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THREE WEAK SOLUTIONS FOR A DEGENERATE NONLOCAL SINGULAR SUB–LINEAR PROBLEM

SHAPOUR HEIDARKHANI*, KIT IAN KOU AND AMJAD SALARI

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Abstract. Based on one recent abstract critical point result for differentiable and parametric functionals which was recently proved by Ricceri, we establish the existence of three weak solutions for a class of degenerate nonlocal singular sub-linear problems when the nonlinear term admits some hypotheses on the behavior at infinitely or perturbation property.

1. Introduction

The aim of this paper is to study the existence of weak solutions for the following degenerate nonlocal problem

$$\begin{cases} -M\left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right) \operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) \\ \qquad = |x|^{-p(a+1)+c}\big(\varepsilon f(u)+\lambda g(u)+\nu h(u)\big), & \text{in}\,\Omega, \\ u=0, & \text{on}\,\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of R^N with Lipschitz boundary, $0 \leqslant a < \frac{N-p}{p}$, $1<p<N$, $c>0$, $M:\mathbb{R}^+\to\mathbb{R}$, $f,g,h:\mathbb{R}\to\mathbb{R}$ are three continuous functions, $\varepsilon>0$, $\lambda>0$ and $\nu \geqslant 0$ are three parameters.

Problem (1.1) is related to the stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \quad 0<x<L,\ t \geqslant 0, \tag{1.2}$$

where $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length and ρ_0 is the initial axial tension. Equation (1.2) was proposed by Kirchhoff [22] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. In the literature, such problems like (1.1) are called nonlocal problems

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* Corresponding author.

because they are no longer pointwise identity, since the function M in problem (1.1) contains an integral over Ω.

Some interesting results can be found, for example in [1, 9], if the length changes of the string produced by transverse vibrations are taken into account in the Kirchhoff's model. On the other hand, Kirchhoff-type boundary value problems appear in several physical and biological systems where u describes a process which depends on the average of itself, for example the population density. It received great attention only after Lions [26] proposed an abstract framework for the problem. The solvability of the Kirchhoff type problems is obvious an important topic and has been studies by many researchers. Some early classical results can be seen in [19, 20, 21, 27, 33] and the references therein. The existence and multiplicity of solutions for stationary higher order problems of Kirchhoff type were also treated recently, via variational methods like the symmetric mountain pass theorem in [17] and a three critical point theorem in [4]. Moreover, in [2, 3], some evolutionary higher order Kirchhoff problems were treated, mainly focusing on the qualitative properties of the solutions.

If we take $a = 0$ and $c = p$, problem (1.1) becomes the well-known Kirchhoff boundary value problem involving the p-Laplacian equation

$$\begin{cases} -M\left(\int_\Omega |\nabla u(x)|^p \mathrm{d}x\right)\Delta_p u = \varepsilon f(u) + \lambda g(u) + \nu h(u), & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega. \end{cases} \tag{1.3}$$

During the last few decades, the existence of multiple nontrivial solutions for p-Laplacian type equations has been studied by many researchers using different methods. See for instance [6, 14, 16, 18, 23, 25]. For example, using the minimization technique and maximum principle, Brézis and Oswald in [6] obtained an existence and uniqueness result for the problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = f(x,u) & \text{in}\,\Omega, \\ u = 0 & \text{on}\,\partial\Omega, \end{cases}$$

when the behaviour of $f(s)/s$ is suitably controlled at infinity. In [16] Chang and Toan used variational methods to show that the p-Laplacian type elliptic problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = h(x)|u|^{q-2}u + g(x), & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega \end{cases} \tag{1.4}$$

has at least one/two solutions when $g = 0/g \neq 0$.

On the other hand, during the last two decades, different kinds of singular differential equations have been studied by many researchers. We refer the reader to [5, 10, 13, 15] and the references therein. Some classical tools have been used to study singular equations in the literature, including variational methods, the method of upper and lower solutions, degree theory and fixed point theorems. We mention the following two works which are related to our problems. Tyagi in [35] considered a singular quasilinear equation with sign changing nonlinearity and used a three critical point theorem to established the existence results. In [37], it was proved that (1.1) has least three distinct weak solutions under some mild assumptions on a and some growth and singularity conditions of f. In [11] the existence and energy estimates of solutions for the problem (1.1), in the case $\lambda = \nu = 0$, while the nonlinear part of the problem admits some hypotheses on the behavior at origin or perturbation property were established. In particular, for a precise localization of the parameter, the existence of a non-zero solution was discussed requiring the sublinearity of nonlinear part at origin and infinity. Also the existence of solutions for the problem under algebraic conditions with the classical Ambrosetti-Rabinowitz was investigated. Then, by combining two algebraic conditions on the nonlinear term guaranteeing the existence of two solutions as well as applying the mountain pass theorem given by Pucci and Serrin, the existence of the third solution for the problem was ensured. In [12] using variational methods and critical point theory, the existence results and energy estimates of solutions for singular p-Laplacian-type equations if the nonlinear term admits some suitable conditions on the behavior at origin or perturbation property, were established. In particular, for a precise localization of the parameter, the existence of a non-zero solution was ensured and the existence of solutions for positive values of the parameter, with requiring $(p-1)$-sublinearity of nonlinear part at the origin and the infinity was deduce.

The novelty of this paper is that we deal with problem (1.1) in which there is singularities not only in the nonlinear term, but also in the diverge term

$$\operatorname{div}(|x|^{-ap}|\nabla u(x)|^{p-2}\nabla u),$$

which will lead to some difficulties in the proof, and as far as we know, there are very few results even for such singular p-Laplacian type equations in the literature. Based on one recent abstract critical point result for differentiable and parametric functionals which was recently proved by Ricceri, we establish the existence of three weak solutions for (1.1) when the nonlinear term admits some hypotheses on the behavior at infinitely or perturbation property.

2. Preliminaries

In this section, we state some preliminary results, which can be found in [7, 8, 36]. First, for all $u \in C_0^\infty(\mathbb{R}^N)$, there exists a constant $C_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bq}|u(x)|^q \mathrm{d}x\right)^{\frac{p}{q}} \leqslant C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x, \tag{2.1}$$

where

$$-\infty < a < \frac{N-p}{p}, \quad a \leqslant b \leqslant a+1, \quad q = p^*(a,b) = \frac{Np}{N-dp}, \quad d = 1+a-b.$$

Let $W_0^{1,p}(\Omega, |x|^{-ap})$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{a,p} = \left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Then $W_0^{1,p}(\Omega,|x|^{-ap})$ is a reflexive Banach space. From the boundedness of Ω and the standard approximation argument, it is easy to see that (2.1) holds for any $u \in W_0^{1,p}(\Omega,|x|^{-ap})$ in the sense that

$$\left(\int_{\mathbb{R}^N} |x|^{-\alpha}|u(x)|^r \mathrm{d}x\right)^{\frac{p}{r}} \leqslant C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x$$

for $1 \leqslant r \leqslant p^* = \frac{Np}{N-p}$, $\alpha \leqslant (1+a)r + N(1-\frac{r}{p})$, that is, the embedding

$$W_0^{1,p}(\Omega,|x|^{-ap}) \hookrightarrow L^r(\Omega,|x|^{-\alpha})$$

is continuous, where $L^r(\Omega,|x|^{-\alpha})$ is the weighted $L(\Omega,|x|^{-\alpha})$ space with the norm

$$|u|_{r,\alpha} := |u|_{L^r(\Omega,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u(x)|^r \mathrm{d}x\right)^{\frac{1}{r}}.$$

In fact, we have the following compact embedding result which is an extension of the classical Rellich-Kondrachov compactness theorem [36].

LEMMA 1. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary, $0 \in \Omega$ and $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $1 \leqslant r < \frac{Np}{N-p}$, $\alpha < (1+a)r + N(1-\frac{r}{p})$. Then the embedding $W_0^{1,p}(\Omega,|x|^{-ap}) \hookrightarrow L^r(\Omega,|x|^{-\alpha})$ is compact.*

DEFINITION 1. [15, Definition 2.1] We say that $u \in W_0^{1,p}(\Omega,|x|^{-ap})$ is a weak solution of (1.1) if

$$M\left(|x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right)\int_\Omega |x|^{-ap}|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\mathrm{d}x$$

$$-\int_\Omega |x|^{-(a+1)p+c}(\varepsilon f + \lambda g + \nu h)(u(x))v(x)\mathrm{d}x = 0$$

for all $v \in C_0^\infty(\Omega)$.

Let E be a non-empty set and $I,\Psi,\Phi : E \to \mathbb{R}$ be three given functions. If $\mu > 0$ and $r \in (\inf_E \Phi, \sup_E \Phi)$, we put

$$\beta(\mu I + \Psi,\Phi,r) := \sup_{u\in\Phi^{-1}((r,+\infty))} \frac{\mu I(u) + \Psi(u) - \inf_{\Phi^{-1}((-\infty,r))}(\mu I + \Psi)}{r - \Phi(u)}.$$

When the map $\Psi + \Phi$ is bounded from below, for each $r \in (\inf_E \Phi, \sup_E \Phi)$ such that

$$\inf_{\Phi^{-1}(-\infty,r]} I(u) \leqslant \inf_{\Phi^{-1}(r)} I(u),$$

we denote

$$\mu^\star(I,\Psi,\Phi,r) := \inf\left\{\frac{\Psi(u) - \gamma + r}{\eta_r - I(u)} : u \in E, \Phi(u) < r, I(u) < \eta_r\right\},$$

where

$$\gamma := \inf_{\mathrm{E}}(\Psi(u)+\Phi(u)), \quad \eta_r = \inf_{u\in\phi^{-1}(r)} I(u).$$

Using the above notations, we present the following Theorem proved by Ricceri, which was successfully employed in [28] to study the existence of three weak solutions for nonlocal fractional equations. We also refer the reader to the papers [29, 30, 31, 32] and the monograph [24] for some related results along this topic.

Besides, we recall that if $\mathscr{I}$ is a C^1-functional, the derivative $\mathscr{I}' : X \to X^*$ admits a continuous inverse on X^* if there exists a continuous operator $\mathscr{K} : X^* \to X$ such that $\mathscr{K}(\mathscr{I}'(u)) = u$ for every $u \in X$.

THEOREM 1. [29, Theorem 3] *Let* $(\mathrm{E}, \|.\|)$ *be a reflexive Banach space;* $I : \mathrm{E} \to \mathbb{R}$ *be a sequentially weakly lower semicontinuous, bounded on each bounded subset of* E, C^1*-functional whose derivative admits a continuous inverse on the topological dual* E^**;* $\Phi, \Psi : \mathrm{E} \to \mathbb{R}$ *two* C^1*-functionals with compact derivative. Assume also that the functional* $\Psi + \lambda\Phi$ *is bounded below for all* $\lambda > 0$ *and*

$$\liminf_{\|u\|\to\infty} \frac{\Psi(u)}{I(u)} = -\infty.$$

Then, for each $r > \sup_S \Phi$*, where* S *is the set of all global minima of* I*, for each* $\mu > \max\{0, \mu^\star(I, \Psi, \Phi, r)\}$*, and each compact interval* $[\bar{a}, \bar{b}] \subset (0, \beta(\mu I + \Psi, \Phi, r))$*, there exists a number* $\rho > 0$ *with the following property: for every* $\lambda \in [\bar{a}, \bar{b}]$ *and every* C^1*-functional* $\Gamma : \mathrm{E} \to \mathbb{R}$ *with compact derivative, there exists* $\delta > 0$ *such that, for each* $\nu \in [0, \delta]$*, the equation*

$$\mu I'(u) + \Psi'(u) + \lambda\Phi'(u) + \nu\Gamma'(u) = 0,$$

has at least three solutions in E *whose norm are less than* ρ.

Throughout this paper we assume the following assumption:

($\mathscr{M}$) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and satisfies

$$m_0 t^{\alpha-1} \leqslant M(t) \leqslant m_1 t^{\alpha-1} \quad \text{for all}\, t \in \mathbb{R}^+,$$

where $m_1 > m_0 > 0$ and $1 < \alpha < \min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\}$.

We also introduce the following notations

$$\widehat{M}(t) = \int_0^t M(\xi)\mathrm{d}\xi \;\; \text{for all } t \geqslant 0,$$

$$F(t) = \int_0^t f(\xi)\mathrm{d}\xi \;\; \text{for all } t \in \mathbb{R},$$

$$G(t) = \int_0^t g(\xi)\mathrm{d}\xi \;\; \text{for all } t \in \mathbb{R}.$$

For every $\mathscr{A} \subset \Omega$ and $\gamma \in \mathbb{R}$, set

$$\mathscr{D}(\mathscr{A},\gamma) := \int_{\mathscr{A}} |x|^{-\gamma} \mathrm{d}x.$$

In the sequel, we set $X := \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ and consider the functionals $T, J_f, J_g : X \to \mathbb{R}$ defined by

$$T(u) = \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap} |\nabla u(x)|^p \mathrm{d}x\right), \tag{2.2}$$

$$J_f(u) = \int_\Omega |x|^{-p(a+1)+c} F(u(x)) \mathrm{d}x, \tag{2.3}$$

and

$$J_g(u) = \int_\Omega |x|^{-p(a+1)+c} G(u(x)) \mathrm{d}x. \tag{2.4}$$

Since

$$T'(u)(v) = M\left(\int_\Omega |x|^{-ap} |\nabla u(x)|^p \mathrm{d}x\right) \int_\Omega |x|^{-ap} |\nabla u(x)|^{p-2} \nabla u . \nabla v \mathrm{d}x,$$

and

$$J_f'(u)(v) = \int_\Omega |x|^{-(a+1)p+c} f(u(x)) v(x) \mathrm{d}x, \tag{2.5}$$

for every $u, v \in X$, T and J_f are continuously Gâteaux differentiable.

PROPOSITION 1. Let $\mathscr{J} := T' : X \longrightarrow X^*$ be the operator defined by

$$\mathscr{J}(u)(v) = M\left(\int_\Omega |x|^{-ap} |\nabla u(x)|^p \mathrm{d}x\right) \int_\Omega |x|^{-ap} |\nabla u(x)|^{p-2} \nabla u . \nabla v \mathrm{d}x$$

for every $u, v \in X$. Then $\mathscr{J}$ admits a continuous inverse on X^*.

Proof. Since $\mathscr{J}$ is the Fréchet derivative of T, $\mathscr{J}$ is continuous and bounded. On the other hand, for all $u, v \in X$ such that $u \neq v$ we have

$$\begin{aligned}
&(\mathscr{J}(u) - \mathscr{J}(v))(u-v) \\
&\geqslant \max\left\{ m_0 \left| \int_\Omega |x|^{-ap} |\nabla u(x)|^p \mathrm{d}x \right|^{\alpha-1}, m_1 \left| \int_\Omega |x|^{-ap} |\nabla v(x)|^p \mathrm{d}x \right|^{\alpha-1} \right\} \\
&\quad \times \int_\Omega |x|^{-ap} \Big(|\nabla u(x)|^p + |\nabla v(x)|^p - |\nabla v|^{p-2} \nabla v . \nabla u - |\nabla u|^{p-2} \nabla u . \nabla v \Big) \mathrm{d}x.
\end{aligned}$$

Using the elementary inequality [34]

$$|x-y|^\eta \leqslant 2^\eta (|x|^{\eta-2} x - |y|^{\eta-2} y)(x-y) \quad \text{if } \eta \geqslant 2,$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $N \geqslant 1$, we obtain for all $u,v \in X$ such that $u \neq v$,

$$\begin{aligned}&\langle \mathscr{J}(u) - \mathscr{J}(v), u - v\rangle \\ &\geqslant \max\left\{ m_0 \left| \int_\Omega |x|^{-ap} |\nabla u(x)|^p \mathrm{d}x \right|^{\alpha-1}, m_1 \left| \int_\Omega |x|^{-ap} |\nabla v(x)|^p \mathrm{d}x \right|^{\alpha-1} \right\} \\ &\quad \times \int_\Omega |x|^{-ap} \left(\frac{1}{2p} |\nabla v - \nabla u|^p \right) \mathrm{d}x \geqslant 0\end{aligned}$$

which means that $\mathscr{J}$ is a monotone operator. Thus $\mathscr{J}$ is injective. Moreover, $\mathscr{J}$ is a coercive operator. Indeed, we have

$$\frac{\langle \mathscr{J}(u), u\rangle}{\|u\|_{a,p}} \geqslant \frac{m_0 \left(\int_\Omega |x|^{-ap} |\nabla u(x)|^p \mathrm{d}x\right)^\alpha}{\|u\|_{a,p}} = m_0 \|u\|_{a,p}^{\alpha p - 1} \to \infty \quad \text{as } \|u\|_{a,p} \to \infty.$$

Consequently, by Minty-Browder theorem [38], the operator $\mathscr{J}$ is an surjection and admits an inverse mapping. Thus it is sufficient to show that $\mathscr{J}^{-1}$ is continuous. For this, let $(v_n)_{n=1}^\infty$ be a sequence in X^* such that $v_n \to v$ in X^*. Let u_n and u in X such that

$$\mathscr{J}^{-1}(v_n) = u_n \quad \text{and} \quad \mathscr{J}^{-1}(v) = u.$$

By the coercivity of $\mathscr{J}$, we conclude that the sequence (u_n) is bounded in the reflexive space X. For a subsequence, we have $u_n \rightharpoonup \widehat{u}$ in X, which implies

$$\lim_{n \to +\infty} \langle \mathscr{J}(u_n) - \mathscr{J}(u), u_n - \widehat{u}\rangle = \lim_{n \to +\infty} \langle v_n - v, u_n - \widehat{u}\rangle = 0.$$

Therefore, by the continuity of $\mathscr{J}$, we have

$$u_n \to \widehat{u} \quad \text{in} X \quad \text{and} \quad \mathscr{J}(u_n) \to \mathscr{J}(\widehat{u}) = \mathscr{J}(u) \quad \text{in} X^*. \qquad \square$$

3. Main results

We denote by $\mathscr{F}$ the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t \in \mathbb{R}} \frac{|f(t)|}{1 + |t|^{s-1}} < \infty$$

for some $s \in [1, \alpha p)$. First we observe that problem (1.1) has the variational structure, indeed it is the Euler-Lagrange equation of the functional $J_K : X \to \mathbb{R}$ defined as $J_K(u) = T(u) - (J_f + J_g + J_h)(u)$, where T, J_f and J_g are given by (2.2), (2.3) and (2.4), respectively. Thus, a critical point of the functional J_K is a function $u \in X$ such that

$$J_K'(u)(v) = T'(u)(v) - (J_f' + J_g' + J_h')(u)(v) = 0$$

for every $v \in X$. Hence, the critical points of the functional J_K are weak solutions of problem (1.1).

Now, let us fix some notations that we will adopt in the sequel. For each $r > 0$ and each pair of functions $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ belonging $\mathscr{F}$ such that $G - F$ is bounded from below, set

$$\widetilde{\mu}(f,g,r) := \alpha p \inf_{u \in X} \left\{ \frac{r - \widetilde{\gamma} - J_f(u)}{m_0 \widetilde{\eta}_r - m_1 \|u\|_{a,p}^{\alpha p}} : J_g(u) < r, \|u\|_{a,p} < \sqrt[\alpha p]{\widetilde{\eta}_r} \right\},$$

where

$$\widetilde{\gamma} := \mathscr{D}\big(\Omega, p(a+1) - c\big). \inf_{\xi \in \mathbb{R}} (G(\xi) - F(\xi)),$$

and

$$\widetilde{\eta}_r := \inf_{u \in J_g^{-1}(r)} \|u\|_{a,p}^{\alpha p}.$$

Finally, for each $\varepsilon \in \left(0, \frac{1}{\max\{0, \widetilde{\mu}(f,g,r)\}}\right)$, we denote $\widetilde{\beta}(\varepsilon, f, g, r)$ by

$$\sup_{u \in J_g^{-1}(r,\infty)} \frac{m_0 \|u\|_{a,p}^{\alpha p} - \alpha p \varepsilon J_f(u) - \inf_{u \in J_g^{-1}((-\infty,r))} (m_1 \|u\|_{a,p}^{\alpha p} - \alpha p \varepsilon J_f(u))}{\alpha p (r - J_g(u))}.$$

Now we are in a position to state and prove the main result of this paper.

THEOREM 2. *Let $\sigma \in (\alpha p, \alpha p^*)$ with $p^* := \frac{Np}{N-p}$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be two continuous functions belonging to $\mathscr{F}$,*

$$\lim_{|\xi| \to \infty} \frac{f(\xi)}{|\xi|^{\alpha p - 1}} = \infty, \quad \limsup_{|\xi| \to \infty} \frac{f(\xi)}{|\xi|^{\sigma - 1}} < \infty$$

and

$$\lim_{|\xi| \to \infty} \frac{g(\xi)}{|\xi|^{\sigma - 1}} = \infty.$$

Then for each $r > 0$, for each

$$\varepsilon \in \left(0, \frac{1}{\max\{0, \widetilde{\mu}(f,g,r)\}}\right)$$

and for each compact interval $[\bar{a}, \bar{b}] \subset]0, \widetilde{\beta}(\varepsilon, f, g, r)[$, there exists a number $\rho > 0$ with the following property: for every $\lambda \in [\bar{a}, \bar{b}]$ and every function h belong to $(\mathscr{F})$ there exists $\delta > 0$ such that for each $\nu \in [0, \delta]$, problem (1.1) has at least three weak solutions whose norms in X are less than ρ.

Proof. Take $\mathrm{E} = X := \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$. According to Section 2, $(X, \|.\|_{a,\alpha})$ is a reflexive Banach space. It is well known that J_f is Gâteaux differentiable and sequentially weakly upper semicontinuous whose Gâteaux derivative at the point $u \in X$ is the functional $J_f'(u)$ given by (2.5) and $J_f' : X \to X^*$ is a compact operator. Moreover, the functional T is C^1, and by Proposition 1, its derivative admits a continuous inverse on X^*. Furthermore, it is clear that T is sequentially weakly lower semicontinuous. Moreover, since M satisfy the condition $(\mathscr{M})$, from (2.2) we have

$$\frac{m_0}{\alpha p}\|u\|_{a,p}^{\alpha p} \leqslant T(u) \leqslant \frac{m_1}{\alpha p}\|u\|_{a,p}^{\alpha p} \tag{3.1}$$

for all $u \in X$. Now, let A be a bounded subset of X. That is, there exists a constant $c > 0$ such that $\|u\|_{a,p} \leqslant c$ for each $u \in A$. Then, by (3.1) we have

$$|T(u)| \leqslant \frac{m_1}{\alpha p}c^{\alpha p}.$$

Hence T is bounded on each bounded subset of X. Now, let us prove that

$$\limsup_{\|u\|\to\infty} \frac{J_f(u)}{\|u\|_{a,p}^{\alpha p}} = \infty. \tag{3.2}$$

Now, since $1 < \alpha < \min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\}$, the embedding

$$X \hookrightarrow \mathrm{L}^{\alpha p}(\Omega, |x|^{-p(a+1)+c})$$

is compact (see Lemma 1). Thus there exists $C_1 > 0$ such that

$$C_1\|u\|_{\mathrm{L}^{\alpha p}(\Omega,|x|^{-p(a+1)+c})} \leqslant \|u\|_{a,p} \quad \text{for all}\, u \in X,$$

or

$$C_1^{\alpha p}\int_\Omega |x|^{-p(a+1)+c}|u(x)|^{\alpha p}\mathrm{d}x \leqslant \left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p\mathrm{d}x\right)^\alpha \quad \text{for all}\, u \in X.$$

It follows that

$$\lambda_\alpha := \inf_{u\in X\setminus\{0\}} \frac{\left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p\mathrm{d}x\right)^\alpha}{\int_\Omega |x|^{-p(a+1)+c}|u(x)|^{\alpha p}\mathrm{d}x} > 0 \tag{3.3}$$

(see [36]). Furthermore, by [36, Corollary 2.2] the eigenfunction $e_\alpha \in X$ is non-negative in Ω and by (3.3) it follows that

$$\|e_\alpha\|_{a,p}^{\alpha p} = \lambda_\alpha\left(\int_\Omega |x|^{-p(a+1)+c}|e_\alpha(x)|^{\alpha p}\mathrm{d}x\right)^\alpha.$$

To get (3.2) it is enough to show that

$$\lim_{k\to\infty} \frac{J_f(ke_1)}{\|ke_1\|_{a,p}^{\alpha p}} = \infty. \tag{3.4}$$

To this end, fix two positive numbers L_1 and L_2 such that $L_1 < \frac{L_2}{\alpha p}$. Since

$$\lim_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|^{\alpha p-1}} = \infty,$$

there exists $\eta > 0$ such that

$$F(\xi) \geqslant \lambda_1 L_2 |\xi|^{\alpha p}$$

for all $\xi \in (\eta, \infty)$. For each $k \in \mathbb{N}$, set

$$A_k := \left\{ x \in \Omega : e_\alpha(x) \geqslant \frac{\eta}{k} \right\}.$$

Taking into account that, for every $k \in \mathbb{N}$, one has $A_k \subseteq A_{k+1}$, the numerical sequence

$$\left\{ \int_{A_k} e_1(x)^{\alpha p} \mathrm{d}x \right\}_{k \in \mathbb{N}}$$

is non-decreasing, i.e.,

$$\int_{A_k} e_1(x)^{\alpha p} \mathrm{d}x \leqslant \int_{A_{k+1}} e_1(x)^{\alpha p} \mathrm{d}x$$

for every $k \in \mathbb{N}$ and

$$\int_{A_k} e_1(x)^{\alpha p} \mathrm{d}x \to \int_\Omega e_1(x)^{\alpha p} \mathrm{d}x$$

as $k \to +\infty$. At this point, fix $\tilde{k} \in \mathbb{N}$ so that

$$\int_{A_{\tilde{k}}} e_1(x)^{\alpha p} \mathrm{d}x > \frac{\alpha p L_1}{L_2} \int_\Omega e_1(x)^{\alpha p} \mathrm{d}x.$$

Moreover, since $f \in \mathscr{F}$, one has

$$\sup_{\xi \in [0,\eta]} |F(\xi)| < +\infty.$$

Indeed, taking into account that $f \in \mathscr{F}$ there exists a constant $c > 0$ such that

$$|F(\xi)| \leqslant c(|\xi| + |\xi|^s)$$

for every $\xi \in \mathbb{R}$. Thus

$$\sup_{\xi \in [0,\eta]} |F(\xi)| \leqslant c(\eta + \eta^s).$$

Then, for each $k \in \mathbb{N}$ satisfying

$$k > \max \left\{ \tilde{k}, \left(\frac{\mathscr{D}(\Omega, p(a+1) - c) \sup_{\xi \in [0,\eta]} |F(\xi)|}{L_1 \|e_1\|_{a,p}^{\alpha p}} \right)^{\frac{1}{\alpha p}} \right\},$$

we have

$$\begin{aligned}
\frac{J_f(ke_1)}{\|ke_1\|_{a,p}^{\alpha p}} &= \frac{\int_{A_k} |x|^{-p(a+1)+c} F(ke_1(x)) \mathrm{d}x}{k^{\alpha p} \|e_1\|_{a,p}^{\alpha p}} + \frac{\int_{\Omega \setminus A_k} |x|^{-p(a+1)+c} F(ke_1(x)) \mathrm{d}x}{k^{\alpha p} \|e_1\|_{a,p}^{\alpha p}} \\
&\geqslant \frac{\lambda_1 L_2 \int_{A_k} |x|^{-p(a+1)+c} e_1(x)^{\alpha p} \mathrm{d}x}{\|e_1\|_{a,p}^{\alpha p}} + \frac{\int_{\Omega \setminus A_k} |x|^{-p(a+1)+c} F(ke_1(x)) \mathrm{d}x}{k^{\alpha p} \|e_1\|_{a,p}^{\alpha p}} \\
&> \frac{\alpha p \lambda_1 L_1 \int_{A_k} |x|^{-p(a+1)+c} e_1(x)^{\alpha p} \mathrm{d}x}{\|e_1\|_{a,p}^{\alpha p}} - \frac{\mathscr{D}(\Omega, p(a+1) - c) \sup_{\xi \in [0,\eta]} |F(\xi)|}{k^{\alpha p} \|e_1\|_{a,p}^{\alpha p}} \\
&> \alpha p L_1 - L_1,
\end{aligned}$$

which shows (3.4). Thus by (3.1) and (3.4) we have

$$\liminf_{\|u\|\to\infty} \frac{-J_f(u)}{T(u)} = -\infty.$$

Since

$$\limsup_{|\xi|\to\infty} \frac{F(\xi)}{|\xi|^\sigma} < \infty,$$

there exists $\kappa > 0$ such that

$$F(\xi) \leqslant \kappa(|\xi|^\sigma + 1), \quad \text{for every}\, \xi \in \mathbb{R}. \tag{3.5}$$

Moreover, since

$$\lim_{|\xi|\to\infty} \frac{G(\xi)}{|\xi|^\sigma} = \infty$$

for each $\iota > 0$, there exists a constant $c_\iota > 0$ such that

$$G(\eta) \geqslant \eta|\xi|^\sigma - c_\iota \quad \text{for every}\, \xi \in \mathbb{R}. \tag{3.6}$$

Of course, by (3.5) and (3.6), for each $\lambda > 0$, the function $G - \lambda F : \mathbb{R} \to \mathbb{R}$ is bounded from below in $\mathbb{R}$. Indeed, fixing $\lambda > 0$, let us consider $\eta > \lambda\kappa$. Hence, by (3.5) and (3.6) it follows that

$$\begin{aligned}(G-\lambda F)(\xi) &\geqslant \eta|\xi|^\sigma - c_\iota - \lambda\kappa(|\xi|^\sigma + 1)\\ &= (\eta - \lambda\kappa)|\xi|^\sigma - (c_\iota + \lambda\kappa)\\ &\geqslant -(c_\iota + \lambda\kappa)\end{aligned}$$

for every $\xi \in \mathbb{R}$. Thus, by the above relation, one has that

$$\int_\Omega |x|^{-p(a+1)+c}(G(u(x)) - \lambda F(u(x)))\mathrm{d}x \geqslant -(c_\iota + \lambda\kappa)\mathscr{D}\big(\Omega, p(a+1)-c\big).$$

Then, the functional $J_g - \lambda J_f$ is bounded from below in X. At this point, the conclusion comes by Theorem 1, taking

$$I(u) := T(u), \quad \Psi(u) := -J_f(u)$$

as well as

$$\Phi(u) = J_g(u) \quad \Gamma(u) = J_h(u),$$

for every $u \in \mathrm{E} := X$. □

Now we give one example to illustrate our result.

EXAMPLE 1. Let $N = 3$, $p = \frac{5}{2}$, $a = \frac{1}{6}$, $c = \frac{59}{12}$, $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 \leqslant 4\} \subset \mathbb{R}^3$, $M(t) = t$ for all $t \in \mathbb{R}$ and consider the problem

$$\begin{cases} -\mathrm{div}\left(\frac{\sqrt{|\nabla u|}\nabla u}{\sqrt[12]{|x|^5}}\right) \int_\Omega \frac{\sqrt{|\nabla u(x)|^5}}{\sqrt[12]{|x|^5}}\mathrm{d}x = |x|^3(\varepsilon f(u) + \lambda g(u) + \nu h(u)) & \text{in}\, \Omega,\\ u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{3.7}$$

with

$$f(t)=\begin{cases}1+\sqrt{t^9} & \text{for} t\geqslant 0,\\ 1-\sqrt{|t|^9} & \text{for} t<0,\end{cases}\quad \text{and}\quad g(t)=1+t^8.$$

We observe that $\min\left\{\frac{N}{N-p},\frac{N-p(a+1)+c}{N-p(a+1)}\right\}=\min\{6,60\}=6$, thus M satisfies the condition $(\mathscr{M})$ with $m_0=m_1=1$ and $\alpha=\frac{12}{5}$. Moreover, by setting $r=\frac{128\pi}{5}$, $s=\frac{11}{2}$ and $\sigma=7$ we have $1<s<6=\alpha p$, $\alpha p=5<\sigma=7<\frac{15}{2}=\alpha p^*$,

$$\sup_{t\in\mathbb{R}}\frac{|f(t)|}{1+|t|^{s-1}}=\sup_{t\in\mathbb{R}}\frac{1+t^{\frac{9}{2}}}{1+t^{\frac{9}{2}}}=1<\infty,$$

$$\lim_{|\xi|\to\infty}\frac{f(\xi)}{|\xi|^{\alpha p-1}}=\begin{cases}\lim_{\xi\to\infty}\frac{1+\xi^{\frac{9}{2}}}{\xi^4}=\infty\\ \lim_{\xi\to-\infty}\frac{1-|\xi|^{\frac{9}{2}}}{\xi^4}=\infty\end{cases},$$

$$\limsup_{|\xi|\to\infty}\frac{f(\xi)}{|\xi|^{\sigma-1}}=\begin{cases}\lim_{\xi\to+\infty}\frac{1+\xi^{\frac{9}{2}}}{\xi^6}=0<\infty\\ \lim_{\xi\to-\infty}\frac{1-|\xi|^{\frac{9}{2}}}{\xi^6}=0<\infty\end{cases},$$

$$\lim_{|\xi|\to\infty}\frac{g(\xi)}{|\xi|^{\sigma-1}}=\frac{1+t^8}{|\xi|^6}=\infty.$$

Then, the conclusion of Theorem 4.1 holds for the problem (3.7).

Now we consider problem (1.3). The following result comes directly from Theorem 2.

THEOREM 3. *Let $f,g:\Omega\times\mathbb{R}\to\mathbb{R}$ be two continuous functions such that*

$$\sup_{t\in\mathbb{R}}\frac{|f(t)|}{1+|t|^{s-1}}<\infty\quad\text{and}\quad\sup_{t\in\mathbb{R}}\frac{|g(t)|}{1+|t|^{s-1}}<\infty$$

for some $1<s<p^*$ with $p^*=\frac{Np}{N-p}$. Moreover, let

$$\lim_{|\xi|\to\infty}\frac{f(\xi)}{|\xi|^{p-1}}=\infty,\quad \limsup_{|\xi|\to\infty}\frac{f(\xi)}{|\xi|^{p^*-1}}<\infty\quad\text{and}\quad\lim_{|\xi|\to\infty}\frac{g(\xi)}{|\xi|^{p^*-1}}=\infty.$$

Then for each $r>0$, for each

$$\varepsilon\in\left(0,\frac{1}{\max\{0,\widetilde{\mu}_1(f,g,r)\}}\right)$$

where $\mu_1(f,g,r)$ is the same as $\mu(f,g,r)$ but with $a=0$ and $p=c$, and for each compact interval $[\bar{a},\bar{b}]\subset]0,\widetilde{\beta}_1(\varepsilon,f,g,r)[$ where $\widetilde{\beta}_1(\varepsilon,f,g,r)$ is the same as $\widetilde{\beta}(\varepsilon,f,g,r)$

but $a=0$ *and* $p=c$, *there exists a number* $\rho>0$ *with the following property: for every* $\lambda\in[\bar{a},\bar{b}]$ *and every function* h *satisfying*

$$\sup_{t\in\mathbb{R}}\frac{h(t)}{1+|t|^s}<\infty$$

for some $1<s<p$, *there exists* $\delta>0$ *such that for each* $\nu\in[0,\delta]$, *problem* (1.3) *has at least three distinct weak solutions* $\{u_i\}_{i=1}^3\subset W_0^{1,p}(\Omega)$, *such that* $u_i=0$ *a.e. in* $\mathbb{R}^N\backslash\Omega$, *and*

$$\int_\Omega|\nabla u_i|^p\mathrm{d}x<\rho^p$$

for every $i\in 1,2,3$.

Finally, we give the following corollary as a consequence of Theorem 3.

COROLLARY 1. *Let* $1<\vartheta<p<\ell<p^*$ *with* $p^*=\frac{Np}{N-p}$. *Then for each* $\varepsilon>0$ *small enough, there exists* λ_ε *such that, for every compact interval* $[\bar{a},\bar{b}]\subset(0,\lambda_\varepsilon)$ *there exists* $\rho>0$ *with the following property: for every* $\lambda\in[\bar{a},\bar{b}]$ *and every continuous function* $h:\mathbb{R}\to\mathbb{R}$ *satisfying*

$$\limsup_{|\xi|\to\infty}\frac{|h(\xi)|}{|\xi|^\upsilon}<\infty,$$

for some $1<\upsilon<p^*$, *there exists* $\delta>0$ *such that for every* $\nu\in[0,\delta]$, *problem*

$$\begin{cases}-M\left(\int_\Omega|\nabla u(x)|^p\mathrm{d}x\right)\Delta_p u=\varepsilon|u|^{\vartheta-1}u+\lambda|u|^{\ell-1}u+\nu h(u), & \text{in}\,\Omega,\\ u=0, & \text{on}\,\partial\Omega,\end{cases}$$

has at least three distinct weak solutions $\{u_i\}_{i=1}^3\subset W_0^{1,p}(\Omega)$ *such that* $u_i=0$ *for a.e. in* $\mathbb{R}^N\backslash\Omega$, *and*

$$\int_\Omega|\nabla u_i|^p\mathrm{d}x<\rho^p$$

for every $i\in 1,2,3$.

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Shapour Heidarkhani
Department of Mathematics
Faculty of Sciences, Razi University
67149 Kermanshah, Iran
e-mail: sh.heidarkhani@razi.ac.ir

Kit Ian Kou
Department of Mathematics
Faculty of Science and Technology, University of Macau
Macau
e-mail: kikou@umac.mo

Amjad Salari
Department of Mathematics
College of Basic Sciences, Fasa University
Fasa, Iran
e-mail: amjads45@yahoo.com