EXTREMAL SOLUTIONS AT INFINITY FOR SYMPLECTIC SYSTEMS ON TIME SCALES I — GENERA OF CONJOINED BASES

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Abstract. In this paper we present a theory of genera of conjoined bases for symplectic dynamic systems on time scales and its connections with principal solutions at infinity and antiprincipal solutions at infinity for these systems. Among other properties we prove the existence of these extremal solutions in every genus. Our results generalize and complete the results by several authors on this subject, in particular by Došlý (2000), Šepitka and Šimon Hilscher (2016), and the author and Šimon Hilscher (2020). Some of our result are new even within the theory of genera of conjoined bases for linear Hamiltonian differential systems and symplectic difference systems, or they complete the arguments presented therein. Throughout the paper we do not assume any normality (controllability) condition on the system. This approach requires using the Moore–Penrose pseudoinverse matrices in the situations, where the inverse matrices occurred in the traditional literature. In this context we also prove a new explicit formula for the delta derivative of the Moore–Penrose pseudoinverse. This paper is a first part of the results connected with the theory of genera. The second part would naturally continue by providing a characterization of all principal solutions of ($\mathbb{S}$) at infinity in the given genus in terms of the initial conditions and a fixed principal solution at infinity from this genus and focusing on limit properties of above mentioned special solutions and by establishing their limit comparison at infinity.

1. Introduction

In this paper we contribute to the qualitative theory of the symplectic dynamic system

$$x^\Delta = \mathscr{A}(t)x + \mathscr{B}(t)u, \quad u^\Delta = \mathscr{C}(t)x + \mathscr{D}(t)u, \quad t \in [a,\infty)_\mathbb{T}, \tag{$\mathbb{S}$}$$

on time scales by providing the analysis of genera of conjoined bases and its relationship with the principal and antiprincipal solutions of ($\mathbb{S}$) at infinity. More precisely, we consider a time scale $\mathbb{T}$, that is, $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ with the standard topology inherited from $\mathbb{R}$. We assume that $\mathbb{T}$ is unbounded from above and bounded from below with $a := \min\mathbb{T}$ and set $[a,\infty)_\mathbb{T} := [a,\infty) \cap \mathbb{T}$ as the time scale interval. The coefficients $\mathscr{A}(t)$, $\mathscr{B}(t)$, $\mathscr{C}(t)$, $\mathscr{D}(t)$ of system ($\mathbb{S}$) are real piecewise rd-continuous $n \times n$ matrices on $[a,\infty)_\mathbb{T}$ such that the $2n \times 2n$ matrices

$$\mathscr{S}(t) := \begin{pmatrix} \mathscr{A}(t) & \mathscr{B}(t) \\ \mathscr{C}(t) & \mathscr{D}(t) \end{pmatrix}, \quad \mathscr{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{1.1}$$

satisfy the identity

$$\mathscr{S}^T(t)\,\mathscr{J} + \mathscr{J}\,\mathscr{S}(t) + \mu(t)\,\mathscr{S}^T(t)\,\mathscr{J}\,\mathscr{S}(t) = 0, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{1.2}$$

Here $\mu(t)$ is the graininess function of $\mathbb{T}$. We consider the solutions of system ($\mathbb{S}$) as piecewise rd-continuously Δ-differentiable functions, i.e., they are continuous functions on $[a,\infty)_{\mathbb{T}}$ and their Δ-derivative is piecewise rd-continuous on $[a,\infty)_{\mathbb{T}}$. The current literature on symplectic dynamic systems on time scales is rather rich. In [1] the authors consider the so-called symplectic flow and alpha-derivatives on generalized time scales. The paper [2] studies non-self-adjoint Hamiltonian systems on Sturmian time scales. In [5, 11, 12, 6] the authors deal with the oscillation of symplectic dynamic systems and with discrete theory of dynamic symplectic systems in general. Variational problems for symplectic systems on time scales are considered in [18, 19, 20, 31, 32]. In [35] the author derives the Rofe-Beketov formula for symplectic systems, which generalizes the well-known d'Alembert formula. Certain aspects of the spectral theory (eigenvalue theory and the Weyl–Titchmarsh theoty) of symplectic systems on time scales, including the oscillation theorems, are derived in [21, 30, 33, 34].

The theory of extremal solutions of system ($\mathbb{S}$), called principal (and nonprincipal) solutions at infinity, was initiated by Došlý in [10] under a certain complete controllability assumption. This theory was further developed in the context of possibly uncontrollable or abnormal system ($\mathbb{S}$) by Šepitka and Šimon Hilscher in [27] by providing the theory of principal solutions of ($\mathbb{S}$) at infinity and by the author and Šimon Hilscher in [14] by setting the grounds for the antiprincipal solutions of ($\mathbb{S}$) at infinity. In this paper we present further development of this theory, namely we consider the concept of a genus of conjoined bases, which is an equivalence class of all conjoined bases (X,U) of ($\mathbb{S}$) with eventually the same image of the first component X. This notion is known in the continuous time setting (i.e., for linear Hamiltonian differential systems) in [23, 24, 26] and in the discrete time setting (i.e., for symplectic difference systems) in [29] and it is used for studying limit behaviour of solutions at infinity. Here we provide the theory of genera of conjoined bases of system ($\mathbb{S}$) on arbitrary time scales as a unification and extension of the results in [23, 24, 26, 25, 29, 27], leading in some situations to new results even for these special cases, or to corrections of the therein presented arguments. More precisely, we emphasize the following main results of this paper in the general time scales environment:

(i) we derive an explicit formula for the Δ-derivative of the Moore–Penrose pseudoinverse (Theorem 2.2), which completes its earlier properties derived in [18, Lemma 2.1],

(ii) we derive the the rank of a special antiprincipal solution of ($\mathbb{S}$) at infinity (Theorem 4.1), which generalizes [14, Theorem 6.5] to arbitrary (possibly nonmiminal) conjoined bases and which is new even in the continuous and discrete case,

(iii) we define the notion of a genus of conjoined bases of system ($\mathbb{S}$), based on the study of the eventual equality of the image of their first component (Definition 5.2),

(iv) we establish the existence of antiprincipal solutions of ($\mathbb{S}$) at infinity and principal solutions of ($\mathbb{S}$) at infinity in the given genus (Theorems 6.1 and 6.2),

(v) we provide a complete analysis of the relation being contained between two conjoined bases of ($\mathbb{S}$) regarding shifting the left endpoint of the considered interval (the

results in Section 7 and in particular in Propositions 7.1 and 7.2), thus providing at the same time (through Proposition 3.12) a correction of the corresponding arguments in the proof of [29, Theorem 5.6],

The above list serves mainly as a tool for the continuation of this work covered in Part II of this paper. However, this preparatory part has to be done inevitably, to make following results understandable. In the continuation of this paper we will

(a) prove a characterization of all principal solutions of ($\mathbb{S}$) at infinity in the given genus in terms of the initial conditions and a fixed principal solution at infinity from this genus, which generalizes and unifies [23, Theorem 7.13] and [29, Theorem 5.6] to arbitrary time scales,

(b) prove a characterization of all antiprincipal solutions of ($\mathbb{S}$) at infinity in the given genus in terms of the initial conditions and a fixed principal solution at infinity from this genus, which generalizes and unifies [24, Theorem 5.13] and [29, Theorem 5.8] to arbitrary time scales,

(b) establish mutual limit properties of principal and antiprincipal solutions of ($\mathbb{S}$) at infinity, which generalize and unify the results in [24, Theorem 6.1] and [29, Theorem 6.1], respectively in [24, Theorem 6.3] and [29, Theorem 6.4], to arbitrary time scales.

This means that we collect (in Sections 2 and 3) all the main needed statements, to which we refer to in our subsequent work. We believe that this makes the results as well as the methods accessible for proper reading and further possible development, namely the continuation of this paper, which will be called Part II of this paper. The remaining parts of the paper (Sections 4–7) then contain the new results. Finally, we also include some open problems, which are related to the presented theory (Section 8).

2. Matrix analysis and time scales calculus

In this section we introduce some basic matrix notation and recall the properties of the Moore–Penrose pseudoinverse. We suppose that the reader is familiar with basic concepts of dynamic equations on time scales, such as in the introductory sections of the monographs [7, 8]. In particular, $f^{\Delta}(t)$ denotes the Δ-derivative of the function $f : \mathbb{T} \to \mathbb{R}$, $\sigma(t)$ is the forward jump operator on $\mathbb{T}$, $\rho(t)$ is the backward jump operator on $\mathbb{T}$, and $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$ is the graininess of $\mathbb{T}$. The function $(f \circ \sigma)(t) := f(\sigma(t))$ for $t \in \mathbb{T}$ is denoted by $f^{\sigma}(t)$. The Δ-derivative and the Δ-integral of a function f are denoted by $f^{\Delta}(t)$ and $\int_a^b f(t) \Delta t$, respectively, where $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$ is the time scale interval with endpoints $a, b \in \mathbb{T}$ such that $a < b$. A point $t < \max \mathbb{T}$ (provided the maximum exists) is called right-dense if $\mu(t) = 0$, while it is called right-scattered if $\mu(t) > 0$. We will use the concept of piecewise rd-continuously Δ-differentiable functions on $\mathbb{T}$ as defined in [17]. For a function $f : \mathbb{T} \to \mathbb{R}$ we will often use the well-known formula

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t) \quad \text{for all } t \in \mathbb{T}, \text{ for which } f^{\Delta}(t) \text{ exists.} \quad (2.1)$$

Recall that the matrix $M \in \mathbb{R}^{2n \times 2n}$ is *symplectic* if $M^T \mathscr{J} M = \mathscr{J}$, where the matrix $\mathscr{J}$ is given in (1.1). Symplectic matrices form a group with respect to the

matrix multiplication, and the inverse of a symplectic matrix M is given by the formula $M^{-1} = -\mathscr{J} M^T \mathscr{J}$. For a matrix $M \in \mathbb{R}^{m \times n}$ we will use the orthogonal decomposition

$$\mathbb{R}^n = (\operatorname{Im} M) \oplus (\operatorname{Ker} M^T), \quad \text{i.e.,} \quad (\operatorname{Im} M)^\perp = \operatorname{Ker} M^T. \tag{2.2}$$

For a linear subspace $V \subseteq \mathbb{R}^n$ we denote by $\mathscr{P}_V$ the orthogonal projector onto V.

For the results of this paper it is essential to use the following properties of the Moore–Penrose pseudoinverse. It is defined by the following four properties, which we often use in the proofs, see e.g. [3]. Let M be a real $m \times n$ matrix. A real $n \times m$ matrix $M^\dagger$ satisfying

$$MM^\dagger M = M, \quad M^\dagger M M^\dagger = M^\dagger, \quad M^\dagger M = (M^\dagger M)^T, \quad MM^\dagger = (MM^\dagger)^T, \tag{2.3}$$

is called the *Moore–Penrose pseudoinverse* of the matrix M. We will use the following properties of the Moore–Penrose pseudoinverse, which can be found e.g. in [3, 4, 9, 15] and [18, Lemma 2.1]. These properties play an essential role in our theory.

REMARK 2.1. For any real matrix $M \in \mathbb{R}^{m \times n}$ there exists a unique matrix $M^\dagger \in \mathbb{R}^{n \times m}$ satisfying the identities in (2.3). Moreover, the following properties hold.

(i) $(M^\dagger)^T = (M^T)^\dagger$, $(M^\dagger)^\dagger = M$, and $\operatorname{Im} M^\dagger = \operatorname{Im} M^T$, $\operatorname{Ker} M^\dagger = \operatorname{Ker} M^T$.

(ii) If $M \in \mathbb{R}^{m \times n}$ is full rank matrix, then

$$M^\dagger = M^T (MM^T)^{-1} \quad \text{when} \quad \operatorname{rank} M = m, \tag{2.4}$$

$$M^\dagger = (M^T M)^{-1} M^T \quad \text{when} \quad \operatorname{rank} M = n. \tag{2.5}$$

These formulas can be verified by checking the four properties in (2.3).

(iii) The matrix $MM^\dagger$ is the orthogonal projector onto $\operatorname{Im} M$, and the matrix $M^\dagger M$ is the orthogonal projector onto $\operatorname{Im} M^T$. Moreover, $\operatorname{rank} M = \operatorname{rank}(MM^\dagger) = \operatorname{rank}(M^\dagger M)$.

(iv) Let $M(t)$ be an $m \times n$ matrix function defined on the interval $[a, \infty)_\mathbb{T}$ such that $\lim_{t \to \infty} M(t) = M$. Then the limit of $M^\dagger(t)$ for $t \to \infty$ exists if and only if there exists a point $t_0 \in [a, \infty)_\mathbb{T}$ such that $\operatorname{rank} M(t) = \operatorname{rank} M$ for all $t \in [t_0, \infty)_\mathbb{T}$. In this case we have $\lim_{t \to \infty} M^\dagger(t) = M^\dagger$.

(v) Let M_1 and M_2 be symmetric and positive semidefinite matrices such that $M_1 \leqslant M_2$. Then the inequality $M_2^\dagger \leqslant M_1^\dagger$ holds if and only if $\operatorname{Im} M_1 = \operatorname{Im} M_2$, or equivalently if and only if $\operatorname{rank} M_1 = \operatorname{rank} M_2$.

(vi) If M is symmetric and positive semidefinite, then also $M^\dagger$ is symmetric and positive semidefinite. That is, $M \geqslant 0$ if and only if $M^\dagger \geqslant 0$.

(vii) For any matrices M and N with suitable dimensions we have

$$(MN)^\dagger = (\mathscr{P}_{\operatorname{Im} M^T} N)^\dagger (M \mathscr{P}_{\operatorname{Im} N})^\dagger = (M^\dagger MN)^\dagger (MNN^\dagger)^\dagger. \tag{2.6}$$

In particular, if one of the matrices M or N is orthogonal, then

$$(MN)^\dagger = N^\dagger M^\dagger \tag{2.7}$$

This formula can also be verified by checking the four properties in (2.3).

(viii) If M is a given matrix and 0 is a matrix with suitable dimension, then

$$\left(0 \; M \right)^{\dagger} = \begin{pmatrix} 0 \\ M^{\dagger} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ M \end{pmatrix}^{\dagger} = \left(0 \; M^{\dagger} \right). \tag{2.8}$$

The following lemma deals with the delta derivative of the Moore-Penrose pseudoinverse. Note that the first part about the existence of Δ-derivative is known in [18, Lemma 2.1], based on the result in [22, Lemma 6]. Here we derive explicit formulas (2.11) and (2.12), which are new on time scales.

THEOREM 2.2. *Let $M(t)$ be an $m \times n$ piecewise rd-continuously Δ-differentiable matrix function defined on the interval $[\alpha, \infty)_{\mathbb{T}}$ such that $\operatorname{Ker} M(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$. Then the matrix function $M^{\dagger}(t)$ is also piecewise rd-continuously Δ-differentiable on $[\alpha, \infty)_{\mathbb{T}}$ and*

$$[M^{\dagger}(t)]^{\Delta} M(t) = -[M^{\dagger}(t)]^{\sigma} M^{\Delta}(t) = -[M^{\sigma}(t)]^{\dagger} M^{\Delta}(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{2.9}$$

$$[M^{\dagger}(t)]^{\Delta} M^{\sigma}(t) = -M^{\dagger}(t) M^{\Delta}(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{2.10}$$

Moreover, for $[M^{\dagger}(t)]^{\Delta}$ on $[\alpha, \infty)_{\mathbb{T}}$ we have the formula (suppressing the argument t)

$$(M^{\dagger})^{\Delta} = -M^{\dagger} M^{\Delta} (M^{\dagger})^{\sigma} + M^{\dagger} M^{\dagger T} (M^{\Delta})^{T} [I - M^{\sigma} (M^{\dagger})^{\sigma}], \tag{2.11}$$

or equivalently the formula

$$(M^{\dagger})^{\Delta} = -(M^{\dagger})^{\sigma} M^{\Delta} M^{\dagger} + (M^{\dagger})^{\sigma} (M^{\dagger T})^{\sigma} (M^{\Delta})^{T} (I - M M^{\dagger}). \tag{2.12}$$

Proof. The proof of the statement that $M^{\dagger}$ is piecewise rd-continuously Δ-differentiable on $[\alpha, \infty)_{\mathbb{T}}$ follows the idea in [22, Lemma 6], but here we add more details. Let D be a constant matrix such that $\operatorname{Ker} M(t) = \operatorname{Im} D$ for $t \in [\alpha, \infty)_{\mathbb{T}}$, which we denote by $D = (d_1, \dots, d_k) \in \mathbb{R}^{n \times k}$ with $d_i \in \mathbb{R}^n$, and we define

$$k := \operatorname{def} M(t) = n - \operatorname{rank} M(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{2.13}$$

By the Gram-Schmidt theorem, it is possible to select the columns of D as the orthonormal basis of $\operatorname{Im} D$, and then to complete this basis by the vectors $d_{k+1}, \dots, d_n$ to an orthonormal basis of $\mathbb{R}^n$. Then the matrix $\tilde{D} := (d_1, \dots, d_k, d_{k+1}, \dots, d_n) \in \mathbb{R}^n$ is orthogonal and the matrix

$$\tilde{M}(t) := M(t) \tilde{D} = (0, N(t)), \quad t \in [\alpha, \infty)_{\mathbb{T}}, \quad \text{where } 0 \in \mathbb{R}^{m \times k}, \tag{2.14}$$

is such that $N(t) \in \mathbb{R}^{m \times r}$ with $r := n - k$ and $N(t) := M(t) \tilde{D} (0, I)^{T}$ is piecewise rd-continuously Δ-differentiable on $[\alpha, \infty)_{\mathbb{T}}$. Moreover, the matrix $N(t)$ has a full (column) rank r by (2.13). Then the matrix $F(t) := N^{T}(t) N(t) \in \mathbb{R}^{r \times r}$ is invertible for

all $t \in [\alpha, \infty)_{\mathbb{T}}$. Thus functions $F(t)$ and $F^{-1}(t)$ are both piecewise rd-continuously Δ-differentiable on the interval $[\alpha, \infty)_{\mathbb{T}}$. Moreover by (2.7), (2.8), and (2.5) we get

$$M^{\dagger}(t) \overset{(2.14)}{=} (\tilde{M}(t)\tilde{D}^T)^{\dagger} = \left((0, N(t))\tilde{D}^T\right)^{\dagger} \overset{(2.7)}{=} \tilde{D}(0, N(t))^{\dagger} \overset{(2.8)}{=} \tilde{D} \begin{pmatrix} 0 \\ N^{\dagger}(t) \end{pmatrix}$$

$$\overset{(2.5)}{=} \tilde{D} \begin{pmatrix} 0 \\ F^{-1}(t)N^T(t) \end{pmatrix}, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

It implies $M^{\dagger}(t)$ is piecewise rd-continuously Δ-differentiable on $[\alpha, \infty)_{\mathbb{T}}$. The expressions in (2.9) and (2.10) now follow directly from the rules for the Δ-derivative of the product and the fact that the matrix $M^{\dagger}(t)M(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$, since it is an orthogonal projector onto the constant subspace $\operatorname{Im} M^T(t) = [\operatorname{Ker} M(t)]^{\perp}$ on $[\alpha, \infty)_{\mathbb{T}}$ by Remark 2.1(iii).

The proof of (2.11) and (2.12) follows from the basic properties of Δ-derivative together with the properties of the Moore-Penrose pseudoinverse. Fix now a point $t \in [\alpha, \infty)_{\mathbb{T}}$, and note that the argument t will be suppressed in the following computation. From the defining property (2.3) and using the product rule $(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta}$ we get

$$M^{\Delta} = (MM^{\dagger}M)^{\Delta} = (MM^{\dagger})^{\Delta}M + (MM^{\dagger})^{\sigma}M^{\Delta}, \tag{2.15}$$

$$M^{\Delta} = (MM^{\dagger}M)^{\Delta} = (MM^{\dagger})^{\Delta}M^{\sigma} + MM^{\dagger}M^{\Delta}. \tag{2.16}$$

It allows us to find that

$$(MM^{\dagger})^{\Delta}M \overset{(2.15)}{=} [I - (MM^{\dagger})^{\sigma}]M^{\Delta}, \tag{2.17}$$

$$(MM^{\dagger})^{\Delta}M^{\sigma} \overset{(2.16)}{=} (I - MM^{\dagger})M^{\Delta}. \tag{2.18}$$

Notice also that

$$[(MM^{\dagger})^{\Delta}MM^{\dagger}]^T = MM^{\dagger}(MM^{\dagger})^{\Delta}, \tag{2.19}$$

$$[(MM^{\dagger})^{\Delta}(MM^{\dagger})^{\sigma}]^T = (MM^{\dagger})^{\sigma}(MM^{\dagger})^{\Delta}. \tag{2.20}$$

Now again using the defining property for the Moore–Penrose pseudoinverse we get

$$(MM^{\dagger})^{\Delta} \overset{(2.3)}{=} (MM^{\dagger}MM^{\dagger})^{\Delta} = (MM^{\dagger})^{\Delta}(MM^{\dagger})^{\sigma} + MM^{\dagger}(MM^{\dagger})^{\Delta}$$

$$\overset{(2.18)}{=} (I - MM^{\dagger})M^{\Delta}(M^{\dagger})^{\sigma} + MM^{\dagger}(MM^{\dagger})^{\Delta}$$

$$\overset{(2.19)}{=} (I - MM^{\dagger})M^{\Delta}(M^{\dagger})^{\sigma} + [(MM^{\dagger})^{\Delta}(MM^{\dagger})^{\sigma}]^T$$

$$\overset{(2.17)}{=} (I - MM^{\dagger})M^{\Delta}(M^{\dagger})^{\sigma} + \left\{[I - (MM^{\dagger})^{\sigma}]M^{\Delta}M^{\dagger}\right\}^T. \tag{2.21}$$

It shows that

$$M^{\dagger}(MM^{\dagger})^{\Delta} \overset{(2.21)}{=} \underbrace{M^{\dagger}(I - MM^{\dagger})}_{=0} M^{\Delta}(M^{\dagger})^{\sigma} + M^{\dagger}\left\{[I - (MM^{\dagger})^{\sigma}]M^{\Delta}M^{\dagger}\right\}^T$$

$$= M^{\dagger}(M^{\dagger})^T(M^{\Delta})^T[I - M^{\sigma}(M^{\dagger})^{\sigma}]. \tag{2.22}$$

By using the last equality we then obtain that

$$(M^\dagger)^\Delta \overset{(2.3)}{=} (M^\dagger M M^\dagger)^\Delta = (M^\dagger)^\Delta M^\sigma (M^\dagger)^\sigma + M^\dagger (M M^\dagger)^\Delta$$
$$\overset{(2.10)}{=} -M^\dagger M^\Delta (M^\dagger)^\sigma + M^\dagger (M M^\dagger)^\Delta$$
$$\overset{(2.22)}{=} -M^\dagger M^\Delta (M^\dagger)^\sigma + M^\dagger (M^\dagger)^T (M^\Delta)^T [I - M^\sigma (M^\dagger)^\sigma],$$

which proves formula (2.11). Similarly, it is possible to show by using the dual product rule $(fg)^\Delta = f^\Delta g^\sigma + f g^\Delta$ and by (2.20) that formula (2.12) also holds. $\square$

REMARK 2.3. Equalities (2.11) and (2.12) are extensions of the known expression for the derivative of $M^\dagger(t)$ on the real interval $[\alpha, \infty)$ from the continuous case, i.e., if $\mathbb{T} = \mathbb{R}$, see [23, Remark 2.3]. When we investigate what follows from (2.11) and (2.12), while hoping we get something new, we find out that these formulas reduce to the trivial identity $\Delta M_k^\dagger = M_{k+1}^\dagger - M_k^\dagger$.

Further, we use the Moore–Penrose pseudoinverse for the construction of the orthogonal projectors. We consider $X(t)$ to be a matrix function $X : [a, \infty)_\mathbb{T} \to \mathbb{R}^{n \times n}$. Then we define the orthogonal projectors onto the image of $X^T(t)$ or onto the image of $X(t)$ on $[\alpha, \infty)_\mathbb{T}$ as follows. For $t \in [a, \infty)_\mathbb{T}$ we put

$$P(t) := \mathscr{P}_{\mathrm{Im}\,X^T(t)} = X^\dagger(t) X(t), \quad R(t) := \mathscr{P}_{\mathrm{Im}\,X(t)} = X(t) X^\dagger(t), \tag{2.23}$$

i.e., matrices $P(t)$ and $R(t)$ are symmetric on $[a, \infty)_\mathbb{T}$ and

$$\mathrm{Im}\,X^T(t) = \mathrm{Im}\,P(t), \quad \mathrm{Im}\,X(t) = \mathrm{Im}\,R(t), \quad t \in [a, \infty)_\mathbb{T}. \tag{2.24}$$

Using the defining properties of the Moore–Penrose pseudoinverse in (2.3) its easy to find out that for $t \in [a, \infty)_\mathbb{T}$ we have

$$P(t) X^\dagger(t) = X^\dagger(t), \quad X^\dagger(t) R(t) = X^\dagger(t), \quad X(t) P(t) = X(t), \quad R(t) X(t) = X(t). \tag{2.25}$$

Recall that for the orthogonal projectors $P(t)$ and $R(t)$ on $[a, \infty)_\mathbb{T}$, as well as for all orthogonal projectors in general, it holds that they are idempotent, i.e.,

$$P(t) P(t) = P(t), \quad R(t) R(t) = R(t), \quad t \in [a, \infty)_\mathbb{T}. \tag{2.26}$$

If matrix functions $X(t)$ has constant kernel on $[\alpha, \infty)_\mathbb{T}$, then the orthogonal projector $P(t)$ defined in (2.23) is constant on $[\alpha, \infty)_\mathbb{T}$, since $[\mathrm{Ker}\,X(t)]^\perp = \mathrm{Im}\,X^T(t)$ is constant on $[a, \infty)_\mathbb{T}$. Then we denote by P the corresponding constant orthogonal projector in (2.23), i.e., we define

$$P := P(t) \quad \text{for } t \in [\alpha, \infty)_\mathbb{T}, \text{ where } \mathrm{Ker}\,X(t) \text{ is constant.} \tag{2.27}$$

3. Symplectic dynamic systems and their properties

In this section we present needed properties of symplectic systems on time scales and their conjoined bases. These results are known in the literature, we refer to [27, 14, 13, 16, 18], and they include the order of abnormality of system $(\mathbb{S})$, properties of the associated matrices $S(t)$ and T (which are used for the definitions of a principal solution at infinity and an antiprincipal solution at infinity), a mutual representation of conjoined bases with specific properties, and (with high importance) the relation being contained between two conjoined bases. We start with basic notation for solutions of system $(\mathbb{S})$. Vector solutions of $(\mathbb{S})$ will be denoted by the small letters, typically (x, u), and $2n \times n$ matrix solutions of system $(\mathbb{S})$ will be denoted by the capital letters, typically (X, U) or with tildes and hats over the involved matrices.

3.1. Conjoined bases and their properties

Notice that identity (1.2) implies that the $2n \times 2n$ matrix $I + \mu(t)\mathscr{S}(t)$ is symplectic, hence invertible, on $[\alpha, \infty)_{\mathbb{T}}$. This implies through [7, Theorem 5.8] that system $(\mathbb{S})$ is uniquely solvable on $[\alpha, \infty)_{\mathbb{T}}$ given any initial point $t_0 \in [\alpha, \infty)_{\mathbb{T}}$ and any initial values (vector or matrix) at t_0. A solution (X, U) of system $(\mathbb{S})$ is called a *conjoined basis*, if the matrix $X^T(t) U(t)$ is a symmetric matrix and $\text{rank}(X^T(t), U^T(t))^T = n$ at some and hence at any point $t \in [a, \infty)_{\mathbb{T}}$. According to [13, Definition 3], a conjoined basis (X, U) of $(\mathbb{S})$ is called *nonoscillatory*, if there exists point $\alpha \in [a, \infty)_{\mathbb{T}}$ such that (X, U) has no focal points in the real interval (α, ∞), i.e., if

$$\text{Ker} X(s) \subseteq \text{Ker} X(t) \quad \text{for all } t, s \in [\alpha, \infty)_{\mathbb{T}} \text{ with } t \leqslant s, \tag{3.1}$$

$$X(t) [X^\sigma(t)]^\dagger \mathscr{B}(t) \geqslant 0 \quad \text{for all } t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.2}$$

We will say that the conjoined basis (X, U) has constant kernel (or constant rank) on the interval $[\alpha, \infty)_{\mathbb{T}}$, if the kernel (or rank) of the matrix $X(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$. As a consequence of (3.1), such properties are always satisfied on intervals $[\beta, \infty)_{\mathbb{T}}$ for sufficiently large $\beta \in [a, \infty)_{\mathbb{T}}$, when the conjoined basis (X, U) is nonoscillatory.

For any two solutions (X, U) and $(\bar{X}, \bar{U})$ of system $(\mathbb{S})$ their *Wronskian matrix*

$$\mathscr{W}[(X, U), (\bar{X}, \bar{U})] := X^T(t) \bar{U}(t) - U^T(t) \bar{X}(t), \quad t \in [a, \infty)_{\mathbb{T}},$$

is constant on $[a, \infty)_{\mathbb{T}}$, as we can verify by the Δ-differentiation. Recall that two conjoined bases (X, U) and $(\bar{X}, \bar{U})$ of $(\mathbb{S})$ are called *normalized* if

$$\mathscr{W}[(X, U), (\bar{X}, \bar{U})] = I. \tag{3.3}$$

It is easy to verify that two conjoined basis (X, U) and $(\bar{X}, \bar{U})$ of $(\mathbb{S})$ on $[a, \infty)_{\mathbb{T}}$ are normalized if and only if the matrix

$$\mathscr{Z}(t) := \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix}, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{3.4}$$

is symplectic on $[a,\infty)_{\mathbb{T}}$. Then by the expression $\mathscr{Z}^{-1}(t) = -\mathscr{J} \mathscr{Z}^T(t) \mathscr{J}$ on $[a,\infty)_{\mathbb{T}}$ for the inverse of a symplectic matrix and the product $\mathscr{Z}(t)\mathscr{Z}^{-1}(t) = I$, it follows that

$$X(t)\bar{U}^T(t) - \bar{X}(t)U^T(t) = I, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.5}$$

This equality is very similar to (3.3) and we will use it later in the proofs. Using the formula for inverse of symplectic matrix it is possible to derive an equivalent expression of the original system ($\mathbb{S}$), the so-called time reversed (or adjoint) system, which has according to [18, Remark 3.1(iii)] the form

$$x^\Delta = -\mathscr{D}^T(t)x^\sigma + \mathscr{B}^T(t)u^\sigma, \quad u^\Delta = \mathscr{C}^T(t)x^\sigma - \mathscr{A}^T(t)u^\sigma, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.6}$$

Recall that by the principal solution of ($\mathbb{S}$) at the point $\alpha \in [a,\infty)_{\mathbb{T}}$, denoted by $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$, we mean the conjoined basis of the system ($\mathbb{S}$) satisfying the initial conditions

$$\hat{X}^{[\alpha]}(\alpha) = 0 \text{ and } \hat{U}^{[\alpha]}(\alpha) = I. \tag{3.7}$$

3.2. Order of abnormality

In this article we deal with a possibly abnormal symplectic system ($\mathbb{S}$), for which we use the order of abnormality. We recall its definition from [18, 27]. For any $\alpha \in [a,\infty)_{\mathbb{T}}$ we denote by $\Lambda[\alpha,\infty)_{\mathbb{T}}$ the linear space of n-vector functions $u : [\alpha,\infty)_{\mathbb{T}} \to \mathbb{R}^n$ such that $\mathscr{B}(t)u(t) = 0$ and $u^\Delta = \mathscr{D}(t)u(t)$ on $[\alpha,\infty)_{\mathbb{T}}$. Therefore, a function $u \in \Lambda[\alpha,\infty)_{\mathbb{T}}$ if and only if the pair $(x \equiv 0, u)$ is a solution of system ($\mathbb{S}$). The number $d[\alpha,\infty)_{\mathbb{T}} := \dim\Lambda[\alpha,\infty)_{\mathbb{T}}$ is called the *order of abnormality* of system ($\mathbb{S}$) on the interval $[\alpha,\infty)_{\mathbb{T}}$. The limit

$$d_\infty := \lim_{t\to\infty} d[t,\infty)_{\mathbb{T}} \quad \text{with } 0 \leqslant d[t,\infty)_{\mathbb{T}} \leqslant d_\infty \leqslant n \text{ for } t \in [a,\infty)_{\mathbb{T}},$$

is then called the *maximal order of abnormality* of system ($\mathbb{S}$). In a similar way we define the order of abnormality $d[\alpha,t]_{\mathbb{T}}$ of system ($\mathbb{S}$) on $[\alpha,t]_{\mathbb{T}}$ and then $d[\alpha,\infty)_{\mathbb{T}} = \lim_{t\to\infty} d[\alpha,t]_{\mathbb{T}}$ holds. In addition, we denote by $\Lambda_0[\alpha,\infty)_{\mathbb{T}}$ the subspace of $\mathbb{R}^n$ of the initial values $u(\alpha)$ of the elements $u \in \Lambda[\alpha,\infty)_{\mathbb{T}}$. Then $\dim\Lambda_0[\alpha,\infty)_{\mathbb{T}} = \dim\Lambda[\alpha,\infty)_{\mathbb{T}}$ holds.

3.3. Auxiliary matrices $S(t)$, T, and $P_{S\infty}$

Let (X,U) be a conjoined basis of system ($\mathbb{S}$) with constant kernel on the interval $[\alpha,\infty)_{\mathbb{T}}$. Then according to Theorem 2.2 the matrix $X^\dagger(t)$ is piecewise rd-continuously Δ-differentiable on $[\alpha,\infty)_{\mathbb{T}}$, and hence we may define the associated S-matrix by

$$S(t) := \int_\alpha^t [X^\sigma(s)]^\dagger \mathscr{B}(s) [X^\dagger(s)]^T \Delta s, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.8}$$

By [18, Lemma 3.1], the matrix

$$X(t)[X^\sigma(t)]^\dagger \mathscr{B}(t) \text{ is symmetric for all } t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.9}$$

when the kernel of $X(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$. This yields that

$$[X^{\sigma}(t)]^{\dagger} \mathscr{B}(t) [X^{\dagger}(t)]^{T} \text{ is also symmetric for all } t \in [\alpha, \infty)_{\mathbb{T}}, \tag{3.10}$$

and the corresponding S-matrix given by (3.8) is symmetric. Moreover, if the matrices P and $S(t)$ are defined by (2.27) and (3.8), then [14, Lemma 3.2] gives that

$$\operatorname{Im} S(t) \subseteq \operatorname{Im} P \quad \text{for all } t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.11}$$

The inclusion of the sets in (3.11) can be equivalently written as

$$PS(t) = S(t) = S(t)P, \quad PS^{\dagger}(t) = S^{\dagger}(t) = S^{\dagger}(t)P, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.12}$$

The following result is proven in [14, Theorem 3.4], see also [27, Lemma 3.1], and it plays a key role in definitions of antiprincipal and principal solutions of ($\mathbb{S}$) at infinity.

PROPOSITION 3.1. *Let (X,U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and let the matrix $S(t)$ given by (3.8). Then the limit of $S^{\dagger}(t)$ as $t \to \infty$ exists. Moreover, the matrix T defined by*

$$T := \lim_{t \to \infty} S^{\dagger}(t) \tag{3.13}$$

is symmetric and positive semidefinite, i.e., $T \geqslant 0$, and there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that

$$\operatorname{rank} T \leqslant \operatorname{rank} S(t) \leqslant \operatorname{rank} X(t) \quad \text{for all} \quad t \in [\beta, \infty)_{\mathbb{T}}.$$

The next proposition is proven in [27, Theorem 3.2] and it brings additional properties of the matrices $S(t)$ and T, which we will often use later.

PROPOSITION 3.2. *Let (X,U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and let the matrices P, $R(t)$, T be defined in (2.27), (2.23), and (3.13). Then*

$$R^{\sigma}(t) \mathscr{B}(t) = \mathscr{B}(t), \quad \mathscr{B}(t) R(t) = \mathscr{B}(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.14}$$

If in addition (X,U) has no focal points in (α, ∞), then

$$PT = T = TP, \quad PT^{\dagger} = T^{\dagger} = T^{\dagger}P. \tag{3.15}$$

According to [14, Remark 3.5], the S-matrix associated with a conjoined basis (X,U) of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) has non-decreasing image on $[\alpha, \infty)_{\mathbb{T}}$ and hence, in this case there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that $\operatorname{Im} S(t)$ is constant on $[\beta, \infty)_{\mathbb{T}}$. On such intervals $[\beta, \infty)_{\mathbb{T}}$ we define the associated constant orthogonal projector

$$P_{S\infty} := \mathscr{P}_{\operatorname{Im} S(t)} = S(t) S^{\dagger}(t) = S^{\dagger}(t) S(t), \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{3.16}$$

The last equality in (3.16) follows from the symmetry of the matrix $S(t)$. From (3.12) we get

$$\operatorname{Im} S(t) \subseteq \operatorname{Im} P_{S\infty} \subseteq \operatorname{Im} P, \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{3.17}$$

The inclusions in (3.17) can be written as

$$P_{S\infty} S(t) = S(t) = S(t) P_{S\infty}, \quad t \in [\beta, \infty)_{\mathbb{T}}, \quad P P_{S\infty} = P_{S\infty} = P_{S\infty} P. \tag{3.18}$$

Using the definition of the Moore–Penrose pseudoinverse in (2.3) and observing the limit $\lim_{t\to\infty} S^\dagger(t)$ we obtain the equalities

$$P_{S\infty} T = T = T P_{S\infty}, \quad \text{i.e.,} \quad \operatorname{Im} T \subseteq \operatorname{Im} P_{S\infty}. \tag{3.19}$$

In some places, e.g. in Proposition 3.9, we will use the time dependent orthogonal projector

$$P_S(t) := \mathscr{P}_{\operatorname{Im} S(t)}, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \quad \lim_{t\to\infty} P_S(t) = P_{S\infty},$$

where the matrix on the right-hand side is given in (3.19).

REMARK 3.3. Let (X, U) be a conjoined basis of $(\mathbb{S})$ with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Let the matrices P, $R(t)$, $S(t)$, $P_{S\infty}$ be defined by (2.27), (2.23), (3.8), (3.16). Then from [27, Proposition 3.9] it follows that

$$\operatorname{rank} P_{S\infty} = n - d[\alpha, \infty)_{\mathbb{T}}, \tag{3.20}$$

$$n - d[\alpha, \infty)_{\mathbb{T}} \leqslant \operatorname{rank} X(t) \leqslant n. \tag{3.21}$$

A conjoined basis (X, U) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) is called *minimal* on the interval $[\alpha, \infty)_{\mathbb{T}}$, if it has the smallest possible rank, i.e., if

$$\operatorname{rank} X(t) = n - d[\alpha, \infty)_{\mathbb{T}} = n - d_\infty, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

On the other hand, if $\operatorname{rank} X(t) = n$ on $[\alpha, \infty)_{\mathbb{T}}$, then (X, U) is called *maximal* on $[\alpha, \infty)_{\mathbb{T}}$. Obviously, the matrix $X(t)$ is invertible on $[\alpha, \infty)_{\mathbb{T}}$ in this case. The existence of conjoined bases of $(\mathbb{S})$ with the range given in (3.21) is discussed in Proposition 3.16 below.

3.4. Antiprincipal and principal solutions at infinity

Now we recall the definitions of an antiprincipal solution of $(\mathbb{S})$ at infinity from [14, Definition 4.1] and a principal solution of $(\mathbb{S})$ at infinity from [27, Definition 6.1], see also [10] for a special case when the matrices $X(t)$ and $S(t)$ are invertible.

DEFINITION 3.4. A conjoined basis (X, U) of $(\mathbb{S})$ is said to be an *antiprincipal solution at infinity* with respect to the interval $[\alpha, \infty)_{\mathbb{T}} \subseteq [a, \infty)_{\mathbb{T}}$ if

(i) the order of abnormality of $(\mathbb{S})$ on the interval $[\alpha, \infty)_{\mathbb{T}}$ is maximal, i.e., $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$,

(ii) the conjoined basis (X, U) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞),

(iii) the matrix T defined in (3.13) corresponding to (X, U) satisfies $\operatorname{rank} T = n - d_\infty$.

DEFINITION 3.5. A conjoined basis $(\hat{X}, \hat{U})$ of $(\mathbb{S})$ is said to be a *principal solution at infinity* with respect to the interval $[\alpha, \infty)_{\mathbb{T}} \subseteq [a, \infty)_{\mathbb{T}}$ if

(i) the conjoined basis $(\hat{X}, \hat{U})$ has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞),

(ii) the matrix $\hat{T}$ defined in (3.13) associated with $(\hat{X}, \hat{U})$ satisfies $\operatorname{rank} \hat{T} = 0$, i.e., $\hat{T} = 0$.

REMARK 3.6. The results in [27, Theorem 6.8] and in [14, Theorem 5.3] states that a nonoscillatory system $(\mathbb{S})$ possesses a principal solution at infinity and an antiprincipal solution at infinity for any admissible rank. More precisely, for any integer value r between the numbers $n - d_\infty$ and n there exists a principal and an antiprincipal solution $(\hat{X}, \hat{U})$ of $(\mathbb{S})$ at infinity with the rank of $\hat{X}(t)$ equal to r for large t.

In the context of Remark 3.6 and Definitions 3.4 and 3.5 we will use the following terminology. A conjoined basis (X, U) of $(\mathbb{S})$ is a minimal (or maximal) antiprincipal solution at infinity, if it is an antiprincipal solution at infinity with respect to some interval $[\alpha, \infty)_{\mathbb{T}}$ (according to Definition 3.4) and at the same time the rank of $X(t)$ is equal to $n - d_\infty$ (or to n) on $[\alpha, \infty)_{\mathbb{T}}$. Similarly, a conjoined basis $(\hat{X}, \hat{U})$ of $(\mathbb{S})$ is a minimal (or maximal) principal solution at infinity, if it is a principal solution at infinity with respect to some interval $[\alpha, \infty)_{\mathbb{T}}$ (according to Definition 3.5) and at the same time the rank of $\hat{X}(t)$ is equal to $n - d_\infty$ (or to n) on $[\alpha, \infty)_{\mathbb{T}}$. The essential uniqueness of the minimal principal solution of $(\mathbb{S})$ at infinity is proven in [27, Theorem 6.9].

PROPOSITION 3.7. *Let $(\hat{X}, \hat{U})$ be a principal solution of $(\mathbb{S})$ at infinity with rank equal to r satisfying $n - d_\infty \leqslant r \leqslant n$. Then $(\hat{X}, \hat{U})$ is unique up to a right nonsingular multiple if and only if $r = n - d_\infty$.*

The following result from [14, Theorem 4.4] reveals that the property of the existence of the limit of the S-matrix associated with a conjoined basis of $(\mathbb{S})$ with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) is a characterizing property of antiprincipal solutions of $(\mathbb{S})$ at infinity.

PROPOSITION 3.8. *Let (X, U) be a conjoined basis of $(\mathbb{S})$ with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), let the matrices $S(t)$ and T be given by (3.8) and (3.13), and assume that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then the following statements are equivalent.*

(i) *The conjoined basis (X, U) is an antiprincipal solution of $(\mathbb{S})$ at ∞.*

(ii) *The limit of $S(t)$ for $t \to \infty$ exists.*

(iii) *The condition $\lim_{t \to \infty} S(t) = T^\dagger$ holds.*

3.5. Mutual representation

Now we recall important concepts regarding mutual representations of conjoined bases of ($\mathbb{S}$) with constant kernel on the interval $[\alpha, \infty)_{\mathbb{T}}$. These are based on finding a suitable conjoined basis $(\bar{X}, \bar{U})$ of ($\mathbb{S}$), which completes a given conjoined basis (X, U) to a normalized pair, thus allowing to construct through (3.4) a special symplectic fundamental matrix of system ($\mathbb{S}$). The following result is from [14, Proposition 3.14].

PROPOSITION 3.9. *Let (X, U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$, let the matrices P and $S(t)$ defined by (2.27) and (3.8). Then there exists a conjoined basis $(\bar{X}, \bar{U})$ of ($\mathbb{S}$) such that (X, U) and $(\bar{X}, \bar{U})$ satisfy*

(i) *the Wronskian $\mathscr{W} := X^T(t)\bar{U}(t) - U^T(t)\bar{X}(t) \equiv I$ on $[a, \infty)_{\mathbb{T}}$, and*

(ii) *$X^\dagger(\alpha)\bar{X}(\alpha) = 0$.*

Moreover, such a conjoined basis $(\bar{X}, \bar{U})$ then satisfies

(iii) *the equality $X^\dagger(t)\bar{X}(t)P = S(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$,*

(iv) *the equalities $\bar{X}(t)P = X(t)S(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$ (in particular $\bar{X}(\alpha)P = 0$) and $\bar{U}(t)P = U(t)S(t) + X^{\dagger T}(t) + U(t)(I - P)\bar{X}^T(t)X^{\dagger T}(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$,*

(v) *the equality $\operatorname{Ker}\bar{X}(t) = \operatorname{Im}[P - P_S(t)] = \operatorname{Im}P \cap \operatorname{Ker}S(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$,*

(vi) *the equality $\bar{P}(t) = I - P + P_S(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$, where $\bar{P}(t) := \bar{X}^\dagger(t)\bar{X}(t)$,*

(vii) *the equalities $S^\dagger(t) = \bar{X}^\dagger(t)X(t)P_S(t) = \bar{X}^\dagger(t)X(t)\bar{P}(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$.*

Note that in the above proposition we displayed only those properties of the conjoined basis $(\bar{X}, \bar{U})$, which are directly needed in this paper and its continuation. Some additional properties are derived in [14, Proposition 3.14]. Based on the result in Proposition 3.9 we can present the following mutual representation of conjoined bases of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), see [27, Proposition 3.6].

PROPOSITION 3.10. *Let (X_i, U_i) for $i \in \{1, 2\}$ be two conjoined bases of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and let P_i be the constant orthogonal projector defined in (2.27) through the function X_i. Let the conjoined basis (X_{3-i}, U_{3-i}) be expressed in terms of (X_i, U_i) via the matrices M_i and N_i, i.e.,*

$$\begin{pmatrix} X_{3-i}(t) \\ U_{3-i}(t) \end{pmatrix} = \begin{pmatrix} X_i(t) & \bar{X}_i(t) \\ U_i(t) & \bar{U}_i(t) \end{pmatrix} \begin{pmatrix} M_i \\ N_i \end{pmatrix}, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{3.22}$$

where $(\bar{X}_i, \bar{U}_i)$ is the conjoined basis of ($\mathbb{S}$) satisfying the properties in Proposition 3.9 with respect to (X_i, U_i). If the equality $\operatorname{Im}X_1(\alpha) = \operatorname{Im}X_2(\alpha)$ is satisfied, then for $i \in \{1, 2\}$

(i) *the matrix $M_i^T N_i$ is symmetric and $N_{3-i} = -N_i^T$,*

(ii) *the matrix M_i is invertible and $M_{3-i} = M_i^{-1}$,*

(iii) *the inclusion $\operatorname{Im} N_i \subseteq \operatorname{Im} P_i$ holds.*

Moreover, the matrices M_i and N_i do not depend on the choice of $(\bar{X}_i, \bar{U}_i)$ with

$$N_i = \mathscr{W}[(X_i, U_i), (X_{3-i}, U_{3-i})]. \tag{3.23}$$

The following properties complement the results in Proposition 3.10 with respect to the associated matrices $S_i(t)$. They are derived in [27, Remark 3.7].

REMARK 3.11. With the notation and the assumptions in Proposition 3.10, we set

$$L_1 := X_1^\dagger(\alpha) X_2(\alpha), \quad L_2 := X_2^\dagger(\alpha) X_1(\alpha),$$

and consider the associated matrix $S_i(t)$, which is defined for $t \in [\alpha, \infty)_\mathbb{T}$ in (3.8) through the matrix $X_i(t)$. Then the following properties hold for $i \in \{1, 2\}$:

$$L_i L_{3-i} = P_i, \quad L_{3-i} = L_i^\dagger, \quad L_i = P_i M_i, \quad N_i = P_i N_i, \tag{3.24}$$

$$P_i = \mathscr{P}_{\operatorname{Im} L_i}, \quad L_i^T N_i = M_i^T P_i N_i = M_i^T N_i \text{ is symmetric}, \tag{3.25}$$

$$X_{3-i}(t) = X_i(t)[L_i + S_i(t) N_i], \quad t \in [\alpha, \infty)_\mathbb{T}, \tag{3.26}$$

$$[L_i + S_i(t) N_i]^\dagger = L_{3-i} + S_{3-i}(t) N_{3-i}, \quad t \in [\alpha, \infty)_\mathbb{T}, \tag{3.27}$$

$$\operatorname{Im}[L_i + S_i(t) N_i] = \operatorname{Im} P_i, \quad t \in [\alpha, \infty)_\mathbb{T}, \tag{3.28}$$

$$S_{3-i}(t) = [L_i + S_i(t) N_i]^\dagger S_i(t) L_i^{\dagger T}, \quad t \in [\alpha, \infty)_\mathbb{T}. \tag{3.29}$$

Note that some additional properties are derived in [27, Remark 3.7], such as the invertibility of the matrix $M_i + S_i(t) N_i$ on $[\alpha, \infty)_\mathbb{T}$, which are not needed in this paper.

We finish this subsection by presenting an additional property of the matrices $S_i(t)$ in Remark 3.11, which corrects the discrete time identity in [29, Eq. (2.13)]. This result will be used in our results, where it plays key role.

PROPOSITION 3.12. *With the notation and assumptions in Proposition 3.10 and Remark 3.11, for $i \in \{1, 2\}$ we have*

$$\operatorname{Im}[P_{3-i} M_{3-i} S_i(t)] = \operatorname{Im} S_{3-i}(t), \quad t \in [\alpha, \infty)_\mathbb{T}. \tag{3.30}$$

Proof. Let us fix an index $i \in \{1, 2\}$. We obtain equality (3.30) as a consequence of formulas (3.24) and (3.29). More precisely, from (3.24) it follows that for $t \in [\alpha, \infty)_\mathbb{T}$

$$S_i(t) = P_i S_i(t) = L_i L_i^\dagger S_i(t) = L_{3-i}^\dagger L_{3-i} S_i(t) = (P_{3-i} M_{3-i})^\dagger P_{3-i} M_{3-i} S_i(t).$$

Then (3.29) implies that for $t \in [\alpha, \infty)_{\mathbb{T}}$ we have

$$
\begin{aligned}
S_i(t) L_{3-i}^T &\overset{(3.29)}{=} [L_{3-i} + S_{3-i}(t) N_{3-i}]^\dagger S_{3-i}(t) L_{3-i}^{\dagger T} L_{3-i}^T \\
&\overset{(3.24)}{=} [L_{3-i} + S_{3-i}(t) N_{3-i}]^\dagger S_{3-i}(t) P_{3-i}^T \overset{(3.12)}{=} [L_{3-i} + S_{3-i}(t) N_{3-i}]^\dagger S_{3-i}(t).
\end{aligned}
\tag{3.31}
$$

Hence, due to the symmetry of $S_i(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ for $i \in \{1, 2\}$ we have that

$$
P_{3-i} M_{3-i} S_i(t) = S_{3-i}(t) [L_{3-i} + S_{3-i}(t) N_{3-i}]^{\dagger T}, \quad t \in [\alpha, \infty)_{\mathbb{T}}.
$$

This shows that the inclusion $\operatorname{Im}[P_{3-i} M_{3-i} S_i(t)] \subseteq \operatorname{Im} S_{3-i}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ holds. On the other hand, for $t \in [\alpha, \infty)_{\mathbb{T}}$ we have

$$
\begin{aligned}
L_{3-i} S_i(t) [L_{3-i} &+ S_{3-i}(t) N_{3-i}]^T \\
&\overset{(3.31)}{=} S_{3-i}(t) [L_{3-i} + S_{3-i}(t) N_{3-i}]^{\dagger T} [L_{3-i} + S_{3-i}(t) N_{3-i}]^T.
\end{aligned}
$$

Notice now that (3.28) implies that

$$
[L_{3-i} + S_{3-i}(t) N_{3-i}][L_{3-i} + S_{3-i}(t) N_{3-i}]^\dagger = P_{3-i}, \quad t \in [\alpha, \infty)_{\mathbb{T}}.
\tag{3.32}
$$

The latter two equalities and the symmetry of P_{3-i} for $i \in \{1, 2\}$ yields that

$$
\begin{aligned}
S_{3-i}(t) &\overset{(3.12)}{=} S_{3-i}(t) P_{3-i} \overset{(3.32)}{=} S_{3-i}(t) [L_{3-i} + S_{3-i}(t) N_{3-i}]^{\dagger T} [L_{3-i} + S_{3-i}(t) N_{3-i}]^T \\
&= P_{3-i} M_{3-i} S_i(t) [L_{3-i} + S_{3-i}(t) N_{3-i}]^T, \quad t \in [\alpha, \infty)_{\mathbb{T}}.
\end{aligned}
$$

The latter equality shows that the inclusion $\operatorname{Im} S_{3-i}(t) \subseteq \operatorname{Im}[P_{3-i} M_{3-i} S_i(t)]$ on $[\alpha, \infty)_{\mathbb{T}}$ is valid. Hence, the proof of (3.30) is complete. $\square$

3.6. Minimal conjoined bases

In this subsection we present two important properties of minimal conjoined bases, which we will need for our further investigations. The first one says that such minimal conjoined bases are characterized by considering the smallest orthogonal projector P in (3.17). This result is from [14, Lemma 3.17].

PROPOSITION 3.13. *Let (X, U) be a conjoined basis of $(\mathbb{S})$ on $[\alpha, \infty)_{\mathbb{T}}$ with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and assume that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then (X, U) is a minimal conjoined basis of $(\mathbb{S})$ on the interval $[\alpha, \infty)_{\mathbb{T}}$ if and only if the orthogonal projectors P and $P_{S\infty}$ defined by (2.27) and (3.16) satisfy*

$$
P = P_{S\infty}.
\tag{3.33}
$$

In the following result we describe all minimal conjoined bases of $(\mathbb{S})$ on some interval $[\alpha, \infty)_{\mathbb{T}}$ by their initial conditions. It is from [14, Theorem 5.1] and it will be used in Remark 5.5 and in our future research.

PROPOSITION 3.14. *Let (X,U) be a minimal conjoined basis of system $(\mathbb{S})$ on the interval $[\alpha,\infty)_{\mathbb{T}}$, let $P_{S\infty}$ and T defined by (3.16) and (3.13), and assume that $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. Then a solution $(\tilde{X},\tilde{U})$ is a minimal conjoined basis on $[\alpha,\infty)_{\mathbb{T}}$ if and only if there exist matrices $M,N \in \mathbb{R}^{n \times n}$ such that*

$$\tilde{X}(\alpha) = X(\alpha)M, \quad \tilde{U}(\alpha) = U(\alpha)M + X^{\dagger T}(\alpha)N, \tag{3.34}$$

$$M \text{ is nonsingular}, \quad M^T N = N^T M, \quad \operatorname{Im} N \subseteq \operatorname{Im} P_{S\infty}, \tag{3.35}$$

$$NM^{-1} + T \geqslant 0. \tag{3.36}$$

In this case the matrix $\tilde{T}$ in (3.13) corresponding to $(\tilde{X},\tilde{U})$ satisfies

$$\tilde{T} = M^T T M + M^T N, \quad \operatorname{rank} \tilde{T} = \operatorname{rank}(NM^{-1} + T). \tag{3.37}$$

3.7. Relation being contained

In this subsection we recall a crucial relation being contained for conjoined bases of system $(\mathbb{S})$ from [27, Section 4]. For this purpose, we also recall the concept of *equivalent solutions* (X_1,U_1) and (X_2,U_2) of $(\mathbb{S})$ on some interval $[\alpha,\infty)_{\mathbb{T}}$, which is defined by the property that $X_1(t) = X_2(t)$ on $[\alpha,\infty)_{\mathbb{T}}$. This notion leads to the following notion, which was introduced in the time scales setting in [27, Definition 4.1]. Let (X,U) be a conjoined basis of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞) and let the matrices P and $P_{S\infty}$ be defined by (2.27) and (3.16). Consider an orthogonal projector P_* satisfying

$$\operatorname{Im} P_{S\infty} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P. \tag{3.38}$$

We say that a conjoined basis (X_*,U_*) of $(\mathbb{S})$ is *contained* in (X,U) on $[\alpha,\infty)_{\mathbb{T}}$ with respect to P_*, or that (X,U) *contains* (X_*,U_*) on $[\alpha,\infty)_{\mathbb{T}}$ with respect to P_*, if the solutions (X_*,U_*) and (XP_*,UP_*) are equivalent, that is, if $X_*(t) = X(t)P_*$ on $[\alpha,\infty)_{\mathbb{T}}$.

The next result from [27, Proposition 4.2] describes the properties of a conjoined basis (X_*,U_*) of $(\mathbb{S})$, which is contained on $[\alpha,\infty)_{\mathbb{T}}$ in a given conjoined basis (X,U) with with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). Such properties of (X_*,U_*) are essentially inherited from (X,U) and they will be frequently used in our analysis.

PROPOSITION 3.15. *Let (X,U) be a conjoined basis of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞) and assume that a conjoined basis (X_*,U_*) of $(\mathbb{S})$ is contained in (X,U) on $[\alpha,\infty)_{\mathbb{T}}$ with respect to an orthogonal projector P_* satisfying (3.38).*

(i) Then (X_*,U_*) has also constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). Moreover, the matrix P_* is then the associated orthogonal projector defined in (2.27) for (X_*,U_*), i.e., $P_* = \mathscr{P}_{\operatorname{Im} X_*^T(t)} = X_*^\dagger(t)X_*(t)$ on $[\alpha,\infty)_{\mathbb{T}}$.

(ii) If $S(t)$ and $S_*(t)$ are the S-matrices corresponding to the conjoined bases (X,U) and (X_*,U_*) on $[\alpha,\infty)_{\mathbb{T}}$, then $S(t) = S_*(t)$ on $[\alpha,\infty)_{\mathbb{T}}$.

The next proposition from [27, Theorem 5.1] guarantees the existence of a conjoined basis of (𝕊) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), which has any given rank between the numbers $n - d_\infty$ and n. Note that the conjoined bases with the given rank r are constructed by the above relation being contained.

PROPOSITION 3.16. *Assume that there exists a conjoined basis (X_*, U_*) of (𝕊) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Then for any integer r between $n - d_\infty$ and n there exists a conjoined basis (X, U) of (𝕊), which has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) too, such that $\operatorname{rank} X(t) = r$ on $[\alpha, \infty)_{\mathbb{T}}$.*

As a combination of Propositions 3.16 with Propositions 3.13 and 3.15 we obtain the existence of a minimal conjoined basis, which is contained in a given conjoined basis (X, U) with with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). These properties are also highlighted in Proposition 3.21 below.

REMARK 3.17. For a given conjoined basis (X, U) of (𝕊) with with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points there always exists some minimal conjoined basis $(X_{\min}, U_{\min}) := (X_*, U_*)$ of (𝕊) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), which is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$. We obtain this conjoined basis by the choice $P_* := P_{S\infty}$ according to (3.33).

In the following three results we analyze the equivalence of two solutions and the relation being contained from the point of view of the initial conditions on the considered solutions of (𝕊). The next proposition is presented in [27, Proposition 3.10]. It introduces different way how to characterize the relation to be contained or to contain for conjoined bases of system (𝕊). All the following details serve to make the relations (3.41) in Remark 3.20 below understandable. The results containing the approach with the matrices G and H presented below aims to state and prove Theorem 6.1.

PROPOSITION 3.18. *Let (X, U) be a conjoined bases of (𝕊) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Let P and $P_{S\infty}$ be defined in (2.23) and (3.16). Then two solutions (X_1, U_1) and (X_2, U_2) of (𝕊) are equivalent on $[\alpha, \infty)_{\mathbb{T}}$ if and only if there exists unique $n \times n$ matrices G and H such that*

$$X_1(\alpha) = X_2(\alpha), \quad U_1(\alpha) - U_2(\alpha) = X^{\dagger T}(\alpha)G + U(\alpha)H,$$
$$\operatorname{Im} G \subseteq \operatorname{Im}(P - P_{S\infty}), \quad \operatorname{Im} H \subseteq \operatorname{Im}(I - P).$$

The following theorem is presented in [27, Theorem 4.3]. Before we state it, we need to introduce the set $\mathscr{M}(P_{**}, P_*, P)$ of pairs (G, H) associated with orthogonal projectors P_{**}, P_*, and P, see [27, page 864] for more details. Let P_{**}, P_*, P satisfy the inclusions

$$\operatorname{Im} P_{**} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P.$$

Then we define the set

$$\mathscr{M}(P_{**}, P_*, P) := \left\{ (G, H) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}, \operatorname{rank}(G^T, H^T, P_*) = n, \right.$$
$$\left. P_{**}G = 0, PG = G, P_*G = G^T P_*, PH = 0 \right\}. \tag{3.39}$$

PROPOSITION 3.19. *Let (X,U) be a conjoined bases of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). Let P and $P_{S\infty}$ be defined in (2.23) and (3.16) and consider an orthogonal projector P_* satisfying (3.38). Then a conjoined basis (X_*,U_*) is contained in (X,U) on $[\alpha,\infty)_{\mathbb{T}}$ with respect to P_* if and only if for some $(G,H) \in \mathscr{M}(P_{S\infty},P_*,P)$ defined in (3.39), we have*

$$X_*(\alpha) = X(\alpha)P_*, \quad U_*(\alpha) = U(\alpha)P_* + [X^\dagger(\alpha)]^T G + U(\alpha)H. \tag{3.40}$$

REMARK 3.20. As the authors of [27] mention, it follows from Propositions 3.18 and 3.19 that the pair (G,H) in Proposition 3.19 is unique. Thus we may say that the conjoined basis (X_*,U_*) is contained in (X,U) on $[\alpha,\infty)_{\mathbb{T}}$ through the pair of matrices (G,H). Notice also that then $(G,H) \in \mathscr{M}(P_{S\infty},P_*,P)$, which means that the pair (G,H) satisfies all the additional properties from (3.39), i.e.,

$$P_{S\infty}G = 0, \quad PG = G, \quad P_*G = G^T P_*, \quad PH = 0. \tag{3.41}$$

We will use this fact later in the proof of Proposition 7.1.

The following proposition is a useful tool when we deal with the images of conjoined bases with constant kernel, as we do while investigating the genera of conjoined bases of $(\mathbb{S})$. It guarantees the existence of other conjoined bases, which are either contained in the first one or which contain the first one. The first part is a consequence of Proposition 3.19 and [27, Remark 4.4], the second part follows from [27, Theorem 4.5]. We will use it in the proof of Theorem 6.1.

PROPOSITION 3.21. *Assume that (X,U) is a conjoined basis of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞) and let $R(t)$ and P be the associated orthogonal projectors defined in (2.23). Then the following statements hold.*

(i) *For every orthogonal projector P_* satisfying condition (3.38) there exists a conjoined basis (X_*,U_*) of $(\mathbb{S})$ which is contained in (X,U) such that $\operatorname{Im}X_*^T(t) = \operatorname{Im}P_*$ for all $t \in [\alpha,\infty)_{\mathbb{T}}$.*
(ii) *For every orthogonal projectors $\tilde{P}_\alpha$ and $\tilde{R}_\alpha$ satisfying*

$$\operatorname{Im}P \subseteq \operatorname{Im}\tilde{P}_\alpha, \quad \operatorname{Im}R(\alpha) \subseteq \operatorname{Im}\tilde{R}_\alpha, \quad \operatorname{rank}\tilde{P} = \operatorname{rank}\tilde{R}_\alpha,$$

there exists a conjoined basis $(\tilde{X},\tilde{U})$ of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ which contains (X,U) and satisfies $\operatorname{Im}\tilde{X}^T(t) = \operatorname{Im}\tilde{P}_\alpha$ on $[\alpha,\infty)_{\mathbb{T}}$ and $\operatorname{Im}\tilde{X}(\alpha) = \operatorname{Im}\tilde{R}_\alpha$.

The following two results reveal that the property to be an antiprincipal solution of $(\mathbb{S})$ at infinity or a principal solution of $(\mathbb{S})$ at infinity remains preserved under the relation being contained. It can be found in [14, Theorems 4.6 and 4.7] and in [27, Proposition 6.5].

PROPOSITION 3.22. *Let (X,U) be an antiprincipal (a principal) solution of $(\mathbb{S})$ at infinity with respect to the interval $[\alpha,\infty)_{\mathbb{T}}$. Then every conjoined basis, which is contained in (X,U) on $[\alpha,\infty)_{\mathbb{T}}$, is also an antiprincipal (a principal) solution of $(\mathbb{S})$ at infinity with respect to the interval $[\alpha,\infty)_{\mathbb{T}}$.*

PROPOSITION 3.23. *Let* (X,U) *be an antiprincipal (a principal) solution of* ($\mathbb{S}$) *at infinity with respect to the interval* $[\alpha,\infty)_{\mathbb{T}}$. *Then every conjoined basis with constant kernel on* $[\alpha,\infty)_{\mathbb{T}}$ *and no focal points in* (α,∞), *which contains* (X,U) *on* $[\alpha,\infty)_{\mathbb{T}}$, *is also an antiprincipal (a principal) solution of* ($\mathbb{S}$) *at infinity with respect to the interval* $[\alpha,\infty)_{\mathbb{T}}$.

In the final result of this subsection we present a characterization of antiprincipal solutions of ($\mathbb{S}$) at infinity on intervals $[\alpha,\infty)_{\mathbb{T}}$, where the left endpoint exceeds a specific bound given by the minimal principal solution $(\hat{X}_{\min},\hat{U}_{\min})$ of ($\mathbb{S}$) at infinity. In particular, following [14, Eq. (6.3)] we define the point

$$\hat{\alpha}_{\min} := \inf \left\{ \alpha \in [a,\infty)_{\mathbb{T}},\ (\hat{X}_{\min},\hat{U}_{\min}) \text{ has constant kernel on } [\alpha,\infty)_{\mathbb{T}} \atop \text{and no focal points in } (\alpha,\infty) \right\}, \qquad (3.42)$$

which satisfies, in view of estimate (3.21) and $\operatorname{rank}\hat{X}_{\min}(t) = n - d_{\infty}$ on $[\alpha,\infty)_{\mathbb{T}}$, the equality

$$d[\hat{\alpha}_{\min},\infty)_{\mathbb{T}} = d_{\infty} = d[\alpha,\infty)_{\mathbb{T}} \quad \text{for every } \alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}. \qquad (3.43)$$

The following result is proven in [14, Theorem 6.3] for antiprincipal solutions of ($\mathbb{S}$) at infinity. An obvious modification of its proof yields the same property for principal solutions of ($\mathbb{S}$) at infinity.

PROPOSITION 3.24. *Assume that system* ($\mathbb{S}$) *is nonoscillatory, let* $\hat{\alpha}_{\min} \in [a,\infty)_{\mathbb{T}}$ *be the point defined in* (3.42). *Then a solution* (X,U) *of* ($\mathbb{S}$) *is an antiprincipal (a principal) solution at infinity if and only if* (X,U) *is a conjoined basis of* ($\mathbb{S}$), *which contains some minimal antiprincipal (a principal) solution of* ($\mathbb{S}$) *at infinity on* $[\alpha,\infty)_{\mathbb{T}}$ *for some* $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$.

4. Improved result regarding normalized conjoined basis

In this section we present an additional property of the conjoined basis $(\bar{X},\bar{U})$ from Proposition 3.9, which is normalized with a given conjoined basis (X,U) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). It is a generalization of [14, Theorem 6.5], where we considered only a minimal conjoined basis (X,U) on $[\alpha,\infty)_{\mathbb{T}}$. This extended result is new even in the special cases of the purely continuous time and the purely discrete time, compare with [28, Proposition 1] and [29, Proposition 7.5], see also [12, Proposition 6.155].

THEOREM 4.1. *Assume that system* ($\mathbb{S}$) *is nonoscillatory and let* (X,U) *be a conjoined basis of* ($\mathbb{S}$) *with constant kernel on an interval* $[\alpha,\infty)_{\mathbb{T}}$ *satisfying* $d[\alpha,\infty)_{\mathbb{T}} = d_{\infty}$ *and no focal points in* (α,∞). *Then the associated conjoined basis* $(\bar{X},\bar{U})$ *from Proposition* 3.9 *is an antiprincipal solution of* ($\mathbb{S}$) *at infinity, and there exists* $\beta \in [\alpha,\infty)_{\mathbb{T}}$ *such that*

$$\operatorname{rank}\bar{X}(t) = 2n - d_{\infty} - \operatorname{rank}X(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \qquad (4.1)$$

Proof. Let the conjoined bases (X,U) and $(\bar{X},\bar{U})$ be as in the theorem. Let P, $S(t)$, $P_{S\infty}$ be the matrices in (2.27), (3.8), (3.16) corresponding to (X,U). Since (X,U) has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞), it follows from (3.16) that $\operatorname{Im}S(t) \equiv \operatorname{Im}P_{S\infty}$ and hence $\operatorname{Ker}S(t) \equiv \operatorname{Ker}P_{S\infty}$ on the interval $[\beta,\infty)_{\mathbb{T}}$ for some $\beta \in [\alpha,\infty)_{\mathbb{T}}$. By Proposition 3.9(v) we then derive that

$$\operatorname{Ker}\bar{X}(t) = \operatorname{Im}P \cap \operatorname{Ker}S(t) = \operatorname{Im}P \cap \operatorname{Ker}P_{S\infty}, \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{4.2}$$

In particular, the kernel of $\bar{X}(t)$ is constant on $[\beta,\infty)_{\mathbb{T}}$, and

$$\operatorname{rank}\bar{X}(t) \overset{(4.2)}{=} \dim[(\mathbb{R}^n \setminus \operatorname{Im}P) \cup (\mathbb{R}^n \setminus \operatorname{Ker}P_{S\infty})] \overset{(3.11)}{=} n - \operatorname{rank}X(t) + n - d_\infty$$

for all $t \in [\beta,\infty)_{\mathbb{T}}$. This proves (4.1). Next we will show that $(\bar{X},\bar{U})$ has no focal points in the interval (β,∞). Recall from Remark 2.1(v) that the matrix $S^\dagger(t)$ is nonincreasing on the interval $[\beta,\infty)_{\mathbb{T}}$ and that, by Theorem 2.2, we know that

$$-[S^\sigma(t)]^\dagger S^\Delta(t) S^\dagger(t) = [S^\dagger(t)]^\Delta \leqslant 0, \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{4.3}$$

Moreover, by Proposition 3.9(vii) we obtain for $t \in [\beta,\infty)_{\mathbb{T}}$ the equality

$$S^\dagger(t) X^\dagger(t) = \bar{X}^\dagger(t) X(t) P_{S\infty} X^\dagger(t) = \bar{X}^\dagger(t) X(t) P X^\dagger(t) = \bar{X}^\dagger(t) R(t). \tag{4.4}$$

Then by (3.14) and by (4.4) with (4.3) we deduce that

$$\begin{aligned}
&[\bar{X}^\sigma(t)]^\dagger \mathscr{B}(t) [\bar{X}^\dagger(t)]^T \\
&= [\bar{X}^\sigma(t)]^\dagger R^\sigma(t) \mathscr{B}(t) R(t) [\bar{X}^\dagger(t)]^T \overset{(4.4)}{=} [S^\dagger(t)]^\sigma [X^\dagger(t)]^\sigma \mathscr{B}(t) [X^\dagger(t)]^T S^\dagger(t) \\
&= [S^\dagger(t)]^\sigma S^\Delta(t) S^\dagger(t) \overset{(4.3)}{=} -[S^\dagger(t)]^\Delta \geqslant 0, \quad t \in [\beta,\infty)_{\mathbb{T}},
\end{aligned} \tag{4.5}$$

and consequently

$$\bar{X}(t) [\bar{X}^\sigma(t)]^\dagger \mathscr{B}(t) = \bar{X}(t) [\bar{X}^\sigma(t)]^\dagger \mathscr{B}(t) [\bar{X}^\dagger(t)]^T \bar{X}^T(t) \overset{(4.5)}{\geqslant} 0, \quad t \in [\beta,\infty)_{\mathbb{T}}.$$

This proves that $(\bar{X},\bar{U})$ has no focal points in the interval (β,∞). We will show that $(\bar{X},\bar{U})$ is an antiprincipal solution of (S) at infinity with respect to the interval $[\beta,\infty)_{\mathbb{T}}$. First we observe that $d[\beta,\infty)_{\mathbb{T}} = d_\infty$, since $\beta \geqslant \alpha$ and we assume that the abnormality $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$ is maximal. According to (3.8), we define the associated matrix $\bar{S}(t)$ by

$$\bar{S}(t) := \int_\beta^t [\bar{X}^\sigma(s)]^\dagger \mathscr{B}(s) \bar{X}^{T\dagger}(s) \Delta s, \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{4.6}$$

Then by using (4.5) and (4.6) we get

$$\bar{S}(t) \overset{(4.5)}{=} -\int_\beta^t [S^\dagger(s)]^\Delta \Delta s = S^\dagger(\beta) - S^\dagger(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{4.7}$$

This implies that the limit

$$\lim_{t\to\infty} \bar{S}(t) \overset{(4.7)}{=} \lim_{t\to\infty}[S^\dagger(\beta) - S^\dagger(t)] = S^\dagger(\beta) - T$$

exists. By Proposition 3.8(ii) (applied to $(X,U) := (\bar{X},\bar{U})$) it then follows that the conjoined basis $(\bar{X},\bar{U})$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. The proof is complete. $\square$

REMARK 4.2. Theorem 4.1 implies that if a conjoined basis (X,U) is a minimal conjoined basis of a nonoscillatory system ($\mathbb{S}$), then the associated conjoined basis $(\bar{X},\bar{U})$ is a maximal antiprincipal solution of ($\mathbb{S}$) at infinity. This is known in [14, Theorem 6.5]. On the other hand, if (X,U) is a maximal conjoined basis of a nonoscillatory system ($\mathbb{S}$), then the associated conjoined basis $(\bar{X},\bar{U})$ is a minimal antiprincipal solution of ($\mathbb{S}$) at infinity according to (4.1).

5. Genus of conjoined bases

In this section we will keep focusing on the conjoined bases (X,U) of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_\mathbb{T}$. We will show that the behaviour of the image of $X(t)$ is a key property for a classification of the set of all conjoined bases (X,U) of ($\mathbb{S}$). We naturally focus our attention on the associated orthogonal projector $R(t)$ defined in (2.23). For a possible future generalization, we present two proofs of the following key result. The first proof follows the idea of the continuous time theory in [23, Lemma 6.1] and it it based on the uniqueness of solutions of a Riccati type dynamic equation. It also uses the derivative of the Moore–Penrose pseudoinverse from Theorem 2.2. The second proof is motivated by the discrete case in [29, Proposition 2.7]. It is based on the unique solvability of certain shifted linear dynamic systems. These systems correspond to the backward recurrence difference systems used in the proof of [29, Proposition 2.7].

LEMMA 5.1. *Let (X_1,U_1) and (X_2,U_2) be two conjoined bases of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_\mathbb{T}$ such that there exists $t_0 \in [\alpha,\infty)_\mathbb{T}$ such that*

$$\mathrm{Im}X_1(t_0) = \mathrm{Im}X_2(t_0). \tag{5.1}$$

Then we have $\mathrm{Im}X_1(t) = \mathrm{Im}X_2(t)$ for all $t \in [\alpha,\infty)_\mathbb{T}$.

Proof. Let $R_1(t)$ and $R_2(t)$ be the orthogonal projectors onto images $\mathrm{Im}X_1(t)$ and $\mathrm{Im}X_2(t)$, respectively, defined according to (2.23). Condition (5.1) can be read as $R_1(t_0) = R_2(t_0)$. We now investigate the delta derivative of the orthogonal projector $R(t)$ associated to any conjoined basis (X,U) of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_\mathbb{T}$. We suppress the argument t in the following computation, while we keep in mind that we work with $t \in [\alpha,\infty)_\mathbb{T}$. Then we can write with the aid of Theorem 2.2, regarding the Δ-derivative of the Moore–Penrose pseudoinverse, that

$$R^\Delta = X^\Delta X^\dagger + X^\sigma (X^\dagger)^\Delta$$
$$\overset{(2.12)}{=} (\mathscr{A}X + \mathscr{B}U)X^\dagger + X^\sigma[-(X^\dagger)^\sigma X^\Delta X^\dagger + (X^\dagger)^\sigma (X^\dagger)^{\sigma T}(X^\Delta)^T(I - XX^\dagger)]$$
$$\overset{(3.6)}{=} \mathscr{A}R + \mathscr{B}UX^\dagger - R^\sigma \mathscr{B}UX^\dagger + (X^\dagger)^{\sigma T}[(U^\sigma)^T\mathscr{B} - (X^\sigma)^T\mathscr{D}](I - R).$$

Thus, using (3.14) we get that the orthogonal projector $R(t)$ satisfies the equation

$$R^\Delta(t) - \mathscr{A}(t)R(t) - R^\sigma(t)[\mathscr{A}(t) - \mathscr{D}(t)]R(t) - R^\sigma(t)\mathscr{D}(t) = 0, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \quad (5.2)$$

Equation (5.2) is a symmetric Riccati-type equation. This is uniquely solvable first order dynamic equation, see [7, page 229]. Then the uniqueness of solution of equation (5.2) together with (5.1) implies that $R_1(t) = R_2(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$, which proves that the statement of the lemma holds. $\quad\square$

Alternative proof of Lemma 5.1. Consider the initial point $t_0 \in [a, \infty)_{\mathbb{T}}$ from condition (5.1). If $t \in [t_0, \infty)_{\mathbb{T}}$, then the result follows directly from Proposition 3.10 and Remark 3.11. If $t \in [\alpha, t_0)_{\mathbb{T}}$, then we need to use a different approach. Define the symmetric Riccati quotients

$$Q_i(t) := U_i(t)X_i^\dagger(t) + X_i^{\dagger T}(t)U_i^T(t)[I - R_i(t)], \quad i \in \{1,2\}, t \in [\alpha, t_0]_{\mathbb{T}},$$

where $R_i(t)$ is the orthogonal projector onto $\mathrm{Im}\, X_i(t)$ according to (2.23). It is easy to verify that, using (2.26) and (3.6), we have

$$U_i(t)X_i^\dagger(t) = Q_i(t)R_i(t), \quad t \in [\alpha, t_0]_{\mathbb{T}}, \quad (5.3)$$

$$X_i^\Delta(t) = [-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_i^\sigma(t)]X_i^\sigma(t), \quad t \in [\alpha, \rho(t_0)]_{\mathbb{T}}. \quad (5.4)$$

Moreover, the Wronskian $N_1 := \mathscr{W}[(X_1, U_1),(X_2, U_2)]$ is constant on $[\alpha, t_0]_{\mathbb{T}}$. Then we have

$$[X_1^\dagger(t)]^T N_1 = [X_1^\dagger(t)]^T[X_1^T(t)U_2(t) - U_1^T(t)X_2(t)]$$
$$\stackrel{(5.3)}{=} R_1(t)U_2(t) - R_1(t)Q_1(t)X_2(t), \quad t \in [\alpha, t_0]_{\mathbb{T}}, \quad (5.5)$$

as well as, using $\mathscr{B}(t) = R_1^\sigma(t)\mathscr{B}(t)$ from (3.14), for $t \in [\alpha, \rho(t_0)]_{\mathbb{T}}$

$$X_2^\Delta(t) \stackrel{(3.6)}{=} -\mathscr{D}^T(t)X_2^\sigma(t) + \mathscr{B}^T(t)U_2^\sigma(t) = -\mathscr{D}^T(t)X_2^\sigma(t) + \mathscr{B}^T(t)R_1^\sigma(t)U_2^\sigma(t)$$
$$\stackrel{(5.5)}{=} -\mathscr{D}^T(t)X_2^\sigma(t) + \mathscr{B}^T(t)[X_1^\dagger(t)^{\sigma T}N_1 + R_1^\sigma(t)Q_1^\sigma(t)X_2^\sigma(t)]$$
$$= [-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_1^\sigma(t)]X_2^\sigma(t) + \mathscr{B}^T(t)[X_1^\sigma(t)]^{\dagger T}N_1. \quad (5.6)$$

Let M_1 be the matrix from Proposition 3.10 and define the function

$$Z(t) := X_1(t)[M_1 - F(t)N_1], \quad t \in [\alpha, t_0]_{\mathbb{T}}, \quad (5.7)$$

where the function $F(t)$ is defined by

$$F(t) := \int_t^{t_0} X_1^{\sigma\dagger}(s)\mathscr{B}(s)[X_1^\dagger(s)]^T \Delta s, \quad t \in [\alpha, t_0]_{\mathbb{T}}. \quad (5.8)$$

Moreover, since $R_1(t_0) = R_2(t_0)$, as a consequence of (5.1), we can use the results and the notation from Proposition 3.10 and Remark 3.11 with the point t_0 instead of α. Thus, we derive that the function $Z(t)$ defined in (5.7) satisfies the initial condition

$$Z(t_0) = X_1(t_0)M_1 = X_1(t_0)P_1M_1 \stackrel{(3.24)}{=} X_1(t_0)L_1 = R_1(t_0)X_2(t_0) = X_2(t_0) \quad (5.9)$$

and the nonhomogeneous linear dynamic equation

$$Z^{\Delta}(t) - [\mathcal{B}^T(t)Q_1^{\sigma}(t) - \mathcal{D}^T(t)]Z^{\sigma}(t) = \mathcal{B}^T(t)[X_1^{\sigma}(t)]^{\dagger T}N_1, \quad t \in [\alpha, \rho(t_0)]_{\mathbb{T}}. \quad (5.10)$$

Equation (5.10) is true since from (3.10) we know that the matrix $[X_1^{\sigma}(t)]^{\dagger}\mathcal{B}(t)[X_1^{\dagger}(t)]^T$ is symmetric on $[\alpha, \rho(t_0)]_{\mathbb{T}}$ and

$$X_1(t)X_1^{\dagger}(t)\mathcal{B}^T(t) = R_1(t)\mathcal{B}^T(t) = \mathcal{B}^T(t), \quad t \in [\alpha, \rho(t_0)]_{\mathbb{T}}. \quad (5.11)$$

Hence, it follows that

$$
\begin{aligned}
Z^{\Delta}(t) &\overset{(5.7)}{=} X_1^{\Delta}(t)[M_1 - F(t)N_1]^{\sigma} + X_1[M_1 - F(t)N_1]^{\Delta} \\
&\overset{(5.4)}{=} [-\mathcal{D}^T(t) + \mathcal{B}^T(t)Q_1^{\sigma}(t)]X_1^{\sigma}(t)[M_1 - F^{\sigma}(t)N_1] - X_1(t)F^{\Delta}(t)N_1 \\
&\overset{(5.7),(5.8)}{=} [-\mathcal{D}^T(t) + \mathcal{B}^T(t)Q_1^{\sigma}(t)]Z_1^{\sigma}(t) + X_1(t)[X_1^{\sigma}(t)]^{\dagger}\mathcal{B}(t)[X_1^{\dagger}(t)]^T N_1 \\
&\overset{(3.10)}{=} [-\mathcal{D}^T(t) + \mathcal{B}^T(t)Q_1^{\sigma}(t)]Z_1^{\sigma}(t) + X_1(t)X_1^{\dagger}(t)\mathcal{B}^T(t)[X_1^{\sigma}(t)]^{\dagger T} N_1 \\
&\overset{(5.11)}{=} [-\mathcal{D}^T(t) + \mathcal{B}^T(t)Q_1^{\sigma}(t)]Z_1^{\sigma}(t) + \mathcal{B}^T(t)[X_1^{\sigma}(t)]^{\dagger T} N_1, \quad t \in [\alpha, \rho(t_0)]_{\mathbb{T}}.
\end{aligned}
$$

Equation (5.10) has a unique solution satisfying the initial condition at t_0, which exists on the whole interval $[\alpha, t_0]_{\mathbb{T}}$, with no additional requirement of regressivity of the coefficients, see e.g. [27, Proposition 2.1]. Since by (5.6) and (5.9) the matrix $X_2(t)$ satisfies the same equation with the same initial condition, it follows that $X_2(t) = Z(t) = X_1(t)[M_1 - F(t)N_1]$ on $[\alpha, t_0]_{\mathbb{T}}$. This shows that $\mathrm{Im}\,X_2(t) \subseteq \mathrm{Im}\,X_1(t)$ for all $t \in [\alpha, t_0]_{\mathbb{T}}$. By interchanging the roles of (X_1, U_1) and (X_2, U_2) we find out that also $\mathrm{Im}\,X_2(t) \supseteq \mathrm{Im}\,X_1(t)$ for all $t \in [\alpha, t_0]_{\mathbb{T}}$ holds. Putting the previous two steps together, we get that $\mathrm{Im}\,X_1(t) = \mathrm{Im}\,X_2(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$, which completes the proof. $\square$

Now we state the key definition of this section. The so-called genus of conjoined bases of ($\mathbb{S}$) turns out to be an important tool for the investigation of the behaviour of special conjoined bases of ($\mathbb{S}$), like the principal or antiprincipal solutions of ($\mathbb{S}$) at infinity.

DEFINITION 5.2. We say that two conjoined bases (X_1, U_1) and (X_2, U_2) of ($\mathbb{S}$) belong to the same genus $\mathscr{G}$, or have the same genus $\mathscr{G}$, if the matrices $X_1(t)$ and $X_2(t)$ have eventually the same images, i.e., if there exists a point $\alpha \in [a, \infty)_{\mathbb{T}}$ such that

$$\mathrm{Im}\,X_1(t) = \mathrm{Im}\,X_2(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

We can see from the Lemma 5.1 that if (X_1, U_1) and (X_2, U_2) are two conjoined bases of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ such that $\mathrm{Im}\,X_1(t_0) = \mathrm{Im}\,X_2(t_0)$ for some point $t_0 \in [\alpha, \infty)_{\mathbb{T}}$, then (X_1, U_1) and (X_2, U_2) belong to the same genus. This is summarized in the following statement, which we use for our future reference.

THEOREM 5.3. *Assume that* ($\mathbb{S}$) *is nonoscillarory. Let* (X_1, U_1) *and* (X_2, U_2) *be conjoined bases of* ($\mathbb{S}$) *with constant kernel on* $[\alpha, \infty)_{\mathbb{T}}$. *Then the following statements are equivalent.*

(i) *The conjoined bases* (X_1, U_1) *and* (X_2, U_2) *belong to the same genus* $\mathscr{G}$.

(ii) *The equality* $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ *holds on some subinterval* $[\beta, \infty)_{\mathbb{T}}$ *of* $[\alpha, \infty)_{\mathbb{T}}$.

(iii) *The equality* $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ *holds on every subinterval* $[\beta, \infty)_{\mathbb{T}}$ *of* $[\alpha, \infty)_{\mathbb{T}}$.

Proof. The statement of this theorem is a direct consequence of Lemma 5.1. $\quad\square$

The relation "to have the same genus" is an equivalence relation on the set of all conjoined bases of a nonoscillatory system $(\mathbb{S})$. Therefore, there exists a partition of this set into disjoint classes of all conjoined bases of a nonoscillatory system $(\mathbb{S})$ according to the equivalence above. Then we naturally interpret each such an equivalence class as a genus itself.

REMARK 5.4. According to [14, Proposition 3.18] and Lemma 5.1 we get that all minimal conjoined bases of $(\mathbb{S})$ with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ are equivalent in the sense of Definition 5.2, i.e., they belong to the same genus. We denote this genus by $\mathscr{G}_{\min}$ and we call it the minimal genus of $(\mathbb{S})$. From the fact that if $\operatorname{rank} X(t) = n$ on the interval $[\alpha, \infty)_{\mathbb{T}}$, then $X(t)$ is regular on $[\alpha, \infty)_{\mathbb{T}}$, we get that also all maximal conjoined bases of nonoscillatory system $(\mathbb{S})$ belong to the same genus. We denote it by $\mathscr{G}_{\max}$ and call it the maximal genus of $(\mathbb{S})$.

Now we focus our attention to the pairs of minimal conjoined bases of $(\mathbb{S})$. Such conjoined bases are mutually representable in the sense of Proposition 3.10, since they belong to the same genus $\mathscr{G}_{\min}$.

REMARK 5.5. From Proposition 3.14 and its proof displayed in [14, Theorem 5.1] it can be seen that any two minimal conjoined bases $(X_{\min}^{(i)}, U_{\min}^{(i)})$ for $i \in \{1, 2\}$ on $[\alpha, \infty)_{\mathbb{T}}$ can be mutually representable in the sense of Proposition 3.10. That is, there exist constant matrices $M_{\min}^{(i)}$ and $N_{\min}^{(i)}$ such that

$$\begin{pmatrix} X_{\min}^{(3-i)}(t) \\ U_{\min}^{(3-i)}(t) \end{pmatrix} = \begin{pmatrix} X_{\min}^{(i)}(t) & \bar{X}_{\max}^{(i)}(t) \\ U_{\min}^{(i)}(t) & \bar{U}_{\max}^{(i)}(t) \end{pmatrix} \begin{pmatrix} M_{\min}^{(i)} \\ N_{\min}^{(i)} \end{pmatrix}, \quad t \in [\alpha, \infty)_{\mathbb{T}},$$

where $(\bar{X}_{\max}^{(i)}, \bar{U}_{\max}^{(i)})$ is the conjoined basis of $(\mathbb{S})$ satisfying the properties in Proposition 3.9 with respect to $(X_{\min}^{(i)}, U_{\min}^{(i)})$. Note that $(\bar{X}_{\max}^{(i)}, \bar{U}_{\max}^{(i)})$ is indeed a maximal antiprincipal solutions of $(\mathbb{S})$ at infinity by Theorem 4.1, and

$$N_{\min}^{(i)} = \mathscr{W}[(X_{\min}^{(i)}, U_{\min}^{(i)}), (X_{\min}^{(3-i)}, U_{\min}^{(3-i)})]. \tag{5.12}$$

Notice that the matrices $M_{\min}^{(i)}$ and $N_{\min}^{(i)}$ satisfy the properties (i)–(iii) from Proposition 3.10 with the associated orthogonal projector $P_{\min}^{(i)}$ from (2.23). Moreover, if $d[\alpha, \infty)_{\mathbb{T}} = d_{\infty}$ and if we denote by $P_{S_i\infty}$ the orthogonal projector from (3.16) associated with $(X_{\min}^{(i)}, U_{\min}^{(i)})$, then $P_{S_i\infty} = P_{\min}^{(i)}$ (see Proposition 3.13).

We continue our investigation of the mutual representation of conjoined basis by the following proposition, which deals with the minimal conjoined bases mentioned in the previous remark. This is an analogy of the discrete case, see [12, Proposition 6.99] and [23, Lemma 6.9]. Relations (5.13) and (5.14) play a key role in our follow-up research.

PROPOSITION 5.6. *Let* (X_1, U_1) *and* (X_2, U_2) *be conjoined bases of* $(\mathbb{S})$ *with constant kernel on* $[\alpha, \infty)_{\mathbb{T}}$ *and no focal points in* (α, ∞) *and let* P_1, P_2 *and* $P_{S_1\infty}$, $P_{S_2\infty}$ *be the corresponding orthogonal projectors from* (2.23) *and* (3.16) *associated with conjoined bases* (X_1, U_1) *and* (X_2, U_2), *respectively. Moreover, let* $(X_{\min}^{(1)}, U_{\min}^{(1)})$ *be a minimal conjoined basis of* $(\mathbb{S})$, *which is contained in* (X_1, U_1) *on* $[\alpha, \infty)_{\mathbb{T}}$ *with respect to* $P_{S_1\infty}$, *and* $(X_{\min}^{(2)}, U_{\min}^{(2)})$ *be a minimal conjoined basis of* $(\mathbb{S})$, *which is contained in* (X_1, U_1) *on* $[\alpha, \infty)_{\mathbb{T}}$ *with respect to* $P_{S_2\infty}$. *Suppose that* (X_1, U_1) *and* (X_2, U_2) *are mutually representable as in Proposition* 3.10 *on* $[\alpha, \infty)_{\mathbb{T}}$ *through the matrices* M_1, N_1, M_2, N_2, *i.e.,* (3.22) *holds. If* $M_{\min}^{(1)}$, $M_{\min}^{(2)}$, $N_{\min}^{(1)}$, $N_{\min}^{(2)}$ *are the corresponding matrices from Remark* 5.5, *then for* $i \in \{1, 2\}$ *we have*

$$P_i M_i P_{S_{3-i}\infty} = P_{S_i\infty} M_{\min}^{(i)}, \tag{5.13}$$

$$N_{\min}^{(i)} (M_{\min}^{(i)})^{-1} = P_{S_i\infty} N_i (M_i)^{-1} P_{S_i\infty}. \tag{5.14}$$

Proof. The proof follows the same way as the proof of [12, Proposition 6.99]. The proof uses Proposition 3.19, namely equation (3.40), which provides the mutual characterization between (X_1, U_1) and $(X_{\min}^{(1)}, U_{\min}^{(1)})$ as well as between (X_2, U_2) and $(X_{\min}^{(2)}, U_{\min}^{(2)})$. Further, it uses Remark 5.4 and the relations in (5.12), and the properties of Remark 3.20. All the arguments remain the same as in the discrete case in the proof of [12, Proposition 6.99]. Therefore, the full details of the proof are omitted. $\square$

6. Antiprincipal and principal solutions at infinity in any genus

The following theorem guarantees that in every genus $\mathscr{G}$ of $(\mathbb{S})$ there exists an antiprincipal solution of $(\mathbb{S})$ in infinity belonging to $\mathscr{G}$. It is a generalization and unification of the corresponding statements in the continuous case in [24, Theorem 5.12] and the discrete case in [29, Theorem 5.5].

THEOREM 6.1. *Assume that system* $(\mathbb{S})$ *is nonoscillatory. Let* $\mathscr{G}$ *be a genus of a conjoined bases of* $(\mathbb{S})$. *Then there exists an antiprincipal solution of* $(\mathbb{S})$ *at infinity belonging to the genus* $\mathscr{G}$.

Proof. Given that $(\mathbb{S})$ is nonoscillarory, let $(X_{\mathscr{G}}, U_{\mathscr{G}})$ be a conjoined basis belonging to a given genus $\mathscr{G}$. Denote by $P_{S_{\mathscr{G}}\infty}$ its associated orthogonal projector from (3.16), and by $R_{\mathscr{G}}(t)$ the R-projector from (2.23). Let $\alpha \in [a, \infty)_{\mathbb{T}}$ be such that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Our aim is to show that there exists an antiprincipal solution $(\tilde{X}, \tilde{U})$ of $(\mathbb{S})$ at infinity such that

$$\operatorname{Im} X_{\mathscr{G}}(\alpha) = \operatorname{Im} \tilde{X}(\alpha). \tag{6.1}$$

Then according to Lemma 5.1 both $(\tilde{X}, \tilde{U})$ and $(X_{\mathscr{G}}, U_{\mathscr{G}})$ will belong to the same genus $\mathscr{G}$. According to Remark 3.6, there exists a minimal antiprincipal solution $(X_{\min}, U_{\min})$ of $(\mathbb{S})$ at infinity, denote by $R_{\min}(t)$ its R-projector from (2.23). Let (X_*, U_*) be a conjoined basis of $(\mathbb{S})$, which is contained in $(X_{\mathscr{G}}, U_{\mathscr{G}})$ on $[\alpha, \infty)_{\mathbb{T}}$ with respect to the orthogonal projector P_* from (3.38), which we choose as

$$P_* := P_{S_{\mathscr{G}}\infty}. \tag{6.2}$$

This choice is possible. Denote by $P_{S_*\infty}$ the orthogonal projector associated with (X_*, U_*) from (3.16). From Proposition 3.15 together with Proposition 3.13, we get

$$P_* = \mathscr{P}_{\operatorname{Im} X_*^T(t)} \overset{(6.2)}{=} P_{S_{\mathscr{G}}\infty} \overset{(3.33)}{=} P_{S_*\infty}, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

The latter equality guarantees that (X_*, U_*) is a minimal conjoined basis of system $(\mathbb{S})$ contained in $(X_{\mathscr{G}}, U_{\mathscr{G}})$ on $[\alpha, \infty)_{\mathbb{T}}$. Thus both (X_*, U_*) and $(X_{\min}, U_{\min})$ belong to the same genus $\mathscr{G}_{\min}$ and since (X_*, U_*) is contained in $(X_{\mathscr{G}}, U_{\mathscr{G}})$ we get for the associated orthogonal projectors defined according to (2.23) that $R_{\min}(\alpha) = R_*(\alpha)$ and $\operatorname{Im} R_*(\alpha) \subseteq \operatorname{Im} R_{\mathscr{G}}(\alpha)$. Put now $\tilde{R}_\alpha := R_{\mathscr{G}}(\alpha)$. Then from Proposition 3.21(ii) we get that there exists a conjoined basis $(\tilde{X}, \tilde{U})$ of $(\mathbb{S})$ with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ which contains $(X_{\min}, U_{\min})$ on $[\alpha, \infty)_{\mathbb{T}}$ and

$$\operatorname{Im} \tilde{X}(\alpha) = \operatorname{Im} R_{\mathscr{G}}(\alpha) = \operatorname{Im} X_{\mathscr{G}}(\alpha).$$

This completes the proof of (6.1). Finally, the results in Lemma 5.1 and Proposition 3.23 reveal that $(\tilde{X}, \tilde{U})$ is an antiprincipal solution of $(\mathbb{S})$ at infinity which belongs to the genus $\mathscr{G}$. $\square$

The following theorem guarantees, similarly to the previous result, that in every genus $\mathscr{G}$ of $(\mathbb{S})$ there exists a principal solution of $(\mathbb{S})$ at infinity belonging to $\mathscr{G}$. It is a generalization and unification of the corresponding statements in the continuous case in [23, Theorem 7.12] and the discrete case in [29, Theorem 5.5].

THEOREM 6.2. *Assume that $(\mathbb{S})$ is nonoscillatory. Let $\mathscr{G}$ be a genus of a conjoined bases of $(\mathbb{S})$. Then there exists a principal solution of $(\mathbb{S})$ at infinity belonging to the genus $\mathscr{G}$.*

Proof. The proof of the theorem follows the same idea as the proof of Theorem 6.1, where we replace the minimal antiprincipal solution $(X_{\min}, U_{\min})$ of $(\mathbb{S})$ at infinity by the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of $(\mathbb{S})$ at infinity on the interval $[\alpha, \infty)_{\mathbb{T}}$ with the property that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$, while considering Remark 3.6. $\square$

7. Shift of interval in relation being contained

In this section we present additional results about the relation being contained, which we defined in Subsection 3.7. They are related to the possibility of shifting the left endpoint of the interval, on which this relation is considered. The first three

statements (Propositions 7.1 and 7.2) are of technical nature. Proving these results in full details is, however, necessary in order to validate their frequent applications in the subsequent theory.

The first proposition shows that the initial point of the interval $[\alpha,\infty)_{\mathbb{T}}$ can be shifted both to right and to the left, while the property of being contained for the conjoined bases from the definition remains valid on the new extended or shortened interval. This result turns out to be useful when we deal with the genera of conjoined bases of system $(\mathbb{S})$. It is a unification and extension of the continuous case in [23, Theorem 6.7] and the discrete case in [29, Proposition 2.11], see also [12, Proposition 6.93].

PROPOSITION 7.1. *Let (X,U) and (X_*,U_*) be two conjoined bases of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). Then the following statements hold.*

(i) *If (X,U) contains the conjoined basis (X_*,U_*) on the interval $[\alpha,\infty)_{\mathbb{T}}$, then (X,U) contains the conjoined basis (X_*,U_*) on the interval $[\beta,\infty)_{\mathbb{T}}$ for all $\beta \in [\alpha,\infty)_{\mathbb{T}}$.*
(ii) *Assume that $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. If (X,U) contains the conjoined basis (X_*,U_*) on the interval $[\beta,\infty)_{\mathbb{T}}$ for some $\beta \in [\alpha,\infty)_{\mathbb{T}}$, then (X,U) contains the conjoined basis (X_*,U_*) on the interval $[\alpha,\infty)_{\mathbb{T}}$.*

Proof. Fix $\beta \in [\alpha,\infty)_{\mathbb{T}}$. Denote by $S(t)$ and $S_*(t)$ the S-matrices defined in (3.8) corresponding to conjoined bases (X,U) and (X_*,U_*) on $[\alpha,\infty)_{\mathbb{T}}$, respectively, and denote by $S_\beta(t)$ and $S_{*\beta}(t)$ the S-matrices defined in (3.8) corresponding to conjoined bases (X,U) and (X_*,U_*) on $[\beta,\infty)_{\mathbb{T}}$, respectively. Then we have

$$S_\beta(t) = S(t) - S(\beta), \quad S_{*\beta}(t) = S_*(t) - S_*(\beta), \quad t \in [\beta,\infty)_{\mathbb{T}},$$

which implies that

$$0 \leqslant S_\beta(t) \leqslant S(t), \quad 0 \leqslant S_{*\beta}(t) \leqslant S_*(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{7.1}$$

Let $P_{S\infty}$ and $P_{S_\beta\infty}$ be the orthogonal projectors associated with a conjoined basis (X,U) on $[\alpha,\infty)_{\mathbb{T}}$ and on $[\beta,\infty)_{\mathbb{T}}$ from (3.16), respectively, and $P_{S_*\infty}$ and $P_{S_{*\beta}\infty}$ be the orthogonal projectors associated with (X_*,U_*) on $[\alpha,\infty)_{\mathbb{T}}$ and on $[\beta,\infty)_{\mathbb{T}}$ from (3.16), respectively. Then the set of inequalities in (7.1) can be read as

$$\operatorname{Im} P_{S_\beta\infty} \subseteq \operatorname{Im} P_{S\infty} \quad \text{and} \quad \operatorname{Im} P_{S_{*\beta}\infty} \subseteq \operatorname{Im} P_{S_*\infty}. \tag{7.2}$$

First we are going to prove part (i). Let P_* be an orthogonal projector from (2.23) associated with (X_*,U_*). Since (X,U) contains (X_*,U_*) on $[\alpha,\infty)_{\mathbb{T}}$, then we get that the defining property in (3.38) holds and (X_*,U_*) and (XP_*,UP_*) are equivalent solutions on $[\alpha,\infty)_{\mathbb{T}}$. Then from (3.38) together with (7.2) we get that $\operatorname{Im} P_{S_\beta\infty} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P$. It means that (X,U) contains (X_*,U_*) also on $[\beta,\infty)_{\mathbb{T}}$, which completes the proof of the part (i).

Now we will prove part (ii) with the aid of part (i) we have already proved. Assume that $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$, thus $d[\beta,\infty)_{\mathbb{T}} = d_\infty$, too. Combining (3.20) and (7.2), which we get that

$$P_{S_\beta\infty} = P_{S\infty} \quad \text{and} \quad P_{S_{*\beta}\infty} = P_{S_*\infty}. \tag{7.3}$$

Suppose that (X,U) contains (X_*,U_*) on the interval $[\beta,\infty)_{\mathbb{T}}$ and let (X_{**},U_{**}) be another conjoined basis of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞) such that (X,U) contains (X_{**},U_{**}) on $[\alpha,\infty)_{\mathbb{T}}$ with respect to the orthogonal projector P_*. Note that this choice can be done due to Proposition 3.21. It is possible to show that then (X_*,U_*) and (X_{**},U_{**}) are equivalent on $[\alpha,\infty)_{\mathbb{T}}$ in the sense of the comment at the beginning of Subsection 3.7, which will imply the desired result. We will show the mentioned relation in the following way. Considering part (i) we see that the conjoined basis (X,U) contains (X_{**},U_{**}) with respect to P_* also on the interval $[\beta,\infty)_{\mathbb{T}}$. This implies that

$$X_*(t) = X_{**}(t) \text{ for all } t \in [\beta,\infty)_{\mathbb{T}}. \tag{7.4}$$

But due to Theorem 5.3 the above equality implies that

$$\operatorname{Im} X_*(t) = \operatorname{Im} X_{**}(t) \text{ for all } t \in [\alpha,\infty)_{\mathbb{T}}. \tag{7.5}$$

Hence, the conjoined bases (X_*,U_*) and (X_{**},U_{**}) and mutually representable on $[\alpha,\infty)_{\mathbb{T}}$ in the sense of Proposition 3.10, specifically

$$\begin{pmatrix} X_{**}(t) \\ U_{**}(t) \end{pmatrix} = \begin{pmatrix} X_*(t) & \bar{X}_*(t) \\ U_*(t) & \bar{U}_*(t) \end{pmatrix} \begin{pmatrix} M_* \\ N_* \end{pmatrix}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{7.6}$$

where $(\bar{X}_*,\bar{U}_*)$ is the conjoined basis of $(\mathbb{S})$ from Proposition 3.9 associated with (X_*,U_*). We stress that we know that M_* is a constant invertible matrix and N_* is the Wronskian of (X_*,U_*) and (X_{**},U_{**}), see formula (3.23), i.e.,

$$N_* := \mathscr{W}[(X_*,U_*),(X_{**},U_{**})].$$

Using (7.6) we obtain

$$X_{**}(t) = X_*(t)M_* + \bar{X}_*(t)P_*N_*, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{7.7}$$

Since we already know that the solution (X_*,U_*) and (X_{**},U_{**}) are equivalent on $[\alpha,\infty)_{\mathbb{T}}$, we can use Proposition 3.18 to derive the additional result about the matrix N_*, which we will use later. If we put $t := \beta$ in (7.5), then we get that $X_*(\beta) = X_{**}(\beta)$, i.e., $\operatorname{Im} X_*(\beta) = \operatorname{Im} X_{**}(\beta)$. Then from Proposition 3.18 we get that there exist unique $n \times n$ matrices G and H such that

$$U_*(\beta) - U_{**}(\beta) = [X_*^\dagger(\beta)]^T G + U_*(\beta)H \quad \text{and} \quad \operatorname{Im} G \subseteq \operatorname{Im}(P_* - P_{S_\beta^*\infty}). \tag{7.8}$$

Now from the symmetry of $U_*^T(\beta)X_*(\beta)$ and from (3.41) it follows

$$N_* = X_*^T(\beta)U_{**}(\beta) - U_*^T(\beta)X_{**}(\beta) = X_*^T(\beta)[U_{**}(\beta) - U_*(\beta)]$$

$$\overset{(7.8)}{=} P_*G + X_*^T(\beta)U_*(\beta) \overset{(3.41)}{=} G + U_*^T(\beta)X_*(\beta)P_*H \overset{(3.41)}{=} G.$$

From the latter equality and from the second equality in (7.8) it now follows that the inclusion $\operatorname{Im} N_* \subseteq \operatorname{Im}(P_* - P_{S_\beta^*\infty})$ holds. When we consider (7.3) we see that the inclusion

$$\operatorname{Im} N_* \subseteq \operatorname{Im}(P_* - P_{S_*\infty}) \tag{7.9}$$

also holds and it can be read as $N_* = (P_* - P_{S_*\infty})N_*$. Now we return to (7.7) and by using $G = P_*G$ from (3.41), the equality $N_* = G$, and by (7.9) we receive that $P_*N_* = N_*$, which implies that $P_{S_*\infty}N_* = 0$. Finally, considering also Proposition 3.9(iv) we get

$$X_{**}(t) = X_*(t)M_* + \bar{X}_*(t)P_*N_* = X_*(t)M_* + \bar{X}_*(t)S_*(t)N_*$$
$$\overset{(3.18)}{=} X_*(t)[M_* + S_*(t)P_{S_*\infty}N_*] \overset{(7.9)}{=} X_*(t)M_*, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

But we already know that (7.4) holds, so that we have $X_*(t) = X_*(t)M_*$ for $t \in [\beta, \infty)_{\mathbb{T}}$. Using this while considering $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ we get $P_* = P_*M_*$, and combining the above equality with $X_{**}(t) = X_*(t)M_*$ on $[\alpha, \infty)_{\mathbb{T}}$ we get $X_{**}(t) = X_*(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$. Therefore, we proved that (X_*, U_*) and (X_{**}, U_{**}) are equivalent on $[\alpha, \infty)_{\mathbb{T}}$, which immediately implies that the conjoined basis (X_*, U_*) is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$. The proof is complete. $\square$

The second proposition is similar to the previous one, but reveals slightly more. Notice that it differs in the assumptions on the interval, where the conjoined basis (X_*, U_*) has constant kernel and no focal points. It is a unification and extension of the continuous case in [23, Theorem 6.8] and the discrete case in [29, Proposition 2.12], see also [12, Proposition 6.93].

PROPOSITION 7.2. *Let (X_*, U_*) be a conjoined basis of system ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) such that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then the following statements hold for every initial point $\beta \in [\alpha, \infty)_{\mathbb{T}}$.*

(i) *If (X_{**}, U_{**}) is a conjoined basis of system ($\mathbb{S}$) with constant kernel on $[\beta, \infty)_{\mathbb{T}}$ and no focal points in (β, ∞) and it is contained in (X_*, U_*) on the interval $[\beta, \infty)_{\mathbb{T}}$, then (X_{**}, U_{**}) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), too.*

(ii) *If (X, U) is a conjoined basis of system ($\mathbb{S}$) with constant kernel on $[\beta, \infty)_{\mathbb{T}}$ and no focal points in (β, ∞) and it contains (X_*, U_*) on the interval $[\beta, \infty)_{\mathbb{T}}$, then (X, U) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), too.*

Proof. Fix the number $\beta \in [\alpha, \infty)_{\mathbb{T}}$. Denote by P_{**} and P_* the orthogonal projectors from (2.23) associated with conjoined bases (X_{**}, U_{**}) and (X_*, U_*) on $[\beta, \infty)_{\mathbb{T}}$, respectively. Let $S_*(t)$ and $S_{*\beta}(t)$ be the S-matrices defined in (3.8) corresponding to (X_*, U_*) on $[\alpha, \infty)_{\mathbb{T}}$ and $[\beta, \infty)_{\mathbb{T}}$, respectively. Then since (X_{**}, U_{**}) is contained in (X_*, U_*) on $[\beta, \infty)_{\mathbb{T}}$, we have

$$X_{**}(t) = X_*(t)P_{**} \text{ for all } t \in [\beta, \infty)_{\mathbb{T}}, \tag{7.10}$$

and according to (3.38) the following holds

$$\operatorname{Im} P_{S_{*\beta}\infty} \subseteq \operatorname{Im} P_{**} \subseteq \operatorname{Im} P_*. \tag{7.11}$$

Our aim is to show that (7.10) holds also on $[\alpha, \infty)_{\mathbb{T}}$. But in the same way as in the first part of the proof of Proposition 7.1 we get that $P_{S_{*\beta}\infty} = P_{S_*\infty}$, and hence (7.11) implies that

$$\operatorname{Im} P_{S_*\infty} \subseteq \operatorname{Im} P_{**} \subseteq \operatorname{Im} P_*. \tag{7.12}$$

Consider the solution $(\tilde{X}, \tilde{U}) := (X_{**}, U_{**}) - (X_*, U_*)P_{**}$ on $[a, \infty)_{\mathbb{T}}$. Then the columns of $\tilde{U}$ belong to the space $\Lambda[\alpha, \infty)_{\mathbb{T}}$, if we use the notation from Section 3. Now, since $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$, we get that also $d[\beta, \infty)_{\mathbb{T}} = d_\infty$ and thus we have that $\Lambda[\alpha, \infty)_{\mathbb{T}} = \Lambda[\beta, \infty)_{\mathbb{T}}$. This implies that the columns of $\tilde{U}$ belong to the space $\Lambda[\beta, \infty)_{\mathbb{T}}$ and hence,

$$X_{**}(t) = X_*(t)P_{**} \quad \text{for all} \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{7.13}$$

Conditions (7.13) and (7.12) imply that (X_*, U_*) contains (X_{**}, U_{**}) on $[\alpha, \infty)_{\mathbb{T}}$ with respect to the orthogonal projector P_{**}. This together with Proposition 3.15 proves part (i).

For the proof of part (ii) note that, as well as above, it is possible to show that

$$X_*(t) = X(t)P_* \quad \text{for all} \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{7.14}$$

Denote by $S(t)$ and $S_\beta(t)$ the S-matrices defined in (3.8) corresponding to a conjoined basis (X, U) on the intervals $[\alpha, \infty)_{\mathbb{T}}$ and $[\beta, \infty)_{\mathbb{T}}$, respectively. Since a conjoined basis (X, U) contains (X_*, U_*) on $[\beta, \infty)_{\mathbb{T}}$, $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$, and (X_*, U_*) is a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), then we have

$$P_{S_*\beta\infty} = P_{S_\beta\infty} = P_{S_\infty}. \tag{7.15}$$

Suppose now that (X, U) contains (X_*, U_*) on the interval $[\beta, \infty)_{\mathbb{T}}$ through the pair of matrices $(G, H) \in \mathcal{M}(P_{S_\infty}, P_*, P)$ defined in (3.39). Then according to Proposition 3.19 we have

$$U_*(\beta) = U(\beta)P_* + [X^\dagger(\beta)]^T G + U(\beta)H.$$

In the same way as in the proof of Proposition 7.1 it is possible to show that the matrix G is the Wronskian of (X, U) and (X_*, U_*), which we shorten as

$$W := \mathcal{W}[(X, U), (X_*, U_*)]$$

in this proof. Denote by P the orthogonal projector associated with (X, U) on $[\beta, \infty)_{\mathbb{T}}$ defined in (2.23). Then according to (3.39) together with (7.15) we have

$$P_{S_*\infty}W = P_{S_*\beta\infty}W = 0, \quad PW = W, \quad P_*W = W^T P_*. \tag{7.16}$$

We will show that

$$\text{Im}[P - S_*(t)W] = \text{Im}\,P = \text{Im}[P - S_*(t)W^T]^T \text{ on } [\alpha, \infty)_{\mathbb{T}}. \tag{7.17}$$

Notice that $S_*(t) = PS_*(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ by (3.38), and then it follows that

$$\text{Im}[P - S_*(t)W] = \text{Im}\left(P[P - S_*(t)W]\right) \subseteq \text{Im}\,P \quad \text{on } [\alpha, \infty)_{\mathbb{T}}. \tag{7.18}$$

On the other hand, if $v \in \text{Ker}[P - S_*(t)W^T]$ on $[\alpha, \infty)_{\mathbb{T}}$, then

$$[P - S_*(t)W^T]v = 0. \tag{7.19}$$

We can write $v = v_1 + v_2$, where $v_1 \in \operatorname{Ker} P$ and $v_2 \in \operatorname{Im} P$. Then using $W^T P = W^T$ from (7.16) we get

$$v_2 = P(v_1 + v_2) = Pv \overset{(7.19)}{=} S_*(t) W^T P v_1 = S_*(t) W^T v_2, \quad t \in [\alpha, \infty)_\mathbb{T}.$$

Hence $v_2 \in \operatorname{Im} S_*(t) \subseteq \operatorname{Im} P_{S_{*^\infty}}$, i.e., $v_2 = P_{S_{*^\infty}} v_2$. But since $P_{S_{*^\infty}} W = 0$ by (7.16), we get

$$v_2 = S_*(t) W^T v_2 = S_*(t) W^T P_{S_{*^\infty}} v_2 = 0, \quad t \in [\alpha, \infty)_\mathbb{T}.$$

Therefore, $v = v_1 \in \operatorname{Ker} P$ holds, and we showed that

$$\operatorname{Ker}[P - S_*(t) W^T] \subseteq \operatorname{Ker} P, \quad t \in [\alpha, \infty)_\mathbb{T}. \tag{7.20}$$

By taking the orthogonal complements in (7.20) we have

$$\operatorname{Im} P \subseteq \operatorname{Im}[P - S_*(t) W^T]^T. \tag{7.21}$$

Inclusions (7.18) and (7.21) show that $\operatorname{rank}[P - S_*(t) W^T] \leqslant \operatorname{rank} P$ for $t \in [\alpha, \infty)_\mathbb{T}$. At the same time

$$\operatorname{rank} P \leqslant \operatorname{rank}[P - S_*(t) W^T]^T = \operatorname{rank}[P - S_*(t) W^T], \quad t \in [\alpha, \infty)_\mathbb{T}.$$

Therefore, this implies that

$$\operatorname{rank}[P - S_*(t) W^T] = \operatorname{rank} P = \operatorname{rank}[P - S_*(t) W^T]^T, \quad t \in [\alpha, \infty)_\mathbb{T}$$

which together with (7.18) and (7.21) yields the result in (7.17). Note also that (7.17) implies that the matrices

$$[P - S_*(t) W^T][P - S_*(t) W^T]^\dagger \quad \text{and} \quad [P - S_*(t) W^T]^\dagger [P - S_*(t) W^T]$$

are for $t \in [\alpha, \infty)_\mathbb{T}$ orthogonal projectors onto $\operatorname{Im} P$, i.e., for $t \in [\alpha, \infty)_\mathbb{T}$ we have

$$P = [P - S_*(t) W^T][P - S_*(t) W^T]^\dagger, \quad P = [P - S_*(t) W^T]^\dagger [P - S_*(t) W^T]. \tag{7.22}$$

Let $Q_*(t)$ be the symmetric matrix defined by

$$Q_*(t) := U_*(t) X_*^\dagger(t) + X_*^{\dagger T}(t) U_*^T(t)[I - R_*(t)], \quad t \in [a, \infty)_\mathbb{T},$$

where $R_*(t)$ is the orthogonal projector associated with (X_*, U_*) on $[\alpha, \infty)_\mathbb{T}$ defined in (2.23). Then, in the same way as we received (5.3), we have

$$U_*(t) P_* = Q_*(t) X_*(t), \quad t \in [\alpha, \infty)_\mathbb{T}. \tag{7.23}$$

We also have

$$X^\Delta(t) = \mathscr{A}(t) X(t) + \mathscr{B}(t) U(t) = \mathscr{A}(t) X(t) + \mathscr{B}(t) R_*(t) U(t), \quad t \in [\alpha, \infty)_\mathbb{T}. \tag{7.24}$$

But since using (7.23) we have

$$R_*(t) Q_*(t) X(t) - R_*(t) U(t) = [X_*^\dagger(t)]^T W^T. \tag{7.25}$$

Then from (7.24), considering also that $P_*(t) = P_*$ is constant on $[\alpha, \infty)_{\mathbb{T}}$, we get

$$X^{\Delta}(t) = [\mathscr{A}(t) + \mathscr{B}(t)Q_*(t)]X(t) - \mathscr{B}(t)X_*^{\dagger T}(t)W^T, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{7.26}$$

and using (3.6) we also get

$$X_*^{\Delta}(t) = [-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^{\sigma}(t)]X_*^{\sigma}(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{7.27}$$

Moreover, using $\mathscr{B}^T(t) = \mathscr{B}^T(t)R_*^{\sigma}(t)$ and the fact that Wronskian W is constant on $[a, \infty)_{\mathbb{T}}$ together with (7.25) we get that (X, U) solves the nonhomogeneous version of (7.27), i.e.,

$$X^{\Delta}(t) = [-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^{\sigma}(t)]X^{\sigma}(t) - \mathscr{B}^T(t)[X_*^{\sigma}(t)]^{\dagger T}W^T, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{7.28}$$

Let $\Phi(t)$ and $\Psi(t)$ be the solutions of the associated homogeneous equations

$$\Phi^{\Delta}(t) = [\mathscr{A}(t) + \mathscr{B}(t)Q_*(t)]\Phi(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{7.29}$$

$$\Psi^{\Delta}(t) = [-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^{\sigma}(t)]\Psi^{\sigma}(t), \quad t \in [\alpha, \rho(\beta)]_{\mathbb{T}}, \tag{7.30}$$

such that $\Phi(\alpha) = X(\alpha)$ and $\Psi(\beta) = X(\beta)[P - S_*(\beta)W^T]^{\dagger}$. Note that in the spirit of [27, Proposition 2.1] we do not need the regressivity of the matrix $\mathscr{A}(t) + \mathscr{B}(t)Q_*(t)$ in (7.29), since the solution $\Phi(t)$ is constructed in the forward time. Further, we also do not need the regressivity of the matrix $-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^{\sigma}(t)$ in (7.30), since the solution $\Psi(t)$ is constructed in the backward time. Note that since $\Phi(\alpha)P_* = X(\alpha)P_* = X_*(\alpha)$, we obtain by the uniqueness of solution of (7.29) that

$$X_*(t) = \Phi(t)P_*, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{7.31}$$

In a similar way, the function

$$F(t) := \Phi(t)[P - S_*(t)W^T], \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{7.32}$$

satisfies the equality

$$F(\alpha) = \Phi(\alpha)[P - S_*(\alpha)W^T] = \Phi(\alpha)P = X(\alpha)P = X(\alpha),$$

while using (7.29), (7.32), and (7.31) we have the following equality:

$$\begin{aligned}
F^{\Delta}(t) = \ & \Phi^{\Delta}(t)[P - S_*(t)W^T] + \Phi^{\sigma}(t)[P - S_*(t)W^T]^{\Delta} \\
& \overset{(7.29)}{=} [\mathscr{A}(t) + \mathscr{B}(t)Q_*(t)]\Phi(t)[P - S_*(t)W^T] - \Phi^{\sigma}(t)S_*^{\Delta}(t)W^T \\
& \overset{(7.32)}{=} [\mathscr{A}(t) + \mathscr{B}(t)Q_*(t)]F(t) - \Phi^{\sigma}(t)[X_*^{\sigma}(t)]^{\dagger}\mathscr{B}(t)X_*^{\dagger}(t)W^T \\
& \overset{(7.31)}{=} [\mathscr{A}(t) + \mathscr{B}(t)Q_*(t)]F(t) - R_*^{\sigma}(t)\mathscr{B}(t)X_*^{\dagger}(t)W^T \\
& = [\mathscr{A}(t) + \mathscr{B}(t)Q_*(t)]F(t) - \mathscr{B}(t)X_*^{\dagger}(t)W^T, \quad t \in [\alpha, \infty)_{\mathbb{T}}.
\end{aligned}$$

Therefore, the function $F(t)$ satisfies equation (7.26) with $F(\alpha) = X(\alpha)$. By the uniqueness of solution of (7.26) we get

$$X(t) = F(t) = \Phi(t)[P - S_*(t)W^T], \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{7.33}$$

Now the functions $\Psi(t)$ and $\Psi(t)P$, as well as the function $\Psi(t)P_*$ and $X_*(t)$, satisfy the linear dynamic equation in (7.30) with the initial conditions

$$\Psi(\beta)P = X(\beta)[P - S_*(\beta)W^T]^\dagger P \overset{(7.22)}{=} X(\beta)[P - S_*(\beta)W^T]^\dagger = \Psi(\beta).$$

And similarly by considering (7.33) and (7.31), we also get the second initial condition

$$\Psi(\beta)P_* = \Phi(\beta)[P - S_*(\beta)W^T][P - S_*(\beta)W^T]^\dagger P_* = \Phi(\beta)PP_* = X_*(\beta).$$

Then by the uniqueness of (7.30) we get

$$\Psi(t)P = \Psi(t), \quad \Psi(t)P_* = X_*(t), \quad t \in [\alpha, \beta]_{\mathbb{T}}. \tag{7.34}$$

Consider the function

$$\mathbb{G}(t) := \Psi(t)[P - S_*(t)W^T], \quad t \in [\alpha, \beta]_{\mathbb{T}}. \tag{7.35}$$

Then by using (7.30) and (7.22) we obtain

$$\mathbb{G}(\beta) = \Psi(\beta)[P - S_*(\beta)W^T] \overset{(7.30),(7.22)}{=} X(\beta)P = X(\beta),$$

and further

$$
\begin{aligned}
\mathbb{G}(t) =\ & \Psi(t)[P - S_*(t)W^T]^\sigma + \Psi(t)[P - S_*(t)W^T]^\Delta \\
\overset{(7.30)}{=}\ & [-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^\sigma(t)]\Psi^\sigma(t)[P - S_*^\sigma(t)W^T] - \Psi(t)[S_*^\Delta(t)]^T W^T \\
\overset{(7.35)}{=}\ & [-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^\sigma(t)]\mathbb{G}^\sigma(t) - \Psi(t)X_*^\dagger(t)\mathscr{B}^T(t)[X_*^\sigma(t)]^{\dagger T} W^T \\
\overset{(7.34)}{=}\ & [-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^\sigma(t)]\mathbb{G}^\sigma(t) - \mathscr{B}^T(t)[X_*^\sigma(t)]^{\dagger T} W^T, \quad t \in [\alpha, \rho(\beta)]_{\mathbb{T}}.
\end{aligned}
$$

Thus, the function $\mathbb{G}(t)$ solves equation (7.28) on $[\alpha, \rho(\beta)]_{\mathbb{T}}$ with the initial condition $\mathbb{G}(\beta) = X(\beta)$. The uniqueness of solutions of equation (7.28) yields that

$$X(t) = \mathbb{G}(t) = \Psi(t)[P - S_*(t)W^T], \quad t \in [\alpha, \beta]_{\mathbb{T}}. \tag{7.36}$$

Now we prove that

$$\text{Ker}\,\Psi(t) = \text{Ker}\,P, \quad \Psi^\dagger(t)\Psi(t) = P, \quad t \in [\alpha, \beta]_{\mathbb{T}}. \tag{7.37}$$

We will prove that space $\text{Ker}\,\Psi(t)$ is nonincreasing on $[\alpha, \beta]_{\mathbb{T}}$ by applying the backward version of the time scale induction principle to the statement

$$\mathbb{A}(t) := \text{Ker}\,\Psi(t) \text{ is nonincreasing on } [t, \beta]_{\mathbb{T}},$$

see [7, Remark 1.8] for the method. Applying the principle we have to check the following.

The initial condition. If $t = \beta$, then the statement $\mathbb{A}(\beta)$ holds automatically. The jump condition. Let $t \in (\alpha, \beta]_{\mathbb{T}}$ be left-scattered and assume that $\mathbb{A}(t)$ holds. From (7.30) and (2.1) we get $\Psi^{\sigma}(t) - \Psi(t) = \mu(t)[-\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^{\sigma}(t)]\Psi^{\sigma}(t)$ for $t \in [\alpha, \rho(\beta)]_{\mathbb{T}}$, thus from here we get

$$\Psi(t) = \left(I + \mu(t)\left[\mathscr{D}^T(t) + \mathscr{B}^T(t)Q_*^{\sigma}(t)\right]\right)\Psi^{\sigma}(t), \quad t \in [\alpha, \rho(\beta)]_{\mathbb{T}}.$$

This implies that $\operatorname{Ker}\Psi(t) \subseteq \operatorname{Ker}\Psi(\rho(\beta))$ holds, i.e., the statement $\mathbb{A}(\rho(\beta))$ is valid.

The closure condition. Let $t \in [\alpha, \beta)_{\mathbb{T}}$ be right-dense and assume that statement $\mathbb{A}(\tau)$ holds for all $\tau \in (t, \beta]_{\mathbb{T}}$. Then this implies that $\operatorname{Ker}\Psi(\tau)$ is nonincreasing on $(t, \beta]_{\mathbb{T}}$. Since $\Psi(t)$ is continuous on $[\alpha, \beta]_{\mathbb{T}}$, we get $\operatorname{Ker}\Psi(\tau) \subseteq \operatorname{Ker}\Psi(t)$ for all $\tau \in (t, \beta]_{\mathbb{T}}$, it proves that $\mathbb{A}(t)$ holds.

The continuation condition. Let $t \in (\alpha, \beta]_{\mathbb{T}}$ be left-dense and assume that $\mathbb{A}(t)$ holds. Our aim is to find some $s \in [\alpha, t)_{\mathbb{T}}$ such that $\mathbb{A}(s)$ holds. The fact that the point t is left-dense guarantees that there exists a point $s \in [\alpha, t)_{\mathbb{T}}$ such that the matrix $I + \mu(\tau)[\mathscr{D}^T(\tau) + \mathscr{B}^T(\tau)Q_*^{\sigma}(\tau)]$ is invertible for $\tau \in [s, t)_{\mathbb{T}}$. This implies by [27, Proposition 2.1] that the fundamental matrix $\Omega(\tau)$ of the homogeneous system

$$\Omega^{\Delta}(\tau) = [\mathscr{B}^T(\tau)Q_*^{\sigma}(\tau) - \mathscr{D}^T(\tau)]\Omega^{\sigma}, \quad \tau \in [s, \rho(t)]_{\mathbb{T}} = [s, t]_{\mathbb{T}}, \quad \Omega(t) = I,$$

is invertible on $[s, t]_{\mathbb{T}}$. Consequently, $\Psi(\tau) = \Omega(\tau)\Psi(t)$ for all $\tau \in [s, t]_{\mathbb{T}}$. This implies that $\operatorname{Ker}\Psi(t) \subseteq \operatorname{Ker}\Psi(\tau)$, where $\tau \in [s, t]_{\mathbb{T}}$. But at the same time

$$\operatorname{rank}\Psi(\tau) = \operatorname{rank}[\Omega(\tau)\Psi(t)] = \operatorname{rank}\Psi(t), \quad t \in [s, t]_{\mathbb{T}},$$

since the matrix $\Omega(t)$ is invertible on the interval $[s, t]_{\mathbb{T}}$. This proves that $\operatorname{Ker}\Psi(t) = \operatorname{Ker}\Psi(\tau)$ holds on $\tau \in [s, t]_{\mathbb{T}}$, i.e., the statement $\mathbb{A}(s)$ holds.

Putting the previous four steps together we see that the statement $\mathbb{A}(t)$ holds for all $t \in [\alpha, \beta]_{\mathbb{T}}$. Using this result, we have the inclusion

$$\operatorname{Ker}\Psi(t) \subseteq \operatorname{Ker}\Psi(\alpha) \overset{(7.34)}{=} \operatorname{Ker}[\Psi(\alpha)P] \overset{(7.36)}{=} \operatorname{Ker}X(\alpha) = \operatorname{Ker}P, \quad t \in [\alpha, \beta]_{\mathbb{T}}.$$

On the other hand, (7.34) implies that $\operatorname{Ker}P \subseteq \operatorname{Ker}\Psi(t)$ for $t \in [\alpha, \beta]_{\mathbb{T}}$. Therefore, we obtain that $\operatorname{Ker}\Psi(t) = \operatorname{Ker}P$ holds on $[\alpha, \beta]_{\mathbb{T}}$. Then also

$$\operatorname{Im}P = (\operatorname{Ker}P)^{\perp} = [\operatorname{Ker}\Psi(t)]^{\perp} = \operatorname{Im}\Psi^T(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

This equality implies that

$$\Psi^{\dagger}(t)\Psi(t) = \mathscr{P}_{\operatorname{Im}\Psi^T(t)} = \mathscr{P}_{\operatorname{Im}P} = P, \quad t \in [\alpha, \beta]_{\mathbb{T}}. \tag{7.38}$$

Finally, we prove that the conjoined basis (X, U) has constant kernel on $[\alpha, \beta]_{\mathbb{T}}$ and no focal points in (α, β). We have

$$\operatorname{Ker}X(t) \overset{(7.36)}{=} \operatorname{Ker}\left(\Psi(t)[P - S_*(t)W^T]\right) \overset{(7.38)}{=} \operatorname{Ker}\left(P[P - S_*(t)W^T]\right)$$

$$\overset{(3.12)}{=} \left(\operatorname{Im}[P - S_*(t)W^T]^T\right)^{\perp} \overset{(7.17)}{=} (\operatorname{Im}P)^{\perp} = \operatorname{Ker}P, \quad t \in [\alpha, \beta]_{\mathbb{T}}.$$

Therefore, equation (7.37) holds, so that the kernel of (X,U) is constant on $[\alpha,\beta]_{\mathbb{T}}$. In addition, since for $t \in [\alpha,\beta]_{\mathbb{T}}$ we have that

$$X_*^\dagger(t) = PP_* X_*^\dagger(t) = X^\dagger(t)X(t)P_* X_*^\dagger(t) \overset{(7.14)}{=} X^\dagger(t)R_*(t), \quad t \in [\alpha,\beta]_{\mathbb{T}}, \qquad (7.39)$$

and $P_* = X_*^\dagger(t)X_*(t) = [X_*^\sigma(t)]^\dagger X_*^\sigma(t)$ for $t \in [\alpha,\beta]_{\mathbb{T}}$. It follows, since (X_*,U_*) has no focal points in (α,∞), that

$$0 \leqslant X_*(t)[X_*^\sigma(t)]^\dagger \mathscr{B}(t) \overset{(7.14)}{=} X(t)P_*[X_*^\sigma(t)]^\dagger \mathscr{B}(t) = X(t)[X_*^\sigma(t)]^\dagger \mathscr{B}(t)$$

$$\overset{(7.39)}{=} X(t)[X^\sigma(t)]^\dagger R_*^\sigma(t)\mathscr{B}(t) = X(t)[X^\sigma(t)]^\dagger \mathscr{B}(t), \quad t \in [\alpha,\rho(\beta)]_{\mathbb{T}}.$$

Hence, (X,U) has no focal points in $(\alpha,\beta]$. Consequently, the conjoined basis (X_*,U_*) is contained in (X,U) also on $[\alpha,\infty)_{\mathbb{T}}$. The proof is complete. $\square$

The next theorem is in some sense the extension of [27, Proposition 6.4], which shows that the definition of a principal solution is independent of the point α, when it is moved to the right. We show that it can be also moved to the left until it reaches $\hat{\alpha}_{\min}$ defined in (3.42).

THEOREM 7.3. *Assume that system* ($\mathbb{S}$) *is nonoscillatory, let* $\hat{\alpha}_{\min} \in [a,\infty)_{\mathbb{T}}$ *be defined in* (3.42). *Then if* $(\hat{X},\hat{U})$ *is a principal solution of system* ($\mathbb{S}$) *at infinity with respect to the interval* $[\alpha,\infty)_{\mathbb{T}}$ *for some* $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$, *then it is a principal solution of* ($\mathbb{S}$) *at infinity with respect to the interval* $[\beta,\infty)_{\mathbb{T}}$ *for all* $\beta \in (\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$.

Proof. Note that $d[\hat{\alpha}_{\min},\infty)_{\mathbb{T}} = d_\infty$. Let $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ be as in the theorem. According to Proposition 3.7 we can be sure that the solution $(\hat{X},\hat{U})$ from the theorem contains some minimal principal solution $(\hat{X}_{\min},\hat{U}_{\min})$ on $[\alpha,\infty)_{\mathbb{T}}$, which is uniquely determined up to the right nonsingular multiple, see [27, Theorem 6.9]. But then, according to Proposition 7.1, $(\hat{X},\hat{U})$ contains $(\hat{X}_{\min},\hat{U}_{\min})$ also on the interval $[\beta,\infty)_{\mathbb{T}}$ for all $\beta \in (\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$. And then by Proposition 3.22 we get that $(\hat{X},\hat{U})$ is really a principal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\beta,\infty)_{\mathbb{T}}$ for all $\beta \in (\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$. $\square$

One may wish we could state an analogy of Theorem 7.3 also for the antiprincipal solutions of ($\mathbb{S}$) at infinity, but it is not possible due to [14, Theorem 6.4] and the fact that the minimal antiprincipal solution of ($\mathbb{S}$) at infinity is not uniquely determined.

8. Conclusions

In this paper we developed the theory of genera of conjoined bases for symplectic dynamic systems on time scales and utilized it for obtaining new results about principal and and antiprincipal solutions at infinity. Main goal of the article is to provide all preparatory results we need for our future research, which is already done and ready to submit as Part II of this paper. In our investigations we did not use any controllability (normality) assumption, which leads in natural way to using the Moore–Penrose

pseudoinverse in the situations, where the considered matrices are not invertible. In particular, as a new result we derived the Δ-derivative of the Moore–Penrose pseudoinverse of a matrix with constant kernel on a given time scale interval.

The article opens a door for future research. In the continuation of this article we would investigate the limit properties of principal and antiprincipal solutions of ($\mathbb{S}$) at infinity. In this approach, we would use many of the results contained in this article, which is also the reason why we call it Part I and Part II of the same topic instead of publishing it as two totally separate articles.

In two theorems in our following research, which will be covered in Part II, we will provide classifications of all principal and antiprincipal solutions of ($\mathbb{S}$) at infinity in the genus $\mathscr{G}$ in terms of some known principal solution of ($\mathbb{S}$) at infinity belonging to the same genus $\mathscr{G}$. The main tools to prove these theorems are Propositions 3.10 and 5.6, namely it is the mutual representation of some special conjoined bases and the relation to be contained and its properties related to the inheritance of the property to be a principal or antiprincipal solution at infinity. It seems to be possible to use those tools for deriving the classifications of all principal and antiprincipal solutions of ($\mathbb{S}$) at infinity in the genus $\mathscr{G}$ in terms of some known antiprincipal solution of ($\mathbb{S}$) at infinity belonging to the same genus $\mathscr{G}$. We leave this topic, letting this kind of classification as an open problem. Note that it is an open problem even in the continuous case and also in the discrete case.

A next natural step could be the investigation of an ordering in the set of equivalences given by the relation to belong to the same genus. Once we know that there exists some minimal genus $\mathscr{G}_{\min}$ and the maximal genus $\mathscr{G}_{\max}$, it seems to be a good idea to investigate what happens in between. In the continuous case the ordering on the set of all genera of conjoined bases is described in [26, Theorem 4.8]. Such the result would be new even in the discrete case.

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