hence,

$$\frac{1}{n!2^{n-1}} \sum_{k=1}^{n} \exp\left[(2^{n-1} p_k(r))^{1/k}\right] \leqslant \frac{1}{n!2^{n-1}} \sum_{k=1}^{n} \exp\left[2^{(n-1)/k}(\beta_k + \alpha_k r^{\rho_0})\right]$$
$$= b_0 e^{a_0 r_0^{\rho}}\ (r > 0).$$

Recall that

$$b_0 = \frac{1}{n!2^{n-1}} \sum_{k=1}^{n} \exp\left[2^{(n-1)/k}\beta_k\right],\ \ a_0 = \max_{k=1,\dots,n} 2^{(n-1)/k}\alpha_k,$$

Now (4.1) yields

COROLLARY 3. *Let $v(z)$ be a solution of problem* (1.1), (1.2). *Then*

$$\hat{v}(r) \leqslant 1 + b_0 e^{a_0 r^{\rho_0}} r^n (g(r) + p_n(r))\ \ (r \geqslant 0).$$

Proof of Theorem 1. Let us apply to $v(z)$ Corollary 2 with $B = a_0$, $r_j = \tilde{r}_j = (\frac{j}{a_0\rho_0})^{1/\rho_0}$, $\chi = \chi_0 = 2(ea_0\rho_0)^{1/\rho_0}$. Since the Taylor coefficients v_j of $v(z)$ equal to zero for $j = 1, \dots, n-1$, due to (3.5),

$$\theta(v, \chi_0) := \Big[\sum_{j=n}^{\infty} \Big(\frac{|v_j|(j!)^{1/\rho_0}}{\chi_0^j}\Big)^2\Big]^{1/2} \leqslant b_0 \Big[\sum_{j=n}^{\infty} \frac{\tilde{r}_j^n (p_n(\tilde{r}_j) + g(\tilde{r}_j))}{2^j}\Big]^{1/2} = \theta_0. \quad (4.2)$$

Now Corollary 2 implies (1.4), as claimed. □

5. Applications of Theorem 1

Again $v(z)$ is a solution of problem (1.1), (1.2). For $\rho_0 > 1$, for the brevity put $\omega = \frac{1}{\rho_0}$. Since $|z_k(v)| \leqslant |z_{k+1}(v)|$, Theorem 1 implies that

$$\frac{j}{|z_j(v)|} \leqslant \chi_0 \Big[\theta_0 + \sum_{k=1}^{j} \frac{1}{(k+1)^{\omega}}\Big]\ \ (j = 1, 2, \dots).$$

But

$$\sum_{k=1}^{j} (k+1)^{-\omega} \leqslant \int_1^{j+1} \frac{dx}{x^{\omega}} = \frac{(1+j)^{1-\omega} - 1}{1-\omega}\ \ (0 < \omega < 1).$$

Thus,

$$\frac{j}{|z_j(v)|} \leqslant \chi_0 \left[\theta_0 + \frac{(1+j)^{1-\omega}-1}{1-\omega}\right] \quad (j=1,2,\ldots).$$

and therefore,

$$|z_j(v)| \geqslant \xi_j(v) \quad (j=1,2,\ldots), \tag{5.1}$$

5.1) where

$$\xi_j(v) = \frac{j}{\chi_0[\theta_0 + \frac{(1+j)^{1-\omega}-1}{1-\omega}]}.$$

If $|z_j(v)| \geqslant a \ (a>0)$, then $v(z)$ has in $\Omega(a) := \{z \in \mathbb{C} : |z| < a\}$ no more than $j-1$ zeros. Denote by $\mu(f,a)$ the number of the zeros of an entire function f inside $\Omega(a)$, i.e. $\mu(f,a)$ is the counting function of the zeros of f. In particular, due to (5.1) $\mu(v,a)=0$ for any positive

$$a < \xi_1(v) = \frac{1}{\chi_0[\theta_0 + \frac{2^{1-\omega}-1}{1-\omega}]}.$$

We thus arrive at

COROLLARY 4. *The counting function of the zeros of a solution $v(z)$ of (1.1), (1.2) satisfies the inequality $\mu(v,a) \leqslant j-1$ for any positive $a \leqslant \xi_j(v) \ (j=1,2,\ldots)$. In particular, the disc $\{z \in \mathbb{C} : |z| < \xi_1(v)\}$ is a zero-free domain of $v(z)$.*

To consider additional applications of Theorem 1 recall the following well-known result, cf. [12, p. 53].

LEMMA 4. *Let $\phi(x) \ (-\infty \leqslant x \leqslant \infty)$ be a convex continuous function, such that*

$$\phi(-\infty) = \lim_{x\to-\infty} \phi(x) = 0,$$

and $a_j, b_j \ (j=1,2,\ldots,l \leqslant \infty)$ be two non-increasing sequences of real numbers, such that

$$\sum_{k=1}^{j} a_k \leqslant \sum_{k=1}^{j} b_k \ \ (j=1,2,\ldots,l).$$

Then

$$\sum_{k=1}^{j} \phi(a_k) \leqslant \sum_{k=1}^{j} \phi(b_k) \ \ (j=1,2,\ldots,l).$$

Furthermore, put

$$\vartheta_1 = \chi_0 \left(\theta_0 + \frac{1}{2^{\omega}}\right) \text{ and } \vartheta_k = \frac{\chi_0}{(k+1)^{\omega}} \ \ (k=2,3,\ldots).$$

Inequality (1.4) and Lemma 4 yield

COROLLARY 5. *Let* $\phi(t)$ $(0 \leqslant t < \infty)$ *be a continuous convex function, such that* $\phi(0) = 0$. *Then*

$$\sum_{k=1}^{j} \phi\left(\frac{1}{|z_k(v)|}\right) \leqslant \sum_{k=1}^{j} \phi(\vartheta_k) \ \ (j = 1, 2, \ldots).$$

In particular, for any $\tau \geqslant 1$ *and* $j = 2, 3, \ldots$, *we have*

$$\sum_{k=1}^{j} \frac{1}{|z_k(v)|^{\tau}} \leqslant \sum_{k=1}^{j} \vartheta_k^{\tau}$$

and therefore, if $\tau > 1/\rho_0$, *then*

$$\sum_{k=1}^{\infty} \frac{1}{|\chi_0 z_k(v)|^{\tau}} \leqslant \vartheta_1^{\tau} + \chi_0^{\tau}\left(\zeta(\tau\omega) - 1 - \frac{1}{2^{\tau\omega}}\right) < \infty,$$

where $\zeta(z)$ *is the zeta Riemann function:*

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \ \ (\mathrm{Re}\ z > 1).$$

Note also that from Corollary 3 it follows

COROLLARY 6. *Let* $G(z)$ *be an entire function of order* $\rho(G) \leqslant \infty$. *Let* $v(z)$ *be a solution of problem* (1.1), (1.2). *Then order of* $v(z)$ *is no more than* $\max\{\rho_0, \rho(G)\}$.

This corollary generalizes the well-known [14, Theorem 8.3].

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