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THREE SOLUTIONS FOR A NEW KIRCHHOFF–TYPE PROBLEM

YUE WANG, QI-PING WEI AND HONG-MIN SUO*

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Abstract. This article concerns on the existence of multiple solutions for a Kirchhoff-type problem with positive and negative modulus. By applying the variational methods and algebraic analysis, we prove that there exist the only three solutions when the parameter is absolutely small than a constant, only two solutions when the parameter is absolutely equals with the constant and an unique solution when the parameter is absolutely greater than the constant. Moreover, we use the algebraic analysis to calculating the constant with the help of one of the Mountain Pass Lemma, Ekeland variational principle, and Minimax principle.

1. Introduction and main results

This paper mainly study the following nonlocal problem

$$\begin{cases} -\left(a-b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=\mu f(x), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the constants $a,b\in\mathbb{R}$ with $|a|+|b|>0$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N\geqslant 1$, $\mu\in\mathbb{R}$ is a parameter and $f(x)$ is a nonnegative-nonzero function. The analysis developed in this paper corresponds to propose an approach based on the idea of considering the nonlocal term with negative modulus, which is presents interesting difficulties.

Problem (1) is related with the Kirchhoff-type equation as following:

$$-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(x,u), \text{ in } \Omega, \tag{2}$$

where $a\geqslant 0$, $b>0$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ or $\Omega=\mathbb{R}^N$, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continual function. Mentioned that Eq. (2) is a steady-state subproblem of model

$$\rho h\frac{\partial^2 u}{\partial t^2}-\left(p_0+\frac{Eh}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=f(x,u) \tag{3}$$

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* Corresponding author.

with $0 < x < L$ and $t \geqslant 0$, where $u = u(x,t)$ is the lateral displacement, ρ the mass density, E the Young's modulus, h the cross-section area, L the length and p_0 the initial axial tension. Eq. (3) named the Kirchhoff problem as an extension of classical D'Alembert's wave equation for free vibration of elastic strings by Kirchhoff in [16] before 1876. When finding the existence of stationary solution, Eq. (3) may be express as Eq. (2) and therefore Eq. (2) was named the Kirchhoff-type problem. Eq. (2) has been studied by many researchers on $\mathbb{R}^N$ and bounded domain with some extra conditions, such as [2, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 19, 20, 21, 27, 37, 39] and their references. Eq. (2) contains a nonlocal coefficient $(a+b\int_\Omega |\nabla u|^2dx)$, this leads to that Eq. (2) is often called nonlocal problem. A nonlocal coefficient $(a-b\int_\Omega |\nabla u|^2dx)$ is included in Kirchhoff-type problem, may be an interesting model (see [28, 31]) since the problem involving the minus Young's modulus. It is possible to restate that the research interesting in [38] is that the nonlocal coefficient $(a+b\int_\Omega |\nabla u|^2dx)$ is bounded below but the nonlocal coefficient $(a-b\int_\Omega |\nabla u|^2dx)$ is not. It is different with [38], the research interesting in [31] is that the Kirchhoff-type equation with the nonlocal coefficient $(a-b\int_\Omega |\nabla u|^2dx)$ is a negative modulus problem. Furthermore, some useful conclusion are concluded in [28], there the research history of negative modulus is summarized. It is worth to paying more attentions to Young's modulus, which can also be used in computing tension, when the atoms are pulled apart instead of squeezed together, the strain is negative because the atoms are stretched instead of compressed, this leads to minus Young's modulus, which is the reason why we call the problem (1) as a Kirchhoff-type problem with negative modulus before.

Indeed, there are some results in Eq. (2) with positive a and negative b. Let Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, denote $2^* = \frac{2N}{N-2}$ as $N \geqslant 3$ and $2^* = +\infty$ as $N = 1,2$ below. For the positive constants a and b, some researchers consider the following problem

$$\begin{cases} -\left(a-b\displaystyle\int_\Omega |\nabla u|^2dx\right)\Delta u = |u|^{p-2}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

A result of the existence of infinitely many solutions for Eq. (4) were got in [33] by Wang and Yang, where $2 \leqslant p < 2^*$ and the tool is the Ljusternik-Schnirelman type minimax method. Moreover, an example with one dimensional case for $p = 2$ is given in [33]. Shi and Qian in [24] replaced $|u|^{p-2}u$ by $\lambda|u|^{p-2}u$ with $1 < p < 2$ on Eq. (4), they got the existence of two positive solutions by using the Nehari manifold if $\lambda > 0$ enough small. Yin and Liu in [38] got that there exist at least a nontrivial non-negative solution and a nontrivial non-positive solution with $2 < p < 2^*$ for Eq. (4).

If there is no boundary condition on Eq. (4), infinitely many interesting classical solutions can be found in [30] with all exponents $p \neq -1$ on bounded domain and in [32] with all exponents $p \in [0, 2^*)$ on unbounded domain, where all functions are constructed by authors skillfully. Wang et al in [31] obtained that there exist at least two positive solutions when $f(x) \in L^{4/3}(\mathbb{R}^4)$ with $\mu > 0$ enough small and infinitely many positive solutions with $\mu = 0$ via variational method mainly for the problem

$$-\left(a-b\int_{\mathbb{R}^4} |\nabla u|^2dx\right)\Delta u = |u|^2u + \mu f(x), \quad \text{in } \mathbb{R}^4.$$

Recent studies shown that the problem or Eq. (2) with positive a and negative b is not only the Kirchhoff problem with negative modulus, but also a conveyor belt boundary vibration problem (see some statement in [36]). Moreover, infinitely many solutions were proved in [36] for subcritical exponents and finite positive solutions were concluded for critical exponent with the help of the Symmetric Mountain Pass Lemma and Nihari manifold. For more details about this kind problem with negative modulus, we refer readers to the papers [17, 18, 22, 23, 29, 34, 35, 40], there include Hardy-Sobolev critical exponent, singularity, ground state solution, sign-changing potential, and so on. The papers above are consider no our results in the problem (1), so our research do not conflict. The conclusion in this article state as following theories mainly via variational method and algebraic analysis.

THEOREM 1. *Assume that* $a,b>0$ *and* $f\in L^{\frac{2^*}{2^*-1}}(\Omega)$ *is positive a.e.* $x\in\Omega$*, then, there exists* $\mu_*>0$*, such that the problem* (1) *has at least three nontrivial solutions for* $\mu\in(0,\mu_*)$ *and a nontrivial solution for* $\mu\in[\mu_*,+\infty)$.

As the proof as Theorem 1, the proof of the existence of solutions for $\mu<0$ may be miscellaneous by variational method, we shall use new method to overcome it.

THEOREM 2. *Assume that* $a,b>0$ *and* $f\in L^{\frac{2^*}{2^*-1}}(\Omega)$ *is positive a.e.* $x\in\Omega$*, then, there exists a constant* $\mu_{**}>0$ *such that problem* (1) *has only three solutions for* $0<|\mu|<\mu_{**}$*, only two solutions for* $\mu=\pm\mu_{**}$ *and unique solution for* $|\mu|>\mu_{**}$*. Moreover, problem* (1) *has infinitely many solutions if* $a,b>0,\mu=0$.

Indeed, the condition of $f(x)$ can be replace weakly by $f\in H^{-1}(\Omega)$ and $f(x)>0$ a.e. $x\in\Omega$, where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ and $L^{\frac{2^*}{2^*-1}}(\Omega)\subsetneq H^{-1}(\Omega)$.

COROLLARY 1. *Assume that* $ab>0$ *and* $f(x)\in H^{-1}(\Omega)$ *is positive a.e., then, for* μ_{**} *defined by Theorem 2, problem* (1) *has only three solutions for* $0<|\mu|<\mu_{**}$*, only two solutions for* $\mu=\pm\mu_{**}$ *and unique solution for any* $|\mu|>\mu_{**}$.

The novelty of our results lies in three aspects for positive a and b. Firstly, we prove the $(PS)_c$ condition by new method and $c\in[-\frac{a^2}{12b},+\infty)\setminus\{\frac{a^2}{4b}\}$. What's more, it is not satisfy $(PS)_c$ condition with $c\in(-\infty,-\frac{a^2}{12b})\cup\{\frac{a^2}{4b}\}$. Secondly, with the help of algebraic analysis, we got that the uniqueness of three, two or one solution although we got that there are at least three nontrivial weak solutions by using one of the Mountain Pass Lemma, Ekeland variational principle, and Minimax principle. Thirdly, through of a basic fundamental result, we obtain the specific form of μ_{**}.

This article is organized as follows. In section 2, we give some basic knowledge which use to solving the problem. Section 3 contains elementary results and proof of theorem 1. In section 4, for the Theorem 1, with the help of algebraic analysis, we prove that the existence of three nontrivial weak solutions only by using one of the Mountain Pass Lemma, Ekeland variational principle, and Minimax principle. By using the similar method, we prove the Theorem 2 and calculate the μ_{**} exactly. In section 5, we give an example for one dimensional case and make a summary.

2. Preliminaries

Throughout this paper we denote by $\to$ (resp. $\rightharpoonup$) the strong (resp. weak) convergence. For any $u,v\in H_0^1(\Omega)$, the inner product is $\langle u,v\rangle=\int_\Omega \nabla u\nabla v dx$ and the norm $\|u\|=\left(\int_\Omega |\nabla u|^2dx\right)^{\frac{1}{2}}$. $\|f\|_{H^{-1}}=\sup_{u\in H_0^1(\Omega)}|\int_\Omega fudx|\|u\|^{-1}$ is the norm in the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$. Denote the L^s-norm $\|u\|_s=\left[\int_\Omega |u|^sdx\right]^{\frac{1}{s}}$ for $0<s<+\infty$. We set, as we known in [3, 26], S be the best Sobolev embedding constant for the embedding $H_0^1(\Omega)\hookrightarrow L^{2^*}(\Omega)$ and λ_1 be the first eigenvalue of $-\Delta$, namely

$$S=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega |\nabla u|^2dx}{\left(\int_\Omega |u|^{2^*}dx\right)^{\frac{2}{2^*}}},\quad \lambda_1=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega |\nabla u|^2dx}{\int_\Omega |u|^2dx}.$$

We recall that a function $u\in H_0^1(\Omega)$ is called a weak solution of Eq. (1) if

$$\left(a-b\int_\Omega |\nabla u|^2dx\right)\int_\Omega \nabla u\nabla vdx=\mu\int_\Omega fvdx,\quad \forall\, v\in H_0^1(\Omega).$$

Let $I(u):H_0^1(\Omega)\mapsto\mathbb{R}$ be the functional defined by

$$I(u)=\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4-\mu\int_\Omega fudx, \tag{5}$$

it can be verify that $I(u)\in C(H_0^1(\Omega),\mathbb{R})$ and the Gâteaux derivative of I given by

$$\langle I'(u),v\rangle=(a-b\|u\|^2)\int_\Omega \nabla u\nabla vdx-\mu\int_\Omega fvdx,\quad \forall\, v\in H_0^1(\Omega). \tag{6}$$

If $u\in H_0^1(\Omega)$ such that $I'(u)=0$, then u is a weak solution of problem (1).

3. Proof of the Theorem 1

LEMMA 1. *Assume that $a,b,\mu>0$, $f\in L^{\frac{2^*}{2^*-1}}(\Omega)$ and $f(x)\geqslant 0$ a.e. $x\in\Omega$, then, I satisfies the $(PS)_c$ condition with $c\in[-\frac{a^2}{12b},+\infty)\setminus\{\frac{a^2}{4b}\}$, and I does not satisfy the $(PS)_c$ condition at $c=\frac{a^2}{4b}$. That is, for $c\in[-\frac{a^2}{12b},+\infty)\setminus\{\frac{a^2}{4b}\}$, every (PS) sequence at c has a converge subsequence.*

Proof. We recall that $\{u_n\}_{n=1}^\infty\subset H_0^1(\Omega)$ is a (PS) sequence at c, so that $I(u_n)\to c$ and $I'(u_n)\to 0$ as $n\to\infty$. Let $\{u_n\}_{n=1}^\infty$ be a $(PS)_c$ sequence, then

$$\begin{cases} I(u_n)=\dfrac{a}{2}\|u_n\|^2-\dfrac{b}{4}\|u_n\|^4-\mu\displaystyle\int_\Omega fu_ndx\to c,\\ \langle I'(u_n),v\rangle=\left(a-b\|u_n\|^2\right)\displaystyle\int_\Omega \nabla u_n\nabla vdx-\mu\int_\Omega fvdx\to 0\end{cases} \tag{7}$$

for any $v\in H_0^1(\Omega)$ as $n\to\infty$. Especially, for any $\varepsilon>0$, taking $v=u_n$ in (7) to find

$$-\varepsilon\|u_n\|<\langle I'(u_n),u_n\rangle=\left(a-b\|u_n\|^2\right)\|u_n\|^2-\mu\int_\Omega fu_ndx<\varepsilon\|u_n\|$$

by $|\langle I'(u_n),u_n\rangle| \leqslant \|I'(u_n)\|_{H^{-1}}\|u_n\|$ and $I'(u_n)\to 0$ as $n\to\infty$. So, we have

$$\begin{aligned}-\varepsilon\|u_n\|+c\leqslant I(u_n)-\langle I'(u_n),u_n\rangle &= \frac{3b}{4}\|u_n\|^4-\frac{a}{2}\|u_n\|^2\\ &\leqslant c+\varepsilon\|u_n\|\leqslant |c|+\frac{\varepsilon^2}{2a}+\frac{a}{2}\|u_n\|^2.\end{aligned}$$

Taking $\varepsilon=\sqrt{2a}$ to find

$$\frac{3b}{4}\|u_n\|^4-a\|u_n\|^2-|c|-1\leqslant 0.$$

This shows that

$$0\leqslant\|u_n\|^2\leqslant\frac{a+\sqrt{a^2+3b(|c|+1)}}{2\cdot\frac{3b}{4}}.$$

Obviously, $\{u_n\}_{n=1}^{\infty}$ is bounded with a bounded c. Therefore, it holds that

$$I(u_n)-\langle I'(u_n),u_n\rangle=\frac{3b}{4}\|u_n\|^4-\frac{a}{2}\|u_n\|^2\to c \tag{8}$$

and we can obtain that

$$\lim_{n\to\infty}\|u_n\|^2=\frac{\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+3bc}}{2\cdot\frac{3b}{4}}=\frac{a\pm a\sqrt{1+\frac{12bc}{a^2}}}{3b}=\frac{a}{3b}\left(1\pm\sqrt{1+\frac{12bc}{a^2}}\right).$$

From the equality above, we have

$$\begin{cases}\lim\limits_{n\to\infty}\|u_n\|^2<0 \;\Leftrightarrow\; c<-\dfrac{a^2}{12b};\\ 0\leqslant\lim\limits_{n\to\infty}\|u_n\|^2\leqslant\dfrac{2a}{3b} \;\Leftrightarrow\; -\dfrac{a^2}{12b}\leqslant c\leqslant 0;\\ \dfrac{2a}{3b}<\lim\limits_{n\to\infty}\|u_n\|^2<\dfrac{a}{b} \;\Leftrightarrow\; 0<c<\dfrac{a^2}{4b};\\ \lim\limits_{n\to\infty}\|u_n\|^2=\dfrac{a}{b} \;\Leftrightarrow\; c=\dfrac{a^2}{4b};\\ \lim\limits_{n\to\infty}\|u_n\|^2>\dfrac{a}{b} \;\Leftrightarrow\; c>\dfrac{a^2}{4b}.\end{cases} \tag{9}$$

Therefore, from (9), we can obtain that $c\geqslant-\frac{a^2}{12b}$ is well defined, and there exists no $\lim\limits_{n\to\infty}\|u_n\|^2$ in $\mathbb{R}$ for $c<-\frac{a^2}{12b}$.

Case $\lim\limits_{n\to\infty}\|u_n\|^2=\frac{a}{b}$, we have $a-b\|u_n\|^2\to 0$ as $n\to\infty$. By (3,1), we can choose a function $v\in H_0^1(\Omega)$ such that $\int_\Omega fv\,dx=1$ and it holds that $\|v\|<+\infty$ and

$$\left(a-b\|u_n\|^2\right)\int_\Omega \nabla u_n\nabla v\,dx\to\mu,$$

this leads to that

$$\lim_{n\to\infty}\|u_n\|\|v\| \geqslant \lim_{n\to\infty}\int_\Omega \nabla u_n \nabla v dx = \mu \lim_{n\to\infty}\left(a-b\|u_n\|^2\right)^{-1} = +\infty,$$

and therefore $\lim\limits_{n\to\infty}\|u_n\| \to +\infty$. Which is a contradiction with $\|u_n\|^2 \to \frac{a}{b}$. So, (7) does not hold with all $v \in H_0^1(\Omega)$ and I has no $(PS)_c$ sequence with $c = \frac{a^2}{4b}$.

Case $\lim\limits_{n\to\infty}\|u_n\|^2 \neq \frac{a}{b}$, we have $a-b\|u_n\|^2 \not\to 0$ and $c \neq \frac{a^2}{4b}$.

Since $\{u_n\}_{n=1}^\infty$ is bounded in $H_0^1(\Omega)$, if necessary, pass to a subsequence (still denoted by $\{u_n\}_{n=1}^\infty$) and u_0 in $H_0^1(\Omega)$ such that, for $n \to \infty$,

$$\begin{cases} u_n \rightharpoonup u_0, & \text{weakly in } \ H_0^1(\Omega), \\ u_n \to u_0, & \text{strongly in } \ L^q(1 \leqslant q < 2^*), \\ u_n(x) \to u_0(x), & \text{a.e.} \quad x \in \Omega. \end{cases}$$

Hence there is

$$\begin{cases} \langle I'(u_n), u_n\rangle = \left(a-b\|u_n\|^2\right)\int_\Omega \nabla u_n \nabla u_n dx - \mu \int_\Omega f u_n dx \to 0, \\ \langle I'(u_n), u_0\rangle = \left(a-b\|u_n\|^2\right)\int_\Omega \nabla u_n \nabla u_0 dx - \mu \int_\Omega f u_0 dx \to 0. \end{cases} \tag{10}$$

Lebesgue's dominated convergence theorem (see [25, pp.27]) leads to

$$\lim_{n\to\infty}\int_\Omega f u_n dx = \int_\Omega f u_0 dx. \tag{11}$$

From (10) and (11), we have

$$\left(a-b\|u_n\|^2\right)\int_\Omega \nabla u_n \nabla (u_n-u_0) dx \to 0.$$

If $a-b\|u_n\|^2 \not\to 0$, one has $u_n \to u_0$ in $H_0^1(\Omega)$. This proof is complete.

In order to prove main results, it is necessary to make some notes, that is

$$\mathscr{D}^+ = \left\{u \in H_0^1(\Omega); \int_\Omega f u dx > 0\right\}, \ \ \mathscr{D}^- = \left\{u \in H_0^1(\Omega); \int_\Omega f u dx < 0\right\}.$$

Setting $u \in \mathscr{D}^\pm$, then, there is $tu \in \mathscr{D}^\pm$ with $t > 0$ and $tu \in \mathscr{D}^\mp$ with $t < 0$. For any $u \in \mathscr{D}^\pm$, it is easy to see that $|\int_\Omega f u dx| > 0$, and therefore, we shall let

$$\Lambda := \min_{u\in\mathscr{D}^+\bigcup\mathscr{D}^-}\left\{\|u\|^{-1}\left|\int_\Omega f u dx\right|\right\} = \min_{u\in\mathscr{D}^+}\left\{\|u\|^{-1}\int_\Omega f u dx\right\} \leqslant \frac{1}{\sqrt{S}}\|f\|_{\frac{2^*}{2^*-1}}.$$

LEMMA 2. *Assume that* $a, b > 0$, $f \in L^{\frac{2^*}{2^*-1}}(\Omega)$ *and* $f(x) \geqslant 0$ *a.e.* $x \in \Omega$, *then,*

(i) there exist $r, \rho, \mu_{*1} > 0$, such that for any $\mu \in (0, \mu_{*1}]$, it holds

$$\inf_{\|u\|=r} I(u) \geqslant \rho \quad \text{and} \quad \inf_{\|u\|\leqslant r} I(u) := c_1 < 0;$$

(ii) There exists $R>0$, such that $\sup_{\|u\|\geqslant R} I(u)\leqslant 0$ for any $\mu>0$.

Proof. (i) By using the Sobolev imbedding inequality on (5), we obtain

$$I(u)\geqslant \frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4-\frac{\mu}{\sqrt{S}}\|f\|_{\frac{2^*}{2^*-1}}\|u\|=\|u\|\Big(\frac{a}{2}\|u\|-\frac{b}{4}\|u\|^3-\frac{\mu}{\sqrt{S}}\|f\|_{\frac{2^*}{2^*-1}}\Big).$$

We can see that for any $\mu<\frac{a}{9b}(6abS)^{\frac{1}{2}}\|f\|^{-1}_{\frac{2^*}{2^*-1}}:=\mu_0$ with $r_0=(\frac{2a}{3b})^{\frac{1}{2}}$, there is

$$\underset{\|u\|=r_0,\ \mu<\mu_0}{I(u)}\geqslant\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}\Big[\frac{a}{3}\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}-\frac{\mu}{\sqrt{S}}\|f\|_{\frac{2^*}{2^*-1}}\Big]:=\rho_0>0.$$

Consequently, in order to calculate its conveniently, it is easy to see that there exist $r=(\frac{2a}{3b})^{\frac{1}{2}}$, $\rho=\frac{a^2}{9b}$, $\mu_{*1}=\frac{a}{18b}(6abS)^{\frac{1}{2}}\|f\|^{-1}_{\frac{2^*}{2^*-1}}$, such that, for any $\mu\in(0,\mu_{*1}]$, one has

$$\begin{aligned}\underset{\|u\|=r,\ \mu\leqslant\mu_*}{I(u)}&\geqslant\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}\Big[\frac{a}{3}\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}-\frac{\mu}{\sqrt{S}}\|f\|_{\frac{2^*}{2^*-1}}\Big]\geqslant\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}\Big[\frac{a}{3}\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}-\frac{\mu_*}{\sqrt{S}}\|f\|_{\frac{2^*}{2^*-1}}\Big]\\&\geqslant\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}\Big[\frac{a}{3}\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}-\frac{a}{18b}(6abS)^{\frac{1}{2}}\|f\|^{-1}_{\frac{2^*}{2^*-1}}\cdot\frac{1}{\sqrt{S}}\|f\|_{\frac{2^*}{2^*-1}}\Big]\\&=\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}\Big[\frac{a}{3}\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}-\frac{a}{6}\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}\Big]=\frac{a^2}{9b}=\rho.\end{aligned}$$

Taking $\tilde{u}\in\mathscr{D}^+\subset H_0^1(\Omega)$ with $\|\tilde{u}\|=r$, it holds $\|t\tilde{u}\|\leqslant r$ with $t\to0^+$, and

$$\begin{aligned}\lim_{t\to0^+}\frac{I(t\tilde{u})}{t}&=\lim_{t\to0^+}\frac{1}{t}\Big\{\frac{a}{2}\|tu\|^2-\frac{b}{4}\|tu\|^4-\mu\int_\Omega f\cdot(tu)dx\Big\}\\&=-\mu\int_\Omega f\tilde{u}dx=-\frac{\mu r}{\|\tilde{u}\|}\int_\Omega f\tilde{u}dx\\&\leqslant-\mu r\Lambda=-\Big(\frac{2a}{3b}\Big)^{\frac{1}{2}}\Lambda\mu<0.\end{aligned}$$

Therefore, $c_1:=\inf_{\|u\|\leqslant r}I(u)\leqslant-(\frac{2a}{3b})^{\frac{1}{2}}\Lambda\mu<0$ is well defined.

(ii) For any $\mu>0$, by Young's inequality, we have

$$\begin{aligned}I(u)&\leqslant\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4+\mu\Big|\int_\Omega fudx\Big|\\&\leqslant\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4+\frac{a}{2}\|u\|^2+\frac{\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}}\\&=a\|u\|^2-\frac{b}{4}\|u\|^4+\frac{\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}}.\end{aligned}\tag{12}$$

So, there exists $R=\frac{2}{b}\Big[a+\big(a^2+\frac{b\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}}\big)^{\frac{1}{2}}\Big]$, such that $I(u)\leqslant0$ with $\|u\|\geqslant R$, thus the conclusion $\sup_{\|u\|\geqslant R}I(u)\leqslant0$ can be true. This completes the proof.

3.1. Proof of the first and second solutions of problem (1)

THEOREM 3. Assume that $a,b>0$ and $f(x)\in L^{\frac{2^*}{2^*-1}}(\Omega)$ is positive a.e. $x\in\Omega$, then, for any $\mu\in(0,\mu_{*1})$ (μ_{*1} is defined by lemma 2), the problem (1) has at least two positive solutions.

Proof.

Existence of the first solution. Taking $B_r:=\{u\in H_0^1(\Omega);\|u\|\leqslant r\}$, where $r=(\frac{2a}{3b})^{\frac{1}{2}}$. By the Lemma 2, there exists $\mu_{*1}>0$ such that

$$\inf I(B_r)\leqslant-\left(\frac{2a}{3b}\right)^{\frac{1}{2}}\Lambda\mu<0$$

for any $\mu\in(0,\mu_{*1})$. By the Ekeland variational principle (see [9, Lemma 1.1]), there is a minimizing sequence $\{u_n\}_{n=1}^{\infty}\subset\overline{B_r}$ such that

$$I(u_n)\leqslant\inf I(B_r)+\frac{1}{n}\ \text{ and }\ I(u)\geqslant I(u_n)-\frac{1}{n}\|u-u_n\| \tag{13}$$

for all $n\in\mathbb{N}$ and for any $u\in\overline{B_r}$. Therefore, we can get $I(u_n)\to c_1$ and $I'(u_n)\to 0$ in dual space of $H_0^1(\Omega)$ as $n\to\infty$. By Lemma 1, there exist a subsequence (still denoted by $\{u_n\}$) and $u_1^*\in B_r$ such that $u_n\to u_1^*$ as $n\to\infty$. Then, $I(u_1^*)=c_1<0$ and $I'(u_1^*)=0$. Moreover, $u_1^*\in\mathscr{D}^+$ and $\|u_1^*\|^2<\frac{a}{3b}$. This implies that u_1^* is a local minimizer for I. Hence u_1^* is a weak solution of problem (1).

Existence of the second solution. For any $\bar{u}\in\mathscr{D}^+$, there exists $\theta\in(0,1)$, such that

$$\begin{aligned}\sup I(\bar{u})&\leqslant\sup\left\{\frac{a}{2}\|\bar{u}\|^2-\frac{b}{4}\|\bar{u}\|^4-\mu\Lambda\|\bar{u}\|\right\}\\&\leqslant\frac{a^2}{4b}-\theta\left(\frac{a}{b}\right)^{\frac{1}{2}}\mu\Lambda<\frac{a^2}{4b}.\end{aligned}$$

According to Lemma 2, I has mountain pass geometry for any $\mu\in(0,\mu_{*1})$. For any $e\in H_0^1(\Omega)$ with $\|e\|\geqslant R$(where R defined in the lemma 2), we set

$$\Gamma=\left\{\tau(t)\in C^1\left([0,1],H_0^1(\Omega)\right);\tau(0)=0,\tau(1)=e\right\}.$$

By (5)–(6), $I\big(\tau(t)\big)$ has continuity, $I\big(\tau(0)\big)=0$, $I\big(\tau(1)\big)\leqslant 0$. So, there is

$$0<\rho\leqslant c_2:=\inf_{\tau\in\Gamma}\sup_{t\in[0,1]}I\big(\tau(t)\big)\leqslant\sup_{u\in\mathscr{D}^+}I(u)<\frac{a^2}{4b}.$$

By the mountain pass theorem (see [1, Theorem 2.1–2.4]), there exist u_2^* and a sequence $\{u_k\}_{k=1}^{\infty}$ in $H_0^1(\Omega)$, moreover in $\mathscr{D}^+$, such that $u_k\rightharpoonup u_2^*$ in $H_0^1(\Omega)$ and $I'(u_k)\to 0$ in dual space of $H_0^1(\Omega)$. Lemma 1 means that $u_k\to u_2^*$ in $H_0^1(\Omega)$, $I(u_k)\to c_2=I(u_2^*)$ and

$I'(u_k) \to 0 = I'(u_2^*)$ in dual space of $H_0^1(\Omega)$. Hence u_2^* is a weak solution of problem (1) with $\|u_2^*\| > (\frac{2a}{3b})^{\frac{1}{2}}$. Since $I(u_1^*) < 0 < I(u_2^*)$, we get $u_2^* \neq u_1^*$.

Proof of that u_1^* and u_2^* are positive. Since $u_i^* \in \mathscr{D}^+ (i = 1,2)$ are the weak solutions of problem (1), we have

$$\left(a - b\int_\Omega |\nabla u_i^*|^2 dx\right) \int_\Omega |\nabla u_i^*|^2 dx = \mu \int_\Omega f u_i^* dx > 0.$$

Hence $a - b\|u_i^*\|^2 > 0$. This means that $-\Delta u_i^* = \mu(a - b\|u_i^*\|^2)^{-1} f(x) \geqslant 0$ and $u_i^* \not\equiv 0$. According to the strong maximum principle, we obtain u_i^* are positive solutions.

3.2. Proof of the third solution of problem (1)

LEMMA 3. *Assume that $a,b,\mu > 0$, $f \in L^{\frac{2^*}{2^*-1}}(\Omega)$ and $f(x) \geqslant 0$ a.e. $x \in \Omega$, then*

$$\frac{a^2}{4b} < \sup I(u) \leqslant \frac{a^2}{b} + \frac{\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}}.$$

Proof. For any $u \in \mathscr{D}^+$, $tu \in \mathscr{D}^-$ with $t < 0$ and we have

$$\begin{aligned}\sup_{t<0} I(tu) &\geqslant \sup_{t=-(\frac{a}{b})^{\frac{1}{2}}\|u_0\|^{-1}} \left\{\frac{a}{2}\|tu\|^2 - \frac{b}{4}\|tu\|^4 - \mu\int_\Omega f\cdot(tu)dx\right\} \\ &= \frac{a^2}{4b} + \left(\frac{a}{b}\right)^{\frac{1}{2}} \frac{\mu}{\|u\|}\left|\int_\Omega f u_0 dx\right| > \frac{a^2}{4b}. \end{aligned} \tag{14}$$

So, $\sup I(u) \geqslant \sup\limits_{u\in\mathscr{D}^-} I(u) > \dfrac{a^2}{4b}$. Moreover, via the Young's inequality, we obtain

$$\begin{aligned}\sup I(u) &\leqslant \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 + \mu\left|\int_\Omega f u dx\right| \\ &\leqslant \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 + \frac{a}{2}\|u\|^2 + \frac{\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}} \\ &\leqslant \max_{t>0}\left\{at^2 - \frac{b}{4}t^4 + \frac{\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}}\right\} \\ &= \frac{a^2}{b} + \frac{\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}}.\end{aligned}$$

Consequently, $\sup I(u) \leqslant \frac{a^2}{b} + \frac{\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}}$. This with the (14), our proof is complete.

THEOREM 4. *Assume that $a,b > 0$ and $f(x) \in L^{\frac{2^*}{2^*-1}}(\Omega)$ is a positive function a.e. $x \in \Omega$, then, for any $\mu > 0$, the problem (1) has at least a negative solution.*

Existence of the third solution. From Lemma 3, the functional I has the supremum. Set

$$\mathscr{F}=\left\{T_t\in C^1\left(H_0^1(\Omega),H_0^1(\Omega)\right);T_t(u)=tu,t\in\mathbb{R}\right\},$$

$$\mathscr{A}=\left\{tu;u\in\mathscr{D}^+\cup\mathscr{D}^-,t\in\mathbb{R}\right\},$$

Then, for all $A\in\mathscr{A}$, $T_t(A)\in\mathscr{A}$ are hold for any $T_t\in\mathscr{F}$. Therefore, there is

$$\frac{a^2}{4b}<\inf_{A\in\mathscr{A}}\max_{t\in\mathbb{R}}\sup_{\tilde{u}\in\mathscr{A}}I(t\tilde{u}):=c_3\leqslant\frac{a^2}{b}+\frac{\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}}.$$

for any $\mu>0$. By applying Lemma 1 and the Minimax principle (see [3, Theorem 1.5 & Corollary 1.3 in Chapter 3]) for I, there exist u_3^* and a sequence $\{u_m\}$ in $H_0^1(\Omega)$, moreover in $\mathscr{D}^-$, such that $u_m\to u_3^*$ in $H_0^1(\Omega)$, $I(u_m)\to c_3=I(u_3^*)$ and $I'(u_m)\to 0=I'(u_3^*)$ in $H^{-1}(\Omega)$. Hence u_3^* is a weak solution of problem (1) and $\|u_3^*\|^2>\frac{a}{b}$.

Indeed, instead of the Minimax principle, taking $B_R:=\{u\in H_0^1(\Omega)\mid\|u\|\leqslant R\}$ and applying the Ekeland variational principle for $-I$, we can obtain the existence of u_3^* by Lemma 1, where $R=\frac{2}{b}\left[a+(a^2+\frac{b\mu^2}{2aS}\|f\|^2_{\frac{2^*}{2^*-1}})^{\frac{1}{2}}\right]$.

Proof of the negativity for u_3^*. Since $u_3^*\in\mathscr{D}^-$ is a weak solution of problem (1) with $\|u_3^*\|^2>\frac{a}{b}$, we have $a-b\|u_3^*\|^2<0$ and

$$\mu\int_\Omega fu_3^*dx=\left(a-b\int_\Omega|\nabla u_3^*|^2dx\right)\int_\Omega|\nabla u_3^*|^2dx<0.$$

Hence $u_3^*\not\equiv 0$ and

$$-\Delta u_3^*=\mu(a-b\|u_3^*\|^2)^{-1}f(x)\leqslant 0.$$

By the strong minimum principle, we obtain that u_3^* is negative. This proof is complete.

3.3. Proof of the Theorem 1

It is clear that problem (1) has at least two positive solutions u_1^* and u_2^* for $\mu\in(0,\mu_{*1}]$ by Theorem 3 and a negative solution u_3^* by Theorem 4. Since

$$I(u_1^*)<0<I(u_2^*)<\frac{a^2}{4b}<I(u_3^*),$$

we get that u_1^*, u_2^* and u_3^* are different solutions with

$$0<\|u_1^*\|^2<\frac{a}{3b}<\frac{2a}{3b}<\|u_2^*\|^2<\frac{a}{b}<\|u_3^*\|^2.$$

For $\mu\in[\mu_{*1},+\infty)$, Theorem 4 leads to that problem (1) admits at least a negative solution. Hence there exists $\mu_{*1}>0$ such that the problem (1) has at least three nontrivial solutions for $\mu\in(0,\mu_{*1})$, and a nontrivial solution for $\mu\in[\mu_{*1},+\infty)$.

Moreover, for $\|u\|^2=\frac{4a}{3b}$ and $\mu \leqslant \frac{a}{72b}(3abS)^{\frac{1}{2}}\|f\|^{-1}_{\frac{2^*}{2^*-1}}$, we have $I(u)\leqslant \frac{a^2}{4b}$, $\sup I(u)$ is achieved on $\frac{a}{b}<\|u\|^2<\frac{4a}{3b}$. So, there exists $\mu_*=\frac{a}{72b}(3abS)^{\frac{1}{2}}\|f\|^{-1}_{\frac{2^*}{2^*-1}}<\mu_{*1}$, such that, problem (1) has at least three nontrivial solutions u_1^*, u_2^* and u_3^* for $\mu\in(0,\mu_*)$, and a nontrivial solution u_3^* for $\mu\geqslant\mu_*$. In addition, it holds that $\frac{a}{b}<\|u_3^*\|^2<\frac{4a}{3b}$ for $\mu\in(0,\mu_*)$. This proof is completed.

4. Proof of main results via algebraic analysis

Step 1. Let u be a solution, we shall give the calculated method to other solutions with the help of algebraic analysis. Since u is a solution of problem (1), one has

$$\left(a-b\|u\|^2\right)\int_\Omega \nabla u\nabla v dx=\mu\int_\Omega f(x)v dx,\ \forall\, v\in H_0^1(\Omega). \tag{15}$$

For given u, we have $\|u\|^2=\int_\Omega|\nabla u|^2dx:=\alpha>0$ and it is easy to see that $\alpha\neq\frac{a}{b}$. The existence of three solutions via algebraic analysis, if and only if there exist three values of t such that $tu\in H_0^1(\Omega)$ and

$$\left(a-b\|tu\|^2\right)\int_\Omega \nabla(tu)\nabla v dx=\mu\int_\Omega f(x)v dx,\ \forall\, v\in H_0^1(\Omega). \tag{16}$$

It follows from (15)–(16) that our goal is equivalent to finding all $t\in\mathbb{R}$ such that

$$t\left(a-bt^2\alpha\right)=a-b\alpha. \tag{17}$$

It is easy to see that $t_0=1$ is a solution of Eq. (17), and for addition, Eq. (17) has the solutions $t_1=\frac{1}{2}\left(-1+\sqrt{\frac{4a}{b\alpha}-3}\right)$ and $t_2=\frac{1}{2}\left(-1-\sqrt{\frac{4a}{b\alpha}-3}\right)$ when $\alpha<\frac{4a}{3b}$, $t_1=t_2=-\frac{1}{2}$ when $\alpha=\frac{4a}{3b}$ and no real solution when $\alpha>\frac{4a}{3b}$. There is $t_1=t_0=1$ and $t_2=-2$ when $\alpha=\frac{a}{3b}$ in addition.

We set $u,\bar{u}$ are different solutions of problem (1) with $\mu\neq 0$, then,

$$-\left(a-b\|u\|^2\right)\Delta u=\mu f(x)=-\left(a-b\|\bar{u}\|^2\right)\Delta\bar{u}.$$

It holds that $a-b\|u\|^2\neq 0$ and $a-b\|\bar{u}\|^2\neq 0$ are constants by (15). Consciously,

$$-(a-b\|u\|^2)\Delta u+(a-b\|\bar{u}\|^2)\Delta\bar{u}=0, \tag{18}$$

and $\left[(a-b\|u\|^2)u-(a-b\|\bar{u}\|^2)\bar{u}\right]\in H_0^1(\Omega)$ by $u,\bar{u}\in H_0^1(\Omega)$. Multiplying the equation (18) by $\left[(a-b\|u\|^2)u-(a-b\|\bar{u}\|^2)\bar{u}\right]$ and integrating over Ω, we obtain that

$$\left\|(a-b\|u\|^2)u-(a-b\|\bar{u}\|^2)\bar{u}\right\|^2=0.$$

Hence $u=\frac{a-b\|\bar{u}\|^2}{a-b\|u\|^2}\bar{u}$. Thus all solutions of problem (1) are linear dependence.

Step 2. A proof will be give for three solutions with $\mu > 0$ enough small only by using one of the Mountain Pass Lemma, Ekeland variational principle, and Minimax principle.

Three solutions of Theorem 1. According to the information we state above, Eq. (1) has at least a nontrivial solution (here we denote by u_1^*, u_2^* or u_3^*) with all $a,b > 0$ and $\mu \in (0,\mu_*)$ by using one of the Mountain Pass Lemma, Ekeland variational principle, and Minimax principle. Moreover, one has

$$0 < \|u_1^*\|^2 < \frac{a}{3b} < \frac{2a}{3b} < \|u_2^*\|^2 < \frac{a}{b} < \|u_3^*\|^2 < \frac{4a}{3b}.$$

Choosing one of the solutions u_1^*, u_2^* and u_3^* denoted by u, then, Eq. (17) has three different solutions t_0,t_1,t_2 and problem (1) has three different solutions u,t_1u,t_2u, them are linear dependence.

REMARK 1. From the step 1, we obtain that Eq. (1) has at most three different solutions. We have been proved that Eq. (1) has at least 3 solutions with $\mu > 0$ enough small in Theorem 1. Therefore, Eq. (1) has only three solutions with

$$\left\{u_1^*,u_2^*,u_3^*\right\} = \left\{u_1^*,t_1u_1^*,t_2u_1^*\right\} = \left\{u_2^*,t_1'u_2^*,t_2'u_2^*\right\} = \left\{u_3^*,t_1''u_3^*,t_2''u_3^*\right\}$$

if $\mu > 0$ small enough, where t_1',t_2',t_1'',t_2'' are four constants.

Step 3. The specific form of μ_{**} is given exactly and the Eq. (1) has only three solutions for $0 < |\mu| < \mu_{**}$, two solutions for $|\mu| = \mu_{**}$ and a solution for $|\mu| > \mu_{**}$.

Proof of Theorem 2 *with* $\mu \neq 0$. Research the following elliptic problem:

$$\begin{cases} -\Delta u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where $f \in L^{\frac{2^*}{2^*-1}}(\Omega)$ and $f(x) \geqslant 0$ a.e. $x \in \Omega$. Then, it is well known that problem (19) has an unique positive solution $U \in H_0^1(\Omega)$. Moreover,

$$\begin{cases} \|U\|^2 = \int_\Omega fU dx \leqslant \frac{1}{\sqrt{S}}\|f\|_{\frac{2^*}{2^*-1}}\|U\| \Longrightarrow \|U\|^{-1} \geqslant \sqrt{S}\|f\|^{-1}_{\frac{2^*}{2^*-1}}, \\ \|U\|^2 = \int_\Omega fU dx \leqslant \|f\|_2\|U\|_2 \leqslant \frac{1}{\lambda_1}\|f\|_2\|U\| \Longrightarrow \|U\|^{-1} \geqslant \lambda_1\|f\|_2^{-1}. \end{cases} \tag{20}$$

Consider the function $g(t)$ as following:

$$g(t) = \left(a - b\|tU\|^2\right)t - \mu, \tag{21}$$

where $t \in \mathbb{R}$ is variable, $\mu > 0$ is a parameter. If $t = T$ is a zero-point of $g(t)$, then

$$\left(a - b\int_\Omega |\nabla(TU)|^2 dx\right)\int_\Omega \nabla(TU)\nabla v dx = \mu\int_\Omega f v dx,\ \forall\, v \in H_0^1(\Omega).$$

This means that TU is a solution of Eq. (1). $g'(t) = a - 3b\|U\|^2t^2$ implies that the extreme-points of $g(t)$ are $t_m = -\sqrt{3ab}(3b\|U\|)^{-1}$ and $t_M = \sqrt{3ab}(3b\|U\|)^{-1}$. Hence, $g(t)$ is decreasing in $(-\infty, t_m)$, increasing in $[t_m, t_M]$ and decreasing in $(t_M, +\infty)$,

$$\begin{cases} \min g(t) = g(t_m) = \left(b\|U\|^2 \cdot \dfrac{a}{3b\|U\|^2} - a\right) \cdot \dfrac{\sqrt{3ab}}{3b\|U\|} - \mu = -\dfrac{2a\sqrt{3ab}}{9b\|U\|} - \mu, \\[2ex] \max g(t) = g(t_M) = \left(a - b\|U\|^2 \cdot \dfrac{a}{3b\|U\|^2}\right) \cdot \dfrac{\sqrt{3ab}}{3b\|U\|} - \mu = \dfrac{2a\sqrt{3ab}}{9b\|U\|} - \mu. \end{cases} \tag{22}$$

Consciously, by (22), it is easy to get that, there exists $\mu_{**} = 2a\sqrt{3ab}(9b\|U\|)^{-1}$ such that Eq. (21) has three solutions T_1, T_2, T_3 for $0 < |\mu| < \mu_{**}$ with

$$T_1 < -\sqrt{3ab}(3b\|U\|)^{-1} < T_2 < \sqrt{3ab}(3b\|U\|)^{-1} < T_3,$$

two solutions T_2 and 'T_1 or T_3' for $|\mu| = \mu_{**}$ with

$$T_1 = -2\sqrt{3ab}(3b\|U\|)^{-1} < T_2 = \sqrt{3ab}(3b\|U\|)^{-1}$$

or

$$T_2 = -\sqrt{3ab}(3b\|U\|)^{-1} < T_3 = 2\sqrt{3ab}(3b\|U\|)^{-1},$$

and a solution T_1 or T_3 for $|\mu| > \mu_{**}$ with

$$T_1 < -\sqrt{3ab}(3b\|U\|)^{-1} \text{ or } T_3 > \sqrt{3ab}(3b\|U\|)^{-1}.$$

Besides, we can state that the problem (1) has

(i) three solutions 'T_1U, T_2U, T_3U' for $0 < |\mu| < \mu_{**}$;

(ii) two solutions '$-2\sqrt{3ab}(3b\|U\|)^{-1}U$, $\sqrt{3ab}(3b\|U\|)^{-1}U$' for $\mu = \mu_{**}$;

(iii) two solutions '$-\sqrt{3ab}(3b\|U\|)^{-1}U$, $2\sqrt{3ab}(3b\|U\|)^{-1}U$' for $\mu = -\mu_{**}$;

(iv) a solution 'T_1U' for $\mu > \mu_{**}$ and a solution 'T_3U' for $\mu < -\mu_{**}$,

where U is the unique positive solution of problem (19) and $\mu_{**} = 2a\sqrt{3ab}(9b\|U\|)^{-1}$.

Comparing with the Step 1, all solutions of problem (1) are linear dependence, there are no more solutions than those mentioned above. Consequently, a conclusion can be get that there exists a constant $\mu_{**} = 2a\sqrt{3ab}(9b\|U\|)^{-1}$ such that problem (1) has only three solutions for $0 < |\mu| < \mu_{**}$, two solutions for $\mu = \pm\mu_{**}$ and a solution for $|\mu| > \mu_{**}$. And by (20), $\mu_{**} = 2a\sqrt{3ab}(9b\|U\|)^{-1} \geqslant 2a\sqrt{3abS}(9b\|f\|_{\frac{2^*}{2^*-1}})^{-1}$.

At last, we prove the existence of infinitely many solutions for $\mu = 0$.

Proof of Theorem 2 with $\mu = 0$. For this case, we fix any $V \in H_0^1(\Omega)$ and let $u := \sqrt{ab}(b\|V\|)^{-1}V \in H_0^1(\Omega)$, it is clear that $a - b\|u\|^2 = 0$ and $\int_\Omega \nabla u \nabla v dx$ are bounded with all $v \in H_0^1(\Omega)$. So, u is a solution of problem (1). According to the arbitrary of $V \in H_0^1(\Omega)$, problem (1) has infinitely many solutions when $\mu = 0$.

We replace the condition $f(x) \in L^{\frac{2^*}{2^*-1}}(\Omega)$ by $f(x) \in H^{-1}(\Omega)$ and $f(x) > 0$ a.e. $x \in \Omega$, we can obtain that the Corollary 1 is clear by $\|U\| \leqslant \|f\|_{H^{-1}}$, where $U \in H_0^1(\Omega)$ is the unique positive solution of problem (19). Similarly, we can obtain the constant $\mu_{**} = 2a\sqrt{3ab}(9b\|U\|)^{-1}$.

5. A example for one dimensional case

EXAMPLE 1. Let $a=b=1$, $\Omega=(0,1)$, $f(x)=1$. Eq. (1) becomes

$$\begin{cases} -\Big(1-\int_0^1 |u'|^2dx\Big)u''=\mu, & \text{in } (0,1),\\ u=0, & \text{on } \{0,1\}. \end{cases} \tag{23}$$

(i) For $\mu>\frac{4}{3}$, $u(x)=\frac{1}{2}\Big[\big(-6\mu+2\sqrt{9\mu^2-16}\big)^{\frac{1}{3}}+\big(-6\mu-2\sqrt{9\mu^2-16}\big)^{\frac{1}{3}}\Big]x(1-x)$ is the unique solution of the Eq. (23); and the next results show that $\mu_{**}=\frac{4}{3}$;

(ii) For $\mu=\frac{4}{3}$, Eq. (23) has only two solutions $u_1(x)=-2x(1-x)$, $u_2(x)=x(1-x)$;

(iii) For $\mu\in(0,\frac{4}{3})$, Eq. (23) has only three solutions $u_i(x)=\frac{t_i}{2}x(1-x)$, $i=1,2,3$, where t_i $(i=1,2,3)$ are the roots of the algebraic equation $t^3-12t+12\mu=0$;

(iv) for $\mu=0$, $u_n(x)=\frac{\sqrt{2}\sin n\pi x}{n\pi}$ are the solutions of Eq. (23), $n\in\mathbb{Z}\setminus\{0\}$, but not all.

Indeed, our conclusion is applicable to all $a,b\in\mathbb{R}$ if $|a|+|b|>0$. If $ab>0$, the fact is in Corollary 1. If $ab\leqslant 0$, Eq. (1) has an unique solution for any $\mu\in\mathbb{R}$.

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REFERENCES

[1] A. AMBROSETTI AND P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14**, 4 (1973), 349–381.

[2] Y. C. AN AND H. M. SUO, *Existence of solutions for the Neumann boundary problem of Kirchhoff type equations*, J. Spectr. Theory, **9**, 2 (2019), 547–568.

[3] K. C. CHANG, *Critical Point Theory and its Applications*, Published by the Shanghai Scientific & Technical Publishers, Shanghai, 1986.

[4] S. T. CHEN, B. L. ZHANG AND X. H. TANG, *Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity*, Adv. Nonlinear Anal., **9**, 1 (2020), 148–167.

[5] C. M. CHU, *Multiplicity of positive solutions for Kirchhoff type problem involving critical exponent and sign-changing weight functions*, Bound. Value Probl., **2014**, 19 (2014), 1–10.

[6] C. M. CHU AND J. J. SUN, *Multiplicity of positive solutions for a class of p-Kirchhoff equation with critical exponent*, Ann. Funct. Anal., **11**, (2020), 1126–1140.

[7] Q. Y. DAI, E. H. LAN AND F. L. SHI, *A priori bounds for positive solutions of Kirchhoff type equations*, Comput. Math. Appl., **76**, 6 (2018), 1525–1534.

[8] Y. B. DENG AND W.T. HUANG, *Least energy solutions for fractional Kirchhoff type equations involving critical growth*, Discrete Contin. Dyn. Syst. Ser. S, **12**, 7 (2019), 1929–1954.

[9] I. EKELAND, *On the variational principle*, J. Math. Anal. Appl., **47**, 2 (1974), 324–353.

[10] H. N. FAN, *Positive solutions for a Kirchhoff-type problem involving multiple competitive potentials and critical Sobolev exponent*, Nonlinear Anal., **198**, ID 111869 (2020), 1–35.

[11] Z. J. GUO AND X. G. ZHANG, *Schrödinger-Kirchhoff equation involving double critical nonlinearities*, J. Math. Anal. Appl., **471**, 1-2 (2019), 358–377.

[12] Y. HE, *Concentrating bounded states for a class of singularly perturbed Kirchhoff type equations with a general nonlinearity*, J. Differential Equations, **261**, 11 (2016), 6178–6220.

[13] Y. HE AND G. B. LI, *Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents*, Calc. Var. Partial Differ. Equ., **54**, 3 (2015), 3067–3106.

[14] Y. HE, G. B. LI AND S. J. PENG, *Concentrating Bound States for Kirchhoff Type Problems in $\mathbb{R}^3$ Involving Critical Sobolev Exponents*, Adv. Nonlinear Stud., **14**, 2 (2014), 483–510.

[15] T. X. HU AND C. L. TANG, *Limiting behavior and local uniqueness of normalized solutions for mass critical Kirchhoff equations*, Cal. Var. Partial Differ. Equ., **60**, 210 (2021), 1–26.

[16] G. R. KIRCHHOFF, *Vorlesungen über Matematische Physik: Mechanik*, Druck und von Benedictus-Gotthelf Teubner, Leipzig, 1876.

[17] C. Y. LEI, C. M. CHU AND H. M. SUO, *Positive solutions for a nonlocal problem with singularity*, Electron. J. Diff. Equ., **2017**, 85 (2017), 1–9.

[18] C. Y. LEI, J. F. LIAO AND H. M. SUO, *Multiple positive solutions for a class of nonlocal problems involving a sign-changing potential*, Electron. J. Diff. Equ., **2017**, 9 (2017), 1–7.

[19] G. B. LI AND Y. H. LIU, *The existence and local uniqueness of multi-peak positive solutions to a class of Kirchhoff Equation*, Acta Math. Sci., **40B**, 1 (2020), 90–112.

[20] F. Y. LI, Y. ZHANG, X. L. ZHU AND Z. P. LIANG, *Ground-state solutions to a Kirchhoff-type transmission problem*, Topol. Methods Nonlinear Anal., **57**, 1 (2021), 201–219.

[21] L. LI AND J. J. SUN, *Existence and multiplicity solutions for a nonlocal equation of Kirchhoff type*, Differ. Equ. Appl., **10**, 4 (2018), 369–386.

[22] X. T. QIAN AND W. CHAO, *Existence of positive solutions for nonlocal problems with indefinite nonlinearity*, Bound. Value Probl., **2020**, 40 (2020), 1–13.

[23] X. T. QIAN AND W. CHAO, *Existence and concentration of ground state solutions for a class of nonlocal problem in $\mathbb{R}^N$*, Nonlinear Anal., **203**, ID 112170 (2020), 1–15.

[24] Z. G. SHI AND X. T. QIAN, *New multiplicity of positive solutions for some class of nonlocal problems*, Bound. Value Probl., **2021**, 55 (2021), 1–13.

[25] W. RUDIN, *Real and Complex Analysis, Third Edition*, McGraw-Hill, Singapore, 1987.

[26] G. TALENTI, *Best constant in Sobolev inequality*, Ann. Mat. Pur. Appl., **110**, 1 (1976), 353–372.

[27] X. H. TANG AND B. T. CHENG, *Ground state sign-changing solutions for Kirchhoff type problems in bounded domains*, J. Differential Equations, **261**, 4 (2016), 2384–2402.

[28] Y. WANG, *A review on Kirchhoff-type problem with negative modulus*, Acta Anal. Funct. Appl., **22**, 4 (2020), 230–258.

[29] Y. WANG, J. P. LIANG AND H. M. SUO, *Existence of multiple solutions for a class of nonlocal near resonance problems*, J. Southwest Univ. (Nat. Sci. Ed.), **40**, 4 (2018), 53–58.

[30] Y. WANG, J. P. LIANG, H. M. SUO, J. LEI, S. H. ZHAO AND H. Y. YE, *Classical solutions for nonlocal problems on different unbounded domains*, Acta Anal. Funct. Appl., **21**, 4 (2019), 325–341.

[31] Y. WANG, H. M. SUO AND C. Y. LEI, *Multiple positive solutions for a nonlocal problem involving critical exponent*, Electron. J. Diff. Equ., **2017**, 275 (2017), 1–11.

[32] Y. WANG, H. M. SUO AND W. WEI, *Classical Solutions for a kind of new Kirchhoff-type problems without boundary constraint*, Acta Math. Sci. Ser. A, **40A**, 4 (2020), 857–868.

[33] Y. WANG AND X. YANG, *Infinitely many solutions for a new Kirchhoff type equation with subcritical exponent*, Appl. Anal., **101**, 3 (2022), 1038–1051.

[34] Y. WANG, H. Y. YE, J. LEI AND H. M. SUO, *Infinitely many classical solutions for Kirchhoff type problem with linear term*, J. Guangxi Normal Univ. (Nat. Sci. Ed.), **38**, 6 (2020), 65–73.

[35] Y. WANG, H. Y. YE AND H. M. SUO, *Existence of positive solution for a nonlocal problem with Hardy-Sobolev critical exponent*, Mathematica Applicata, **32**, 2 (2019), 452–456.

[36] Y. WANG, R. H. ZHONG, X. Y. HAO AND Q. P. WEI, *Research on the existence of solutions for a transmission problem with critical exponent*, Mathematica Applicata, **35**, 2 (2022), 317–326.

[37] H. Y. YE, *Positive high energy solution for Kirchhoff equation in $\mathbb{R}^3$ with superlinear nonlinearities via Nehari-Pohožaev manifold*, Discrete Contin. Dyn. Syst. Ser. A, **35**, 8 (2015), 3857–3877.

[38] G. S. YIN AND J. S. LIU, *Existence and multiplicity of nontrivial solutions for a nonlocal problem*, Bound. Value Probl., **2015**, 26 (2015), 1–7.

[39] X. Y. ZENG AND Y. M. ZHANG, *Existence and uniqueness of normalized solutions for the Kirchhoff equation*, Appl. Math. Lett., **74**, (2017), 52–59.

[40] J. ZHANG AND Z. Y. ZHANG, *Existence of nontrivial solution for a nonlocal problem with subcritical nonlinearity*, Adv. Differ. Equ., **2018**, 359 (2018), 1–8.

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Yue Wang
School of Mathematics and Statistics
Guizhou University
Guiyang 550025, People's Republic of China
e-mail: yuewn@sina.cn, eztf@qq.com
https://orcid.org/0000-0003-0292-5413

Qi-Ping Wei
School of Data Science and Information Engineering
Guizhou Minzu University
Guiyang 550025, People's Republic of China
e-mail: 1250973747@qq.com

Hong-Min Suo
School of Data Science and Information Engineering
Guizhou Minzu University
Guiyang 550025, People's Republic of China
e-mail: gzmysxx88@sina.com