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ANALYSIS OF STAGNATION POINT FLOW OVER A STRETCHING/SHRINKING SURFACE

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Abstract. In this article we analyze the boundary value problem governing stagnation-point flow of a fluid with a power law outer flow over a surface moving with a speed proportional to the outer flow. The flow is characterized by two physical parameters; ε, which measures the stretching ($\varepsilon > 0$) or shrinking ($\varepsilon < 0$) of the sheet relative to the outer flow, and $n > 0$, the power law exponent. In the case of aiding flow ($\varepsilon > 0$), where the (stretching) surface and the outer flow move in the same direction, we prove existence of a solution for all values of n. For opposing flow ($\varepsilon < 0$), where the (shrinking) surface and the outer flow move in opposite directions, the situation is much more complicated. For $-1 < \varepsilon < 0$ and all n we prove a solution exists. However, for $\varepsilon \leqslant -1$, we prove there exists a value, $\varepsilon_{crit}(n) \leqslant -1$, such that no solutions exist for $\varepsilon \leqslant \varepsilon_{crit}$. For $n = 1/7$ and $n = 1/3$ we prove that $\varepsilon_{crit} = -1$. For other values of n, we derive bounds which illustrate the complicated nature of the existence/nonexistence boundary for opposing ($\varepsilon < 0$) flows.

1. Introduction

The study of flow over a stretching or shrinking surface is motivated by many industrial processes involving extrusion. Early work on such problems include Crane [1] who studied flow over a two-dimensional stretching sheet and Wang [10] who studied the axisymmetric stretching case. Miklavčič and Wang [8] and Wang [11] also considered a shrinking sheet.

Recently, the behavior of micropolar fluids in flows over stretching or shrinking surfaces has garnered much interest [4], [5], [9], [12], [13]. These models incorporate the microstructure and micromotions of a fluid, which effect the flow in ways not captured by standard theory [2], [13]. Recently, Zaimi and Ishak [13] considered the case of two-dimensional stagnation-point flow and heat transfer of a micropolar fluid over a nonlinearly stretching/shrinking sheet. Using a similarity transformation they reduce the governing PDEs to the following ODE boundary value problem involving the dimensionless stream function $f(\eta)$ and the dimensionless temperature $\theta(\eta)$. (See [4], [6], [12] and [13] for full details on the physical derivation of the model):

$$f''' + \frac{n+1}{k+2} f f'' - \frac{2n}{k+2}\left(f'^2 - 1\right) = 0, \tag{1.1}$$

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$$\frac{1}{Pr}\theta'' + \frac{n+1}{2} f\theta' - 2nf'\theta = 0, \tag{1.2}$$

subject to

$$f(0) = 0, \quad f'(0) = \varepsilon, \quad f'(\infty) = 1, \tag{1.3}$$

$$\theta(0) = 1, \quad \theta(\infty) = 0. \tag{1.4}$$

The stretching/shrinking rate of the sheet in the horizontal direction is given by $u_w(x) = bx^n$ and the free stream velocity is given by $u_e(x) = ax^n$ $(a > 0)$. The velocity ratio parameter, $\varepsilon = b/a$, measures the stretching ($\varepsilon > 0$) or shrinking ($\varepsilon < 0$) of the sheet relative to the free stream, and $n > 0$ is the rate exponent. $Pr > 0$ is the Prandtl number, $k > 0$ is the micropolar parameter and the similarity variable η is proportional to $y\sqrt{u_e/x}$. Zaimi and Ishak [13] investigated this BVP numerically and draw several conclusions regarding the nature of the solutions. The purpose of this article is to analytically investigate the existence or nonexistence of solutions to this BVP as a function of the model parameters and compare the results with the numerical solutions of [13]. The results proved here also apply to a model recently investigated by Merkin [7] for the same flow configuration (but without the micropolar effects).

The most interesting case, both physically and mathematically, is that of opposing flow ($\varepsilon < 0$), where the free stream and the (shrinking) sheet move in opposite directions. Physically, one would expect that steady flow solutions would cease to exist for $\varepsilon < 0$ and $|\varepsilon|$ sufficiently large. We prove that this is indeed the case, and we give fairly sharp bounds on the critical parameter values at the existence/nonexistence interface. (In some cases, the exact critical values of the parameters are obtained.) We also present new results regarding multiplicity of solutions. The plan of the paper is as follows:

As the BVP for $f(\eta)$ is decoupled from that of $\theta(\eta)$, we first study the solutions for $f(\eta)$. In Section 2 we prove the existence of a solution to the BVP for $f(\eta)$ given by equations (1.1) and (1.3) for all $\varepsilon > 1$, $n > 0$ and $k > 0$. This solution is shown to satisfy $f > 0$, $1 < f' < \varepsilon$ and $f'' < 0$ for all $\eta > 0$. For $-1 < \varepsilon < 1$, $n > 0$ and $k > 0$ we prove existence of a solution with the properties $\varepsilon < f' < 1$ and $f'' > 0$ for all $\eta > 0$. Thus, solutions are guaranteed to exist for at least some range of opposing flow $(-1 < \varepsilon < 0)$. For $\varepsilon = 1$, $n > 0$ and $k > 0$ the problem has the closed-form solution $f(\eta) = \eta$.

In Section 3 we consider the case of moderately large opposing flow ($\varepsilon \leqslant -1$) and prove that solutions cease to exist at a finite value of ε. For $n > 1/3$ and $k > 0$ we show that there exists an $\varepsilon_0(n,k) < -\sqrt{3}$ such that no solution exists for $\varepsilon < \varepsilon_0$. When $0 < n < 1/3$, we prove that no solutions exist for $\varepsilon \leqslant \varepsilon_0 = -\sqrt{(1-n)/(2n)}$. For $n = 1/7$ and $n = 1/3$ and $k > 0$ we prove that no solutions exists for $\varepsilon \leqslant \varepsilon_0 = -1$.

In Section 4, for the parameter range $\varepsilon > 1$, $n > 0$ and $k > 0$, we prove that there cannot be two solutions which both satisfy $f' > 1$ and $f'' < 0$ for all $\eta > 0$. We then argue, based on physical grounds, that solutions without these conditions on f' and f'' cannot exist. Thus for $\varepsilon > 1$ the solution is unique. (For $\varepsilon < 1$ the solution need not be unique.) Section 5 considers the BVP given by equations (1.2) and (1.4) governing the temperature $\theta(\eta)$. Finally, in Section 6, we discuss how the results proved here compare with the numerical results reported in [7] and [13] and discuss the implications regarding the physical situation. We also list several open problems.

2. Existence results

To prove existence of solutions to the BVP (1.1,1.3) we will study a family of related initial value problems:

$$f''' + \frac{n+1}{k+2} f f'' - \frac{2n}{k+2}\left(f'^2 - 1\right) = 0, \tag{2.1}$$

subject to

$$f(0) = 0, \tag{2.2}$$

$$f'(0) = \varepsilon, \tag{2.3}$$

$$f''(0) = \alpha, \tag{2.4}$$

where α is a free parameter. By standard existence and uniqueness theory for initial value problems, the IVP given by equations (2.1–2.4) will have a unique local solution, denoted $f(\eta;\alpha)$, for any value of α. Using a routine topological shooting argument we will show that there exists an α^* such that the solution $f(\eta;\alpha^*)$ to the IVP (2.1–2.4) exists for all $\eta > 0$ and satisfies the condition

$$f'(\infty) = 1, \tag{2.5}$$

giving a solution to the BVP (1.1,1.3).

THEOREM 1. *For any $n > 0$, $k > 0$ and $\varepsilon > 1$ there exists a solution to the BVP (2.1–2.3,2.5). Further, this solution satisfies $1 < f'(\eta) < \varepsilon$ and $f''(\eta) < 0$ for all $\eta > 0$.*

Proof. The existence proof will involve the following subsets of $(-\infty, 0)$:

$$\mathscr{A} = \left\{\alpha < 0 \; : \; f''(\eta;\alpha) = 0 \text{ before } f'(\eta;\alpha) = 1\right\} \tag{2.6}$$

and

$$\mathscr{B} = \left\{\alpha < 0 \; : \; f'(\eta;\alpha) = 1 \text{ before } f''(\eta;\alpha) = 0\right\}. \tag{2.7}$$

We will show that both of these sets are non-empty and open. For $\mathscr{A}$ this is just a matter of continuity of the solutions of the IVP (2.1–2.4) in its initial conditions. We claim that for all $\alpha < 0$ sufficiently close to zero, $\alpha \in \mathscr{A}$. To see this, first note that

$$f'''(0;\alpha) = \frac{2n}{k+2}\left(\varepsilon^2 - 1\right) > 0. \tag{2.8}$$

If we take $f''(0) = \alpha = 0$, then for small $\eta > 0$ we have $f'(\eta;0) > \varepsilon > 1$ and $f''(\eta;0) > 0$; say on $(0,b]$ for some $b > 0$. By continuity of the solutions of the initial value problem in its initial conditions, for $\alpha < 0$ sufficiently close to zero,

$f'(\eta;\alpha)$ will stay close to $f'(\eta;0)$, i. e. we can arrange that $f'(\eta;\alpha) > 1$ on $(0,b]$ with $f''(b;\alpha) > 0$. But as $f''(0;\alpha) < 0$, there must exist a first $\eta_0 \in (0,b)$ such that $f''(\eta_0;\alpha) = 0$ with $f'(\eta;\alpha) > 1$ on $[0,\eta_0]$. Thus for $\alpha < 0$ sufficiently close to zero we have $\alpha \in \mathscr{A}$. To show that $\mathscr{A}$ is open, consider $\overline{\alpha} \in \mathscr{A}$. We will show that all α sufficiently close to $\overline{\alpha}$ are also in $\mathscr{A}$. Let $\eta_0(\alpha)$ denote the dependence of the root η_0 on α. At $\eta_0(\overline{\alpha})$ we have $f'(\eta_0;\overline{\alpha}) > 1$ and $f''(\eta_0;\overline{\alpha}) = 0$. Evaluating (2.1) at $\eta_0(\overline{\alpha})$ implies that

$$f'''(\eta_0;\overline{\alpha}) = \frac{2n}{k+2}\left(f'(\eta_0;\overline{\alpha})^2 - 1\right) \neq 0. \tag{2.9}$$

Thus, by continuity of the solutions of the IVP in its initial conditions, for α sufficiently close to $\overline{\alpha}$, $f''(\eta;\alpha)$ will also have a root, $\eta_0(\alpha)$, near $\eta_0(\overline{\alpha})$ with $f'(\eta_0;\alpha) > 1$. Thus $\alpha \in \mathscr{A}$ and $\mathscr{A}$ is open.

We next prove that $\mathscr{B}$ is nonempty by showing that for $\alpha < 0$, $|\alpha|$ sufficiently large, $f' = 1$ in the interval $[0,1]$ strictly before $f'' = 0$. Suppose not. Then one of the following must occur: (i) $f'' = 0$ at some first point in $\eta_1 \in (0,1]$ with $f' > 1$ on $[0,\eta_1]$, (ii) $f'' < 0$ and $f' > 1$ for all $\eta \in (0,1]$, or (iii) $f'' = 0$ and $f' = 1$ simultaneously. We eliminate each of these in turn. To begin with (i), suppose that there exists a first $\eta_1 \in (0,1]$ with

$$f''(\eta_1) = 0 \tag{2.10}$$

and $1 < f' \leqslant \varepsilon$ for $\eta \in [0,\eta_1]$. An integration of the ODE (2.1) from 0 to $\eta > 0$ results in

$$f''(\eta) = \alpha - \frac{n+1}{k+2} f(\eta) f'(\eta) - \frac{2n\eta}{k+2} + \frac{3n+1}{k+2}\int_0^{\eta} f'(t)^2\, dt. \tag{2.11}$$

Since $f > 0$ and $1 < f' \leqslant \varepsilon$ for $\eta \in [0,\eta_1] \subset [0,1]$, we have from (2.11) that

$$f''(\eta) \leqslant \alpha + \frac{3n+1}{k+2}\varepsilon^2 \quad \eta \in [0,\eta_1]. \tag{2.12}$$

Choosing $\alpha < -(3n+1)\varepsilon^2/(k+2)$ we have that $f''(\eta_1) < 0$, contradicting (2.10).

A similar argument shows that if $\alpha < 1 - \varepsilon - (3n+1)\varepsilon^2/(k+2)$ then we cannot have case (ii), $f'' < 0$ and $f' > 1$ on all of $[0,1]$, since $f'(1)$ would then be less than 1. This leaves only case (iii) $f' = 1$ and $f'' = 0$ simultaneously; however, substituting this information into (2.1) gives $f''' = 0$ implying that $f'(\eta) \equiv 1$, contradicting the basic existence and uniqueness theorem for initial value problems, as $f'(0) = \varepsilon \neq 1$. Thus if $\alpha < 1 - \varepsilon - (3n+1)\varepsilon^2/(k+2)$ then we must have $f' = 1$ strictly before $f'' = 0$ and therefore $\alpha \in \mathscr{B}$. An argument similar to that for the set $\mathscr{A}$ shows that $\mathscr{B}$ is also open.

Thus, the sets $\mathscr{A}$ and $\mathscr{B}$ are non-empty and open. They are also obviously disjoint. But the interval $(-\infty,0)$ is connected and thus $\mathscr{A} \cup \mathscr{B} \neq (-\infty,0)$. Therefore, there exists some α^* such that $\alpha^* \notin \mathscr{A}$ and $\alpha^* \notin \mathscr{B}$. For such a value of α^* the only possibility is for the solution $f(\eta;\alpha^*)$ to exist for all $\eta > 0$ with $1 < f'(\eta;\alpha^*) < \varepsilon$ and $f''(\eta;\alpha^*) < 0$. Thus $f'(\infty,\alpha^*) = L \geqslant 1$ exists, and from the ODE (2.1) we see that the only possibility is $L = 1$, giving a solution to the BVP (2.1–2.3,2.5) and proving the theorem. $\square$

COROLLARY 1. *For any $n > 0$, $k > 0$ and $-1 < \varepsilon < 1$ there exists a solution to the BVP (2.1–2.3,2.5). Further, this solution satisfies $\varepsilon < f'(\eta) < 1$ and $f''(\eta) > 0$ for all $\eta > 0$.*

Proof. The proof is analogous to the argument given in Theorem 1 and uses the sets

$$\mathscr{C} = \{\alpha > 0 \ : \ f''(\eta;\alpha) = 0 \text{ before } f'(\eta;\alpha) = 1\} \tag{2.13}$$

and

$$\mathscr{D} = \{\alpha > 0 \ : \ f'(\eta;\alpha) = 1 \text{ before } f''(\eta;\alpha) = 0\}. \quad \square \tag{2.14}$$

3. Nonexistence results

THEOREM 2. *Let $n > 0$ and $k > 0$ be given. Define $\varepsilon_0 < -\sqrt{3}$ to be the root of $p(\varepsilon,n,k) = 4n\varepsilon(\varepsilon^2-3)(k+2)+3(3n+k+3)^2 = 0$. Then for $\varepsilon < \varepsilon_0$, the BVP (2.1–2.3,2.5) has no solution.*

Proof. We will show that there is no value of $f''(0) = \alpha$ for which the solution of the IVP (2.1–2.4), $f(\eta;\alpha)$, satisfies $f'(\infty;\alpha) = 1$. The proof will proceed by contradiction. Suppose a value of $f''(0) = \alpha$ exists which does give a solution to the BVP (2.1–2.3,2.5). Since $f'(0) = \varepsilon < 0$ and $f'(\infty) = 1$, there must exist a first $\eta_2 > 0$ such that $f'(\eta_2) = 0$ with $f''(\eta_2) \geqslant 0$.

Multiplying the ODE (2.1) by f'' and integrating from 0 to η_2 we obtain:

$$f''(\eta_2)^2 = \alpha^2 + \frac{4n\varepsilon(3-\varepsilon^2)}{3(k+2)} - \frac{2(n+1)}{k+2}\int_0^{\eta_2} f(t)f''(t)^2\,dt. \tag{3.1}$$

Since $f' < 0$ on $[0,\eta_2)$ and $f(0) = 0$ we have $f < 0$ on $(0,\eta_2]$ and thus from (3.1) we obtain the bound

$$f''(\eta_2)^2 > \frac{4n\varepsilon(3-\varepsilon^2)}{3(k+2)}, \tag{3.2}$$

which implies (using $f''(\eta_2) \geqslant 0$) that, for $\varepsilon < -\sqrt{3}$,

$$f''(\eta_2) > \sqrt{\frac{4n\varepsilon(3-\varepsilon^2)}{3(k+2)}} > 0. \tag{3.3}$$

Next, integrating (2.1) from η_2 to $\eta > \eta_2$ we have,

$$f''(\eta) = f''(\eta_2) - \frac{n+1}{k+2}f(\eta)f'(\eta) - \frac{2n(\eta-\eta_2)}{k+2} + \frac{3n+1}{k+2}\int_{\eta_2}^{\eta} f'(t)^2\,dt. \tag{3.4}$$

Using an argument similar to the proof of Theorem 1, we will show that for $\varepsilon < 0$ and $|\varepsilon|$ sufficiently large, f' must increase through 1 in the interval $(\eta_2,\eta_2+1]$. If not, then one of the following must occur: (i) $f'' = 0$ at some first point in $\eta_3 \in (\eta_2,\eta_2+1]$ with $f' < 1$ on $(\eta_2,\eta_3]$, (ii) $f'' > 0$ and $f' < 1$ for all $\eta \in (\eta_2,\eta_2+1]$, or (iii) $f'' = 0$ and $f' = 1$ simultaneously. As we have already seen, (iii) cannot occur.

If (i) occurs, then by integrating $0 < f' < 1$ and noting that $f(\eta_2) < 0$, see that $f < 1$ on $(\eta_2, \eta_3] \subset (\eta_2, \eta_2 + 1]$. Using these bounds on f and f' and the inequality (3.3) in (3.4) we obtain

$$f''(\eta) > \sqrt{\frac{4n\varepsilon(3-\varepsilon^2)}{3(k+2)}} - \frac{3n+1}{k+2}, \tag{3.5}$$

for all $\eta \in (\eta_2, \eta_3]$. Choosing $\varepsilon < 0$ sufficiently negative so that

$$\sqrt{\frac{4n\varepsilon(3-\varepsilon^2)}{3(k+2)}} - \frac{3n+1}{k+2} > 0, \tag{3.6}$$

we conclude that $f''(\eta_3) > 0$ contradicting $f''(\eta_3) = 0$. If (ii) occurs, then the bound (3.5) holds on all of $(\eta_2, \eta_2 + 1]$ and by choosing ε so that

$$\sqrt{\frac{4n\varepsilon(3-\varepsilon^2)}{3(k+2)}} - \frac{3n+1}{k+2} > 1, \tag{3.7}$$

we have that $f'' > 1$ for $\eta \in (\eta_2, \eta_2 + 1]$ and on integration we have $f'(\eta_2 + 1) > 1$. The inequality (3.7) holds for all $\varepsilon < \varepsilon_0$ where $\varepsilon_0 < -\sqrt{3}$ is defined as the root of the function $p(\varepsilon, n, k) = 4n\varepsilon(\varepsilon^2 - 3)(k+2) + 3(3n+k+3)^2$ given in the statement of the proof.

Thus, for our proposed solution, f' must increase above 1. But from the ODE (2.1), we see that f' cannot have a maximum above 1. Therefore, f' cannot satisfy the boundary condition $f'(\infty) = 1$ and the BVP (2.1–2.3,2.5) has no solution, proving the theorem. □

For certain parameter values, the nonexistence range can be extended.

COROLLARY 2. *Let $n = 1/3$, $k > 0$ and $\varepsilon \leqslant -1$. Then the BVP (2.1–2.3,2.5) has no solution.*

Proof. First, consider the range $\varepsilon < 1$. Differentiate the ODE (2.1) to obtain

$$f^{(4)} + \frac{n+1}{k+2} f f''' + \frac{1-3n}{k+2} f' f'' = 0. \tag{3.8}$$

Next, set $n = 1/3$ and integrate (3.8) to obtain

$$f'''(\eta) = \frac{2n(\varepsilon^2 - 1)}{k+2} \exp\left(-\frac{n+1}{k+2} \int_0^\eta f(t)\,dt\right) > 0, \tag{3.9}$$

from which we conclude that $f'(\infty) \neq 1$.

Last, consider $\varepsilon = -1$. Then $f'''(0) = 0$ and using (2.1) with $n = 1/3$ we see that $f(\eta) = -\eta + f''(0)\eta^2/2$ and $f'(\infty) = 1$ cannot be satisfied. □

COROLLARY 3. *Let $n = 1/7$, $k > 0$ and $\varepsilon \leqslant -1$. Then the BVP (2.1–2.3,2.5) has no solution.*

Proof. We will assume that a solution exists and derive a contradiction. We start with the case $\varepsilon < -1$. Multiplying (3.8) by f'' and integrating (by parts where necessary) from a value η_A (to be specified below) to $\eta > \eta_A$ we have

$$\begin{aligned} f'''(\eta)f''(\eta) - f'''(\eta_A)f''(\eta_A) - \int_{\eta_A}^{\eta} f'''(t)^2\,dt \\ + \frac{n+1}{2(k+2)}\left(f''(\eta)^2 f(\eta) - f''(\eta_A)^2 f(\eta_A)\right) \\ + \frac{1-7n}{2(k+2)}\int_{\eta_A}^{\eta} f'(t)f''(t)^2\,dt = 0. \end{aligned} \tag{3.10}$$

Before employing (3.10), we discuss some properties that our proposed solution must satisfy. Since $f'(0) = \varepsilon < -1$ and since f' cannot have a maximum at or above 1, to satisfy the boundary condition $f'(\infty) = 1$, f' must approach 1 from below. Also, since f' cannot have a minimum in the range $-1 < f' < 1$, there must exist an $\overline{\eta} > 0$ such that for all $\eta > \overline{\eta}$, we have $f > 0$, $0 < f' < 1$ and $f'' > 0$. From (2.1) we also conclude that $f''' < 0$ for all $\eta > \overline{\eta}$. For the proposed solution, let $f''(0) = \alpha$.

First consider the possibility that $f''(0) = \alpha < 0$. Then $f' < 0$ is initially decreasing and must achieve a first minimum. Let this minimum occur at η_A. Using (3.10) (with $n = 1/7$) we then have

$$f''(\eta)^2 f(\eta) = \frac{2(k+2)}{n+1}\left(\int_{\eta_A}^{\eta} f'''(t)^2\,dt - f'''(\eta)f''(\eta)\right). \tag{3.11}$$

Choosing the upper limit of integration $\eta > \overline{\eta}$ in (3.11) and using $f'' > 0$ and $f''' < 0$, there then exists a constant $K > 0$ such that

$$f''(\eta)^2 f(\eta) > \frac{2(k+2)}{n+1}\int_{\eta_A}^{\eta} f'''(t)^2\,dt > K > 0, \quad \forall\, \eta > \overline{\eta}. \tag{3.12}$$

Now since $f' < 1$ we have $0 < f < f(\overline{\eta}) + \eta - \overline{\eta}$ and so from (3.12) we obtain

$$f''(\eta) > \frac{\sqrt{K}}{\sqrt{f(\overline{\eta}) + \eta - \overline{\eta}}}, \quad \forall\, \eta > \overline{\eta}. \tag{3.13}$$

Integrating (3.13) from $\overline{\eta}$ to $\eta > \overline{\eta}$ we have

$$f'(\eta) > f'(\overline{\eta}) + 2\sqrt{K}\sqrt{f(\overline{\eta}) + \eta - \overline{\eta}} - 2\sqrt{Kf(\overline{\eta})}, \quad \forall\, \eta > \overline{\eta}, \tag{3.14}$$

which implies that $f' \to \infty$ as $\eta \to \infty$, contradicting $f'(\infty) = 1$.

For the case $\alpha \geqslant 0$, let $\eta_A = 0$ in (3.10) to obtain

$$f''(\eta)^2 f(\eta) = \frac{2(k+2)}{n+1}\left(\int_0^{\eta} f'''(t)^2\,dt - f'''(\eta)f''(\eta) + \frac{2n(\varepsilon^2-1)\alpha}{k+2}\right), \tag{3.15}$$

where $(2n(\varepsilon^2-1)\alpha)/(k+2) \geqslant 0$, and we again obtain the bound (3.12) and the proof proceeds as above. Thus for $n=1/7$, $k>0$ and $\varepsilon<-1$, no solution to the BVP (2.1–2.3,2.5) exists. Finally, for the case $\varepsilon=-1$, if $\alpha>0$ or $\alpha<0$, then the proof above holds. If $\alpha=0$, then $f'(\eta)\equiv -1$ and the boundary condition $f'(\infty)=1$ cannot hold. □

THEOREM 3. *Let $k>0$, $\varepsilon<-1$ and $1/(2\varepsilon^2+1)\leqslant n<1/3$. Then the BVP (2.1–2.3,2.5) has no solution.*

Proof. We will assume that there exists a value $f''(0)=\alpha$ which gives a solution to the BVP and derive a contradiction.

Case 1: $f''(0)=\alpha\geqslant 0$. We first claim that there exists a first point $\eta_5>0$ such that $f'(\eta_5)=0$ with $f''>0$, $f'''>0$, and $f^{(4)}>0$ on $(0,\eta_5]$. To prove this, note that $f'''(0)=2n(\varepsilon^2-1)/(k+2)>0$ and $f^{(4)}(0)=(3n-1)\varepsilon\alpha/(k+2)>0$ if $\alpha>0$. (If $\alpha=0$, then $f^{(5)}(0)=4n(n-1)\varepsilon(\varepsilon^2-1)/(k+2)^2>0$.) Suppose that $f^{(4)}$ were to vanish at some first point $\eta_4\in(0,\eta_5]$. Then at this point, $f(\eta_4)<0$, $f'(\eta_4)\leqslant 0$, $f''(\eta_4)>0$, $f'''(\eta_4)>0$, and $f^{(4)}(\eta_4)=0$. However, evaluating (3.8) at η_4 results in

$$\frac{n+1}{k+2}f(\eta_4)f'''(\eta_4)+\frac{1-3n}{k+2}f'(\eta_4)f''(\eta_4)=0, \tag{3.16}$$

which gives a contradiction since the left hand side of this equation is strictly negative. Thus, since f'', f''' and $f^{(4)}$ are all positive as long as $f'\leqslant 0$, there must exist a first point $\eta_5>0$ such that $f'(\eta_5)=0$ with $f''>0$, $f'''>0$, and $f^{(4)}>0$ on $(0,\eta_5]$.

Next, since f' cannot have a maximum at or above 1, there must exist a point $\eta_6>\eta_5$ such that $0<f'(\eta_6)<1$, $f''(\eta_6)>0$ and $f'''(\eta_6)=0$. Evaluating (2.1) at η_6 we conclude that $f(\eta_6)<0$, which in turn implies that $f<0$ on $(0,\eta_6]$.

Using the integrating factor $\exp((n+1)/(k+2)\int f)$ and integrating (3.8) from η_5 to η_6 we obtain

$$f'''(\eta_5)=\frac{1-3n}{k+2}\int_{\eta_5}^{\eta_6} f'(t)f''(t)e^{\frac{n+1}{k+2}\int_{\eta_5}^{t} f(u)\,du}\,dt. \tag{3.17}$$

Using $f<0$ and $0<f'<1$ on $(\eta_5,\eta_6]$ in (3.17), we obtain

$$f'''(\eta_5)<\frac{1-3n}{k+2}\int_{\eta_5}^{\eta_6} f''(t)\,dt=\frac{1-3n}{k+2}\left[f'(\eta_6)-f'(\eta_5)\right]=\frac{1-3n}{k+2}f'(\eta_6)<\frac{1-3n}{k+2}. \tag{3.18}$$

But since $f^{(4)}>0$ on $(0,\eta_5]$ we conclude that

$$\frac{2n}{k+2}(\varepsilon^2-1)=f'''(0)<f'''(\eta_5)<\frac{1-3n}{k+2}, \tag{3.19}$$

which cannot hold if

$$\frac{1}{2\varepsilon^2+1}\leqslant n<\frac{1}{3}, \tag{3.20}$$

giving the desired contradiction.

Case 2: $f''(0) = \alpha < 0$. In this case f' initially decreases below $f'(0) = \varepsilon < -1$. In order to satisfy the boundary condition $f'(\infty) = 1$, f' must increase through ε at some first point, call it η^* with $f''(\eta^*) > 0$. Since $f(0) = 0$ and $f' < 0$ on $[0, \eta^*]$, we conclude that $f < 0$ on $(0, \eta^*]$. Evaluating the ODE (2.1) at η^* we have

$$f'''(\eta^*) = \frac{2n}{k+2}(\varepsilon^2 - 1) - \frac{n+1}{k+2} f(\eta^*) f''(\eta^*) > \frac{2n}{k+2}(\varepsilon^2 - 1). \qquad (3.21)$$

Similarly, evaluating (3.8) at η^* gives $f^{(4)}(\eta^*) > 0$. The argument of Case 1 can now be applied starting at the point η^* instead of 0 and the theorem is proved. □

The nonexistence results of this section are summarized in Figure 1. The form of the ODE (3.8) and the integral expression (3.10) strongly suggest that a nonexistence result should be possible for all of the range $\varepsilon \leqslant -1$ and $1/7 \leqslant n \leqslant 1/3$, but we have not been able to obtain a proof.

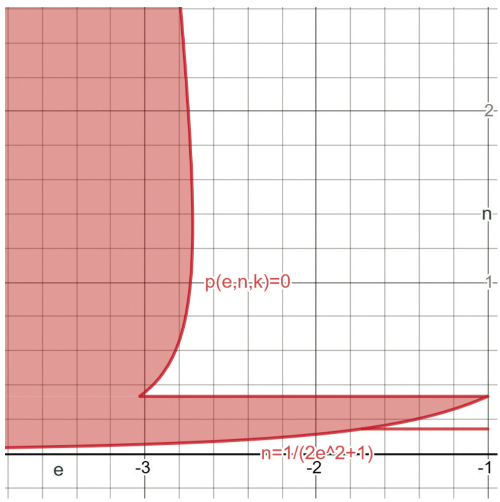

Figure 1: *Region of nonexistence of solutions depicted in the $\varepsilon - n$ parameter plane for $k = 1$. The bounding curves are defined in the text and consist of $p(\varepsilon, n, k) = 0$, $n = 1/(2\varepsilon^2 + 1)$, $n = 1/7$, and $n = 1/3$.*

4. Uniqueness results

In section 2, we proved that for $\varepsilon > 1$ there exists a solution to the BVP with the properties $1 < f'(\eta) < \varepsilon$ and $f''(\eta) < 0$ for all $\eta > 0$. In this section we prove that for $\varepsilon > 1$ there cannot exist two solutions *with these properties.*

THEOREM 4. *For any $n > 0$, $k > 0$ and $\varepsilon > 1$ there cannot exist two solutions to the BVP (2.1–2.3,2.5), both of which satisfy satisfy $1 < f'(\eta) < \varepsilon$ and $f''(\eta) < 0$ for all $\eta > 0$.*

Proof. To prove this theorem we consider the function $u(\eta;\alpha) = \partial f(\eta;\alpha)/\partial\alpha$. Differentiating (2.1–2.4) with respect to α we obtain

$$u''' + \frac{n+1}{k+2} f u'' - \frac{4n}{k+2} f' u' + \frac{n+1}{k+2} f'' u = 0, \tag{4.1}$$

subject to

$$u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 1, \tag{4.2}$$

where the primes still denote differentiation with respect to η.

Suppose there exist two solutions to the BVP with the conditions stated in the theorem, say for values $\alpha_1 < \alpha_2 < 0$. Consider the IVP given by equations (4.1–4.2) for $\alpha_1 \leqslant \alpha \leqslant \alpha_2$ and $\eta > 0$. From the initial data (4.2) we see that both u and u' are initially positive and increasing. We first show that u' cannot have a maximum. For contradiction, suppose a first maximum exists at some point η_7. At this point we have $u(\eta_7) > 0$, $u'(\eta_7) > 0$, $u''(\eta_7) = 0$ and $u'''(\eta_7) \leqslant 0$. Further

$$1 < f'(\eta;\alpha_1) \leqslant f'(\eta;\alpha) \leqslant f'(\eta;\alpha_2), \tag{4.3}$$

and

$$f''(\eta;\alpha_1) \leqslant f''(\eta;\alpha) \leqslant f''(\eta;\alpha_2) < 0, \tag{4.4}$$

for all $\eta \in [0, \eta_7]$ and $\alpha_1 \leqslant \alpha \leqslant \alpha_2$.

Using this information in (4.1) we have

$$u'''(\eta_7) = \frac{4n}{k+2} f'(\eta_7) u'(\eta_7) - \frac{n+1}{k+2} f''(\eta_7) u(\eta_7) > 0, \tag{4.5}$$

contradicting the fact that $u'''(\eta_7) \leqslant 0$ at a maximum.

Thus, since u' cannot have a maximum, there exists $C > 0$ and $\eta_8 > 0$ such that $u' > C$ for all $\eta > \eta_8$. By the Mean Value Theorem we then obtain

$$f'(\eta;\alpha_2) - f'(\eta;\alpha_1) = \left(\frac{\partial f'}{\partial\alpha}\right)_{\alpha=\hat{\alpha}} (\alpha_2 - \alpha_1), \tag{4.6}$$

where $\alpha_1 < \hat{\alpha} < \alpha_2$. Thus as $\eta \to \infty$ in (4.6) we have

$$0 = 1 - 1 = f'(\infty;\alpha_2) - f'(\infty;\alpha_1) = u'(\infty;\hat{\alpha})(\alpha_2 - \alpha_1) > C(\alpha_2 - \alpha_1) > 0, \tag{4.7}$$

giving a contradiction and proving that for $\varepsilon > 1$, there cannot be two solutions, both of which satisfy $f' > 1$ and $f'' < 0$. $\square$

5. Analysis of temperature BVP

Given the existence of a solution for the stream function, $f(\eta)$, of the form established in Section 2, the problem for the temperature, $\theta(\eta)$, reduces to the following linear boundary value problem:

$$\frac{1}{Pr}\theta'' + \frac{n+1}{2} f\theta' - 2nf'\theta = 0, \tag{5.1}$$

subject to

$$\theta(0) = 1, \tag{5.2}$$

and

$$\theta(\infty) = 0. \tag{5.3}$$

For this problem we have the following result:

THEOREM 5. *For any $Pr > 0$, $n > 0$ and $\varepsilon > 1$ there exists a solution to the BVP (5.1–5.3).*

Proof. The proof is similar to that given earlier for $f(\eta)$. Consider the family of initial value problems given by the ODE (5.1) and the initial condition (5.2) along with

$$\theta(0) = \gamma, \tag{5.4}$$

where γ is a free parameter. Since the coefficients f and f' are continuous for all $\eta > 0$, the IVP (5.1–5.2, 5.4) will have a unique global solution for any value of γ. It remains to show that γ can be chosen so that (5.3) is satisfied. To establish this, consider the sets

$$\mathscr{F} = \{\gamma < 0 \,:\, \theta'(\eta;\gamma) = 0 \text{ before } \theta(\eta;\gamma) = 0\} \tag{5.5}$$

and

$$\mathscr{G} = \{\gamma < 0 \,:\, \theta(\eta;\gamma) = 0 \text{ before } \theta'(\eta;\gamma) = 0\}. \tag{5.6}$$

Analysis similar to that of Section 2 can be used to show $\mathscr{F}$ and $\mathscr{G}$ are nonempty and open and that $\gamma^* \notin \mathscr{F}$ and $\gamma^* \notin \mathscr{G}$ then gives a solution to the BVP. □

A similar analysis establishes:

THEOREM 6. *For any $Pr > 0$, $n > 0$ and $-1 < \varepsilon < 1$ there exists a solution to the BVP (5.1–5.3).*

6. Comparison with numerical results and open questions

In this article we establish analytical results regarding the existence and nonexistence of solutions for the boundary value problem (1.1–1.4) proposed in [13] governing stagnation point flow of a micropolar fluid over a stretching or shrinking sheet.

Both the numerics of [13] and the results proved here confirm that a solution exists for the case of a stretching sheet, $\varepsilon > 0$. For a shrinking sheet, where the free stream and the bounding surface move in opposite directions, one would expect that steady solutions would cease to exist once the velocity difference between the two becomes too great. The results of Section 3 proved that this is the case, and bounds were obtained on the exact value of critical parameter value ε_{crit}, for which, no solutions exist when $\varepsilon < \varepsilon_{crit}$. For $n = 1/7$ and $n = 1/3$, we prove that this critical value is exactly $\varepsilon_{crit} = -1$ with nonexistence holding for $\varepsilon \leqslant \varepsilon_{crit} = -1$. For all $n > 1/3$ and $k > 0$, we proved that $\varepsilon_0 < \varepsilon_{crit} \leqslant -1$ where $\varepsilon_0 < -\sqrt{3}$ is the root of $p(\varepsilon,n,k) = 4n\varepsilon(\varepsilon^2-3)(k+2)+$

$3(3n+k+3)^2 = 0$. For $n = 1$ and various values of k, the numerical results of [13] indicate that $\varepsilon_{crit} \approx -1.24657$. For these values of n and k, our bounds put ε_{crit} in the range $-2.45954 < \varepsilon_{crit} < -1$, which contains the value of ε_{crit} reported in [13]. As mentioned earlier, the form of the ODE (3.8) and the integral expression (3.10) strongly suggest:

Open Conjecture 1. If $1/7 < n < 1/3$ and $k > 0$ then the BVP (2.1–2.3,2.5) has no solution for $\varepsilon \leqslant \varepsilon_{crit} = -1$.

Uniqueness of a physically relevant solution can be established for $\varepsilon > 1$ as follows. Section 2 proved the existence of a solution for the stream function, $f(\eta)$, with the properties $f' > 1$ and $f'' < 0$ for all $\eta > 0$. In Section 4 it was proved that for $\varepsilon > 1$ there could not be two solutions, both with these properties. Thus, for $\varepsilon > 1$, if second solution were to exist, it would have to violate at least one of these inequalities. In fact, since f' cannot have a maximum above 1, any second solution would have to violate both inequalities. Since f' cannot have a minimum in the range $-1 < f' < 1$, any second solution would have to have f' decrease below -1 and achieve at least one minimum before eventually approaching 1 from below. When $\varepsilon > 1$, the free stream motion and the stretching wall both move in the same direction. However, a solution for which f' becomes negative would indicate a region of flow reversal, which can be discounted on physical grounds in this case. Thus for $\varepsilon > 1$ there is a unique, physically relevant, solution.

The results obtained here also apply to a model recently considered by Merkin [7] governing boundary layer flow due to an outer flow proportional to x^m and a surface velocity proportional to λx^m, where x measures position along the wall and λ can be positive (aiding flow/stretching surface) or negative (opposing flow/shrinking surface). Specifically, the model that Merkin [7] considers is

$$f'''(\eta) + f(\eta)f''(\eta) + \beta(1 - f'(\eta)^2) = 0, \tag{6.1}$$

subject to

$$f(0) = 0, \quad f'(0) = \lambda, \quad f'(\infty) = 1. \tag{6.2}$$

The Zaimi and Ishak [13] model investigated here can be transformed into the model considered by Merkin [7] by setting $k = n - 1$, $\varepsilon = \lambda$, and $n = \beta/(2-\beta)$. We note that all of the results proved here hold for $k > -2$ (not just $k > 0$). The results given here for $n > 0$ apply to the range $0 < \beta < 2$ in Merkin's model.

For various values of $\beta \in (0,2)$ Merkin [7] finds a range of $\lambda < 0$ where multiple (two or three) solutions exist. The critical value, λ_c, corresponding to turning points of the solution branches, is tracked numerically as a function of β. Merkin notes that there "appears to be a gap in the curve between $\beta = \beta_1 \approx 0.139$ and $\beta = \beta_2 \approx 0.5$." Our conjecture that no solutions exist for $\varepsilon \leqslant -1$ and $n \in (1/7, 1/3)$ for the BVP (1.1,1.3) would correspond to $\lambda \leqslant -1$ and $\beta \in (0.25, 0.5) \subset (\beta_1, \beta_2)$ for the BVP (6.1–6.2). The absence of a turning point bifurcation and the discussion above suggest:

Open Conjecture 2. If $1/7 < n < 1/3$, $k > 0$, and $-1 < \varepsilon < 1$, then the BVP (2.1–2.3,2.5) has a unique solution.

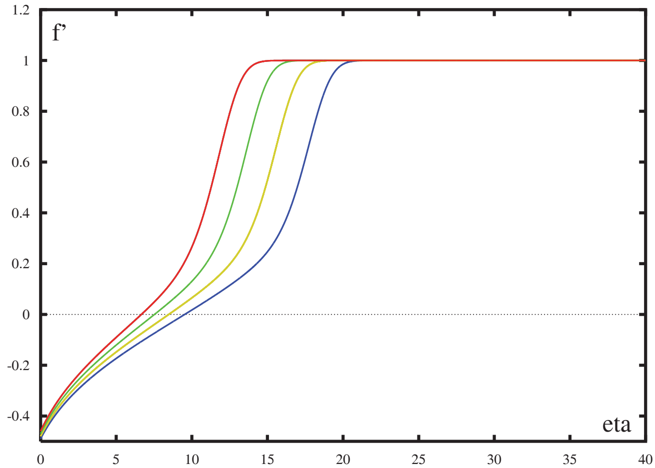

Figure 2: *Plot of* f' *as a function of* η *for* $\beta = 0.07$*. From left to right,* $\lambda = -0.46,\ -0.47,\ -0.48,\ -0.49$.

The delicate nature of the numerical computations for the problem is illustrated by Figure 4 in [7]. For $\beta = 0.07$ (corresponding to $n = 0.36$) Merkin finds a small range of λ for which three solutions exist, with turning points at $\lambda_c \approx -0.47427$ and $\lambda_c \approx -0.43824$. The upper branch exists for $\lambda \geqslant -0.47427$, the middle branch for $-0.47427 < \lambda < -0.43824$, and the lower branch for $\lambda \leqslant -0.43824$ and ending somewhere *before* -0.47427. However, the results of Section 2 prove that for $\beta = 0.07$, solutions exist for all $\lambda > -1$. Thus, either this lower branch can be continued all the way down to $\lambda = -1$ or another branch of solutions exists. We have been able to decrease this lower branch to about $\lambda \approx -0.49$, after which, the solutions become very hard to track. See Figure 2. This suggests the possibility of a hysteresis loop, where the solution could jump from the upper branch to the lower branch as λ decreases through -0.47427, and jump from the lower branch back to the upper branch as λ increases through -0.43824. The numerical solutions of Figure 2 were obtained using the dynamical systems package *XPPAUT* [3] using a fourth order Runge-Kutta scheme with step size $h = 0.011$.

λ	$f''(0)$
-0.46	0.12783014
-0.47	0.12361411
-0.48	0.11972917
-0.49	0.11608943

Table 1: *Initial data for the graphs in Figure 2.*

The case of β outside the range $0 < \beta < 2$ (corresponding to $n < 0$) has not been considered in this paper. However, the results or Merkin [7] indicate a rich variety of behavior in this parameter range which we plan to explore in future work.

REFERENCES

[1] L. J. CRANE, *Flow past a stretching plate*, Z. Ange. Math. Phys., **21** (1970), pp. 645-647.
[2] A. C. ERINGEN, *Theory of thermomicrofluids*, J. Math. Anal. Appl., **38** (1972), pp. 480-496.
[3] B. ERMENTROUT, *XPPAUT*, http://www.math.pitt.edu/$\sim$bard/xpp/xpp.html.
[4] A. ISHAK, Y. Y. LOK AND I. POP, *Stagnation-point flow over a shrinking sheet in a micropolar fluid*, Int. J. Eng. Sci., **197** (2010), pp. 1417–1427.
[5] A. ISHAK, R. NAZAR AND I. POP, *Stagnation flow of a micropolar fluid towards a vertical permeable surface*, Int. Comm. Heat Mass Trans., **35** (2008), pp. 276–281.
[6] M. KHADER AND R. SHARMA, *Evaluating the unsteady MHD micropolar fluid flow past a stretching/shrinking sheet with heat source and thermal radiation: implementing fourth order predictor-corrector FDM*, Math. and Comp. in Sim., **181** (2021), 333–350.
[7] J. H. MERKIN, *Multiple similarity solutions in boundary-layer flow on a moving surface*, Acta Mech., **229** (2018), pp. 4279–4294.
[8] M. MIKLAVČIČ AND C. Y. WANG, *Viscous flow due to a shrinking sheet*, Q. Appl. Math., **64** (2006), pp. 283–290.
[9] R. NAZAR, N. AMIN, D. FILIP AND I. POP, *Stagnation point flow of a micropolar fluid towards a stretching sheet*, Int. J. Nonlinear Mech., **39** (2004), pp. 1227–1235.
[10] C. Y. WANG, *The three-dimensional flow due to a stretching plate*, Phys. Fluids, **27** (1984), pp. 1915–1917.
[11] C. Y. WANG, *Stagnation-point flow towards a shrinking sheet*, Int. J. Nonlinear Mech., **43** (2008), pp. 377–382.
[12] Z. ZIABAKHSH, G. DOMAIRRY AND H. BARARNIA, *Analytical solutions of non-Newtonian micropolar fluid flow with uniform suction/blowing and heat generation*, J. Taiwan Inst. Chem. Eng., **40** (2009), pp. 443–451.
[13] K. ZAIMI AND A. ISHAK, *Stagnation-point flow and heat transfer over a nonlinearly stretching/shrinking sheet in a micropolar fluid*, Abs. Appl. Anal., (2014), Article ID: 261630.

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