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NONTRIVIAL SOLUTIONS FOR A NONLINEAR νTH ORDER ATICI–ELOE FRACTIONAL DIFFERENCE EQUATION SATISFYING DIRICHLET BOUNDARY CONDITIONS

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Dedicated to my friend and collaborator, Professor Paul W. Eloe, honoring him upon his retirement, for his service and contributions to mathematics and academia

(*Communicated by F. Atici*)

Abstract. For $1 < \nu \leqslant 2$ a real number and $T \geqslant 2$ a natural number, by an application of a Krasnosel'skii-Zabreiko fixed point theorem, nontrivial solutions are established for a nonlinear νth order Atıcı-Eloe fractional difference equation, $\Delta^{\nu} u(t) + f(u(t+\nu-1)) = 0$, $t \in \{1,2,\dots,T+1\}$, satisfying the Dirichlet boundary conditions $u(\nu-2) = u(\nu+T+1) = 0$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{|r|\to\infty} \frac{f(r)}{r}$ exists.

1. Introduction

In this paper, for $1 < \nu \leqslant 2$ a real number and $T \geqslant 2$ a natural number, we are concerned with the existence of nontrivial solutions of the nonlinear νth order Atıcı-Eloe fractional difference equation,

$$\Delta^{\nu} u(t) + f(u(t+\nu-1)) = 0, \quad t \in \{1,2,\dots,T+1\}, \tag{1.1}$$

satisfying the Dirichlet boundary conditions

$$u(\nu-2) = u(\nu+T+1) = 0, \tag{1.2}$$

where Δ^{ν} is the Atıcı-Eloe fractional difference and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

There is much current interest in fractional difference equations devoted to both their theoretical development and their applications. Much of this interest is spawned by the definitions, in the context of discrete domains, of fractional sums and fractional differences in the pioneering papers by Atıcı and Eloe [1, 2]. Those papers were further developed and extended in the seminal papers by Goodrich [8, 9, 10]. In their aforementioned papers, Atıcı and Eloe dealt first with a theory for initial value problems for

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fractional difference equations, followed in their second paper by applications of their definitions in obtaining positive solutions for Dirichlet boundary value problems for fractional difference equations. Their second work centered on a Guo-Krasnosel'skii fixed point argument which required the construction of a Green's function for their fractional problem. Other subsequent research by Goodrich that focused on questions expanding the Atıcı and Eloe work can be found in [11, 12, 13, 14, 15]. A couple of recent papers by Henderson [17] and Henderson and Neugebauer [20] were devoted to the existence of local solutions for boundary value problems for Atıcı-Eloe fractional difference equations, and other recent works devoted to Atıcı-Eloe difference equations can be found in, to cite a few, [6, 19, 29].

Discrete fractional calculus and fractional difference equations frequently appear in modeling natural processes such as found in the paper by Atıcı and S. Şengül [3] and in the papers by Magin [23] and Metzler *et al.* [24]. Especially prominent is the current use of boundary value problems for discrete fractional difference equations in their applications to discrete control processes; see, for example, the monographs devoted to discrete fractional control [4, 5, 25, 26, 27].

In this paper, we apply a Krasnosel'skii and Zabreiko fixed point theorem [22] in establishing the existence of nontrivial solutions of (1.1), (1.2). Effective use has been made of that fixed point theorem in showing existence of solutions of boundary value problems (for ordinary differential equations, for difference equations and for dynamic equations) in the context of when the nonlinearity is almost linear at infinity; we cite [7, 16, 18, 21, 28].

2. Some preliminaries and the Green's function

We begin this section with the Atıcı-Eloe definitions of fractional sum and fractional difference in the context of a discrete domain.

DEFINITION 1. Let $n \in \mathbb{N}$ and $n-1 < \kappa \leqslant n$ be a real number, and let $a \in \mathbb{R}$. For $t \in \{a+\kappa, a+\kappa+1, \dots\}$, the κth order *Atıcı-Eloe fractional sum*, $\Delta^{-\kappa}u$, of the function u is defined by

$$\Delta^{-\kappa}u(t) := \frac{1}{\Gamma(\kappa)} \sum_{s=a}^{t-\kappa} (t-s-1)^{(\kappa-1)} u(s),$$

where $t^{(\kappa)} := \frac{\Gamma(t+1)}{\Gamma(t+1-\kappa)}$ is the falling function.

The κth order *Atıcı-Eloe fractional difference*, $\Delta^{\kappa}u$, of the function u is defined by

$$\Delta^{\kappa}u(t) := \Delta^{n-(n-\kappa)}u(t) := \Delta^{n}(\Delta^{-(n-\kappa)}u(t)),$$

where Δ is the forward difference defined by $\Delta u(t) = u(t+1) - u(t)$, and $\Delta^{i}u(t) = \Delta(\Delta^{i-1}u(t))$, $i = 2, 3, \dots$.

REMARK 1. We note that, for u defined on $\{a, a+1, \dots\}$, then $\Delta^{-\kappa}u$ is defined on $\{a+\kappa, a+\kappa+1, \dots\}$.

In [2], for $1 < \nu \leqslant 2$, Atıcı and Eloe, by direct computation, constructed the Green's function, $G(t,s)$, for

$$-\Delta^{\nu} y(t) = 0,\ t \in \{1,2,\dots,T+1\}, \tag{2.1}$$

satisfying the Dirichlet boundary conditions (1.2). $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t^{(\nu-1)}(\nu+T-s)^{(\nu-1)}}{(\nu+T+1)^{(\nu-1)}}, & t-\nu+1 \leqslant s \leqslant T+1, \\ \frac{t^{(\nu-1)}(\nu+T-s)^{(\nu-1)}}{(\nu+T+1)^{(\nu-1)}} - (t-s-1)^{(\nu-1)}, & s < t-\nu+1 \leqslant T+1. \end{cases} \tag{2.2}$$

They also obtained the following properties of $G(t,s)$ which will be of importance to us:

(a) For each $s \in \{1,\dots,T+1\}$,

$$G(\nu-2,s) = 0 \text{ and } G(\nu+T+1,s) = 0.$$

(b) $G(t,s) > 0$, for $(t,s) \in \{\nu-1,\dots,\nu+T\} \times \{1,\dots,T+1\}$.

(c) $\max_{t\in\{\nu-2,\dots,\nu+T+1\}} G(t,s) = G(s+\nu-1,s)$, for $s \in \{1,\dots,T+1\}$.

As stated in the Introduction, we will apply a Krasnosel'skii-Zabreiko fixed point theorem [22] in establishing the existence of nontrivial solutions of (1.1), (1.2). We now state that fixed point theorem.

THEOREM 1. *Let X be a Banach space and $F : X \to X$ be a completely continuous operator. If there exists a bounded linear operator $A : X \to X$ such that 1 is not an eigenvalue and*

$$\lim_{||u||\to\infty} \frac{||F(u) - A(u)||}{||u||} = 0,$$

then F has a fixed point in X.

We will apply Theorem 1 to a nonlinear summation operator whose kernel is the Green's function, $G(t,s)$.

For our setting, let a Banach space $(X, ||\cdot||)$ be defined by

$$X := \{h : \{\nu-2,\dots,T+\nu+1\} \to \mathbb{R} \mid h(\nu-2) = h(T+\nu+1) = 0\}, \tag{2.3}$$

with norm

$$||h|| := \max_{x\in\{\nu-2,\dots,T+\nu+1\}} |h(x)|. \tag{2.4}$$

It is standard that $u \in X$ is a fixed point of the completely continuous operator $F : X \to X$ defined by

$$(Fu)(t) := \sum_{s=1}^{T+1} G(t,s) f(u(s+\nu-1)),\ \ t \in \{\nu-2,\dots,T+\nu+1\}, \tag{2.5}$$

if and only if u is a solution of (1.1), (1.2) (for detailed proofs, see the papers [2] and [17]).

3. Existence results

In this section, we apply Theorem 1 to the operator F defined in (2.5) and to an associated linear operator to obtain solutions of (1.1), (1.2).

THEOREM 2. *Assume* $f:\mathbb{R}\to\mathbb{R}$ *and* $\lim_{|r|\to\infty}\frac{f(r)}{r}=m$. *If*

$$|m| < b := \frac{1}{\sum_{s=1}^{T+1} G(s+\nu-1,s)},$$

then the boundary value problem (1.1), (1.2) *has a solution* u, *and moreover,* $u\neq 0$, *when* $f(0)\neq 0$.

Proof. Let the Banach space $(X,||\cdot||)$ and the completely continuous operator $F:X\to X$ be as defined in Section 2 in (2.3), (2.4) and (2.5), respectively.

Associated with (1.1), (1.2), we consider a linear νth order equation,

$$\Delta^{\nu}u(t)+mu(t+\nu-1)=0,\ \ t\in\{1,\dots,T+1\}, \tag{3.1}$$

satisfying the boundary conditions (1.2), and we define a completely continuous linear operator $A:X\to X$ by

$$(Au)(t):=m\sum_{s=1}^{T+1}G(t,s)u(s+\nu-1),\ \ t\in\{\nu-2,\dots,T+\nu+1\}. \tag{3.2}$$

Of course, solutions of (3.1), (1.2) are fixed points of A, and conversely.

Our first claim is that 1 is not an eigenvalue of A. There are two cases to consider for this claim: (a) $m=0$, and (b) $m\neq 0$.

For (a), if $m=0$, since the boundary value problem (2.1), (1.2) has only the trivial solution, it is immediate that 1 is not an eigenvalue of A.

For (b), if $m\neq 0$ and (3.1), (1.2) has a nontrivial solution, u, then $||u||>0$. And so, we have

$$\begin{aligned}
||u|| &= ||Au|| \\
&= \max_{t\in\{\nu-2,\dots,T+\nu+1\}}\left|m\sum_{s=1}^{T+1}G(t,s)u(s+\nu-1)\right| \\
&= |m|\max_{t\in\{\nu-2,\dots,T+\nu+1\}}\left|\sum_{s=1}^{T+1}G(t,s)u(s+\nu-1)\right| \\
&\leqslant |m|||u||\sum_{s=1}^{T+1}G(s+\nu-1,s) \\
&< b||u||\frac{1}{b} \\
&= ||u||,
\end{aligned}$$

which is a contradiction. So, again, 1 is not an eigenvalue of A.

Next, we exhibit that

$$\lim_{||u||\to\infty} \frac{||F(u)-A(u)||}{||u||} = 0.$$

To that end, let $\varepsilon > 0$ be given. Since $\lim_{|r|\to\infty} \frac{f(r)}{r} = m$, there exists an $M_1 > 0$ such that, for $|r| > M_1$,

$$|f(r) - mr| < \varepsilon|r|. \tag{3.3}$$

Set

$$M = \sup_{|r|\leqslant M_1} |f(r)|,$$

and let $L > M_1$ be such that

$$\frac{M+|m|M_1}{L} < \varepsilon.$$

If we choose $u \in X$ with $||u|| > L$, then for $s \in \{1,\dots,T+1\}$,

(i) if $|u(s+\nu-1)| \leqslant M_1$, we have

$$\begin{aligned} |f(u(s+\nu-1)) - mu(s+\nu-1)| &\leqslant |f(u(s+\nu-1))| + |m||u(s+\nu-1)| \\ &\leqslant M + |m|M_1 \\ &< \varepsilon L \\ &< \varepsilon||u||, \end{aligned}$$

and

(ii) if $|u(s+\nu-1)| > M_1$, we have from (3.3) that

$$|f(u(s+\nu-1)) - mu(s+\nu-1)| < \varepsilon|u(s+\nu-1)| \leqslant \varepsilon||u||.$$

Thus, from (i) and (ii), for all $s \in \{1\dots,T+1\}$,

$$|f(u(s+\nu-1)) - mu(s+\nu-1)| \leqslant \varepsilon||u||. \tag{3.4}$$

It follows from (3.4) that, for $u \in X$ with $||u|| > L$,

$$\begin{aligned} ||F(u)-A(u)|| &= \max_{t\in\{\nu-2,\dots,T+\nu+1\}} \left| \sum_{s=1}^{T+1} G(t,s)[f(u(s+\nu-1)) - mu(s+\nu-1)] \right| \\ &\leqslant \max_{t\in\{\nu-2,\dots,T+\nu+1\}} \sum_{s=1}^{T+1} G(t,s)\,|f(u(s+\nu-1)) - mu(s+\nu-1)| \\ &\leqslant \varepsilon||u|| \sum_{s=1}^{T+1} G(s+\nu-1,s) \\ &= \frac{\varepsilon}{b}||u||. \end{aligned}$$

As a consequence,

$$\lim_{||u||\to\infty} \frac{||F(u)-A(u)||}{||u||} = 0.$$

By Theorem 1, F has a fixed point $u \in X$, and in turn, u is a desired solution of (1.1), (1.2). Moreover, if in addition, $f(0) \neq 0$, it is immediate that $u \neq 0$. The proof is complete. □

As a corollary, we will show that when $f \geqslant 0$, then (1.1), (1.2) has positive solutions.

COROLLARY 1. *Assume $f : [0,\infty) \to [0,\infty)$ is continuous and $\lim_{r\to\infty} \frac{f(r)}{r} = 0$. Then the boundary value problem* (1.1)*,* (1.2) *has a nonnegative solution u, and moreover, u is a positive solution, when $f(0) \neq 0$.*

Proof. Let $\overline{f} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\overline{f}(r) := \begin{cases} f(r), & r \geqslant 0, \\ f(-r), & r < 0. \end{cases}$$

Then, $\overline{f}$ is continuous on $\mathbb{R}$ and $\lim_{|r|\to\infty} \frac{\overline{f}(r)}{r} = 0$. It follows from Theorem 2 that the fractional equation

$$\Delta^{\nu} u(t) + \overline{f}(u(t+\nu-1)) = 0, \ \ t \in \{1,\dots,T+1\}, \tag{3.5}$$

satisfying the boundary conditions (1.2) has a solution u. In particular, u satisifes

$$u(t) = \sum_{s=1}^{T+1} G(t,s)\overline{f}(u(s+\nu-1)), \ \ t \in \{\nu-2,\dots,T+\nu+1\},$$

and hence $u(t) \geqslant 0$, $t \in \{\nu-2,\dots,T+\nu+1\}$. In view of that, $\overline{f}(u(s+\nu-1)) = f(u(s+\nu-1))$, $s \in \{1,\dots,T+1\}$, and so u satisfies (1.1), (1.2). That is, u is a nonnegative solution of (1.1), (1.2). And as before, if $f(0) \neq 0$, then $u \neq 0$. □

REFERENCES

[1] F. M. ATICI AND P. W. ELOE, *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc., **137**, 3 (2009), 981–989.

[2] F. M. ATICI AND P. W. ELOE, *Two-point boundary value problems for finite fractional difference equations*, J. Difference Equ. Appl., **17**, 4 (2011), 445–456.

[3] F. M. ATICI AND S. ŞENGÜL, *Modeling with fractional difference equations*, J. Math. Anal. Appl., **369**, (2010), 1–9.

[4] S. DAS, *Functional Fractional Calculus for System Identification and Controls*, Springer-Verlag, Berlin-Heidelberg, 2009.

[5] S. DAS AND I. PAN, *Fractional Order Signal Processing: Introductory Concepts and Applications*, Springer Briefs in Applied Sciences and Technology, Springer, Heidelberg, 2012.

[6] P. W. ELOE, C. M. KUBLIK, AND J. T. NEUGEBAUER, *Comparison of Green's functions for a family of fractional boundary value problems for fractional difference equations*, J. Difference Equ. Appl., **25**, 6 (2019), 776–787.

[7] F. GENG, D. ZHU AND Q. LU, *A new existence result for impulsive dynamic equations on time scales*, Appl. Math. Lett., **20**, 2 (2007), 206–212.

[8] C. S. GOODRICH, *Continuity of solutions to discrete fractional problems*, Comput. Math. Appl., **59**, (2010), 3489–3499.

[9] C. S. GOODRICH, *Existence of a positive solution to a class of fractional differential equation*, Appl. Math. Lett., **23**, (2010), 1050–1055.

[10] C. S. GOODRICH, *A comparison result for the fractional differential operator*, Int. J. Difference Equ., **1**, 6 (2011), 17–37.

[11] C. S. GOODRICH, *Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions*, Comput. Math. Appl., **61**, (2011), 191–202.

[12] C. S. GOODRICH, *Existence of a positive solution to a system of discrete fractional boundary value problems*, Appl. Math. Comput., **217**, (2011), 4740–4753.

[13] C. S. GOODRICH, *Coercivity of linear functionals on finite dimensional spaces and its application to discrete BVPs*, J. Difference Equ. Appl., **22**, 5 (2016), 623–636.

[14] C. S. GOODRICH, *Summation equations with sign changing kernels and applications to discrete fractional boundary value problems*, Comment. Math. Univ. Carolin., **57**, 2 (2016), 201–229.

[15] C. S. GOODRICH, *Coercive nonlocal elements in fractional differential equations*, Positivity, **21**, 1 (2017), 377–394.

[16] J. HENDERSON, *Nontrivial solutions to a nonlinear boundary value problem on a time scale*, Comm. Appl. Nonlinear Anal., **11**, 1 (2004), 65–71.

[17] J. HENDERSON, *Existence of local solutions for fractional difference equations with Dirichlet boundary conditions*, J. Difference Equ. Appl., **25**, 6 (2019), 751–756.

[18] J. HENDERSON AND B. A. LAWRENCE, *Existence of solutions for even ordered boundary value problems on a time scale*, Difference Equations, Special Functions and Orthogonal Polynomials, 369–377, World Sci. Publ., Hackensack, NJ, 2007.

[19] J. HENDERSON AND J. T. NEUGEBAUER, *Smallest eigenvalues for a fractional difference equation with right focal boundary conditions*, J. Difference Equ. Appl., **23**, 8 (2017), 1317–1323.

[20] J. HENDERSON AND J. T. NEUGEBAUER, *Existence of local solutions for fractional difference equations with left focal boundary conditions*, Frac. Calc. Appl. Anal., **24**, 1 (2021), 324–331.

[21] B. KARNA AND B. A. LAWRENCE, *An existence result for a multipoint boundary value problem on a time scale*, Adv. Difference Equ., **2006**, Art. ID 63208, 8 pp.

[22] M. A. KRASNOSEL'SKII AND P. P. ZABREIKO, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, New York, 1984.

[23] R. L. MAGIN, *Fractional Calculus in Bioengineering*, Begell House, Inc., Redding, CT, 2006.

[24] F. METZLER, W. SCHICK, H. G. KILIAN AND T. F. NONNENMACHER, *Relaxation in filled polymers: a fractional calculus approach*, J. Chem. Phys., **103** (1995), 7180–7186.

[25] P. OSTALCVZYK, *Discrete Fractional Calculus: Applications in Control and Image Processing*, World Scientific Publishing Co. Pte. Ltd., Singapore, Hackensack, NJ, 2016.

[26] A. OUSTALOUP, *Diversity and Non-integer Differentiation for System Dynamics (Control, Systems and Industrial Engineering)*, Wiley and Sons, Inc., Hoboken, NJ, 2014.

[27] I. PETRÁS, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, Series in Nonlinear Physical Science, Springer, Heidelberg, 2011.

[28] J.-P. SUN, *A new existence theorem for right focal boundary Value problems on a measure chain*, Appl. Math Lett., **18**, 1 (2005), 41–47.

[29] A. YANG, L. ZHANG AND J. HENDERSON, *Comparison of smallest eigenvalues for right focal Atıcı-Eloe fractional difference equations*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math., **24**, 4 (2017), 191–200.

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