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OPTIMAL CONTROL FOR AN ORDINARY DIFFERENTIAL EQUATION ONLINE SOCIAL NETWORK MODEL

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Dedicated to Professor Paul Eloe on the occasion of his retirement

(*Communicated by J. Lyons*)

Abstract. In this paper, we propose a set of ordinary differential equation models for online social networks and then consider the optimal control problem subject to a type of objective functions. Numerical simulations are conducted to demonstrate the applications as well.

1. Introduction

Nowadays, online social networks (OSNs, or online social media) have greatly changed the way we live our lives by providing important virtual platforms for socialization, entertainment, and commercial activities [13, 14]. There is an urgent demand of the reliable mathematical models on the dynamics of OSN user adoption and abandonment due to their impacts on the business decisions of an OSN. A search of the literature shows that most of the results on OSNs concentrated on the mathematical models of information diffusion, see for example, [6, 10, 11, 12, 16]; and there appears to be little work on the dynamics of OSN user adoption and abandonment. To the best of our knowledge, the only published papers studying OSN user adoption and abandonment are [4, 8]. There is clearly a demand to develop reliable mathematical models to analyze and understand the dynamics of OSN user adoption and abandonment. Motivated by this demand, mathematical models have been developed in [5, 9] based on the analogy between the OSN user adoption and abandonment process and the infectious disease transmission dynamics.

In [9], the authors developed and investigated a set of ordinary differential equation (ODE) models for OSN user dynamics. The total population eligible to use an OSN product, denoted by N, was divided into three compartments, potential users $\mathcal{S}$, current users $\mathcal{I}$, and OSN opponents $\mathcal{R}$ with

$$N \equiv \mathcal{S}+\mathcal{I}+\mathcal{R}.$$

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Following a transmission scheme described by Figure 1 with the parameters given in Table 1, the following ODE model on the numbers of users in three compartments was obtained

$$\begin{cases} \frac{d\mathcal{S}}{dt} = \Lambda N - \delta \frac{\mathcal{S}\mathcal{I}}{N} - \mu \mathcal{S}, \\ \frac{d\mathcal{I}}{dt} = \delta \frac{\mathcal{S}\mathcal{I}}{N} - \eta \frac{\mathcal{I}\mathcal{R}}{N} - \nu \mathcal{I} - \mu \mathcal{I}, \\ \frac{d\mathcal{R}}{dt} = \eta \frac{\mathcal{I}\mathcal{R}}{N} + \nu \mathcal{I} - \mu \mathcal{R}. \end{cases} \tag{1.1}$$

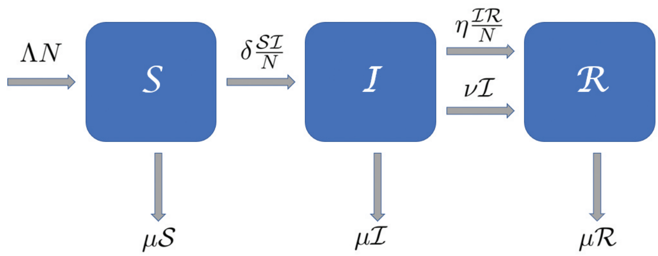

Figure 1: *The instantaneous changes among three compartments.*

Table 1: *Descriptions of the model parameters.*

Notations	Meaning	Units
$\mathcal{S}$	Potential OSN users	Number of people
$\mathcal{I}$	Current OSN users	Number of people
$\mathcal{R}$	Population opposed to OSN use	Number of people
N	The summation of $\mathcal{S}$, $\mathcal{I}$, and $\mathcal{R}$	Number of people
Λ	Recruitment/migration rate of $\mathcal{S}$	[Unit of time]$^{-1}$
δ	Transmission rate at which potential users join OSN	[Unit of time]$^{-1}$
η	OSN infectious abandonment rate as a result of interactions with people who opposed to OSN use	[Unit of time]$^{-1}$
μ	Per capita removal rate of population due to natural death, migration and other reasons	[Unit of time]$^{-1}$
ν	OSN noninfectious abandonment rate without being influenced by other users who opposed to OSN use	[Unit of time]$^{-1}$

REMARK 1. It is notable that model (1.1) significantly distinguishes itself from the classic epidemiological SIR models in the compartment transmission mechanism: for infectious epidemiological models, an infected person can recover without the need of interacting with recovered persons. However, the infectious abandonment dynamics

$\eta \frac{\mathcal{I}\mathcal{R}}{N}$ does exist in modelling OSNs. This feature raises new mathematical challenges in the study of compartment modeling, and clearly demonstrates the necessity and significance of the research problem.

The authors further defined the relative proportionals of each compartment with respect to the total population N by

$$S = \frac{\mathcal{S}}{N}, \quad I = \frac{\mathcal{I}}{N}, \quad \text{and} \quad R = \frac{\mathcal{R}}{N},$$

and derived an ODE model of the relative proportionals

$$\begin{cases} \frac{dS}{dt} = \Lambda - \delta IS - \Lambda S, \\ \frac{dI}{dt} = \delta IS - \eta IR - \nu I - \Lambda I, \\ \frac{dR}{dt} = \eta IR + \nu I - \Lambda R. \end{cases} \tag{1.2}$$

This model was then reduced to a planar system

$$\begin{cases} \frac{dI}{dt} = \delta I(1 - I - R) - \eta IR - \nu I - \Lambda I, \\ \frac{dR}{dt} = \eta IR + \nu I - \Lambda R, \end{cases} \tag{1.3}$$

by the fact that $S + I + R \equiv 1$. A series of criteria on the existence, uniqueness, and positivity of solutions as well as the stability of the user-free equilibrium for models (1.2) and (1.3) were obtained. Numerical simulations and a case study utilizing the real-world data were also conducted. The reader is referred to [9] for the details of the development and qualitative analysis of the models.

Besides the long-term asymptotic behaviors considered in [9], another important research topic on those models is to study how the model states change subject to artificial interference, i.e. the control problems. The reader is referred to [1, 2, 3, 7, 15, 17, 18] and the references therein for some classic results and applications of control theory. As a continuation of the work in [9], in this paper, we will concentrate on the short-term behavior of the OSN models under artificial interference. Particularly, we will extend models (1.2) and (1.3) by introducing a control term and investigate the optimal control problem subject to a type of objective functions. Numerical simulations will also be carried out to demonstrate the applications. This work will showcase the feasibility to boost the OSN usage in a short period by exerting an external effort and how to achieve the optimal solution.

This paper is organized as follows: after this introduction, the main results are presented in Section 2. Numerical simulations are given as demonstrations in Section 3. Section 4 contains a summary and discussion.

2. Main results

Throughout this paper, we assume all the parameters given in Table 1 are positive. We first modify model (1.2) by adding a new transmission term $c(t)R$,

$$\begin{cases} \frac{dS}{dt} = \Lambda - \delta IS - \Lambda S, \\ \frac{dI}{dt} = \delta IS - \eta IR - \nu I - \Lambda I + c(t)R, \\ \frac{dR}{dt} = \eta IR + \nu I - \Lambda R - c(t)R, \end{cases} \tag{2.1}$$

with the initial condition

$$\begin{cases} S(0) = S_0, \\ I(0) = I_0, \\ R(0) = R_0, \end{cases}$$

where

$$c \in \hat{\mathcal{U}} := \{c(\cdot) \text{ measurable}, c(t) \in [0,1],\ t \in \mathbb{R}_+ := [0,\infty)\}.$$

REMARK 2. The term cR represents the effort of some actionable policies to attract the people from Compartment R. By Figure 1 and Table 1, Compartment R consists of the people who quit the OSN due to various reasons. Therefore, cR can be implemented by launching marketing events targeting the previous OSN users. In this paper, the function c will be the control. We also assume that the cost of those marketing events is positively proportional to the value of c. Therefore, the cost can be reduced by minimizing c.

We now consider the existence, uniqueness, and positivity of solutions of model (2.1).

THEOREM 1. *For any initial condition* $(S_0, I_0, R_0) \in \Delta := \{(x,y,z) \in \mathbb{R}^3_+ \mid x+y+z = 1\}$ *and* $c \in \hat{\mathcal{U}}$, *model* (2.1) *has a unique solution, denoted by* (S,I,R). *Moreover,* $(S(t), I(t), R(t)) \in \Delta$ *for any* $t \geqslant 0$.

Proof. Let $Y = S+I+R$. By (2.1), $\frac{dY}{dt} = \Lambda - \Lambda Y$. This implies $Y(t) = 1 + (Y(0)-1)e^{-\Lambda t}$, $t \geqslant 0$. By the initial condition $Y(0) = S_0 + I_0 + R_0 = 1$, we have $Y(t) = S(t)+I(t)+R(t) \equiv 1$, $t \geqslant 0$. Hence $(S(t), I(t), R(t))$ exists on $\mathbb{R}_+$.

It is easy to see that the vector field of model (2.1) is measurable in t for any $(S,I,R) \in \mathbb{R}^3_+$ and continuously differentiable with respect to S, I, and R for all $t \in \mathbb{R}_+$. Therefore, model (2.1) has a unique solution for any initial condition $(S_0, I_0, R_0) \in \Delta$.

We claim that for any $(S_0, I_0, R_0) \in \Delta$, (S,I,R) will stay in the first octant. It is sufficient to investigate three cases at the boundaries, i.e., $S = 0$, $I = 0$, or $R = 0$.

When $S=0$, $I\geqslant 0$, and $R\geqslant 0$, the first equation of (2.1) implies that $\frac{dS}{dt}=\Lambda>0$; when $I=0$, $S\geqslant 0$, and $R\geqslant 0$, the second equation of (2.1) implies that $\frac{dI}{dt}=c(t)R\geqslant 0$; when $R=0$, $S\geqslant 0$, and $R\geqslant 0$, the third equation of (2.1) implies that $\frac{dR}{dt}=\nu I\geqslant 0$. Therefore, (S,I,R) will stay in the first octant. Therefore, $(S(t),I(t),R(t))\in\Delta$ for any $t\geqslant 0$. $\square$

In the rest of the paper, we will only consider the initial values $(S_0,I_0,R_0)\in\Delta$. Theorem 1 allows us to reduce model (2.1) to the following planar system

$$\begin{cases}\frac{dI}{dt}=\delta I(1-I-R)-\eta IR-\nu I-\Lambda I+c(t)R,\\ \frac{dR}{dt}=\eta IR+\nu I-\Lambda R-c(t)R,\end{cases}\tag{2.2}$$

by letting $S=1-I-R$. We now introduce an equivalent vector form of model (2.2). Let

$$x=\begin{bmatrix}I\\R\end{bmatrix}\quad\text{and}\quad f(x,c)=\begin{bmatrix}\delta I(1-I-R)-\eta IR-\nu I-\Lambda I+c(t)R\\ \eta IR+\nu I-\Lambda R-c(t)R\end{bmatrix}.\tag{2.3}$$

Then model (2.2) is equivalent to

$$\frac{dx}{dt}=f(x,c).\tag{2.4}$$

By Theorem 1, it is clear that $(I(t),R(t))\in\Delta_2:=\{(x,y)\in\mathbb{R}^2_+\mid 0\leqslant x+y\leqslant 1\}$ for any $t\geqslant 0$. We will only consider the initial values $(I_0,R_0)\in\Delta_2$ and use $c(t)$ as the control to minimize the problem

$$\min_{c\in\mathcal{U}}\int_0^T\left[-I(s)+w(s)(c(s))^2\right]ds\tag{2.5}$$

over a finite time interval $t\in[0,T]$, where $w\in C[0,T]$ is a nonnegative weight function and

$$c\in\mathcal{U}:=\{c(\cdot)\text{ measurable, } c(t)\in[0,1],\ t\in[0,T]\}.\tag{2.6}$$

Clearly, (2.5) implies to maximize I with the minimum effort c and w can be used to specify various demands on c.

By Theorem 1, all admissible trajectories of model (2.2) remain uniformly bounded as $t\in[0,T]$. Then we have the following result on the existence of optimal control.

THEOREM 2. *The minimization problem* (2.5) *for the model* (2.2) *with any initial condition* $(I_0,R_0)\in\Delta_2$ *has an optimal solution.*

Theorem 2 is proven by modifying the proof of [2, Theorem 5.2.1]. We only give the sketch of the proof below. The reader is referred to [2, Theorem 5.2.1] for the details.

Proof of Theorem 2. Let $L : \mathbb{R}^3 \to \mathbb{R}$ be the integrand in (2.5):

$$L(t, I(t), c(t)) = -I(t) + w(t)[c(t)]^2, \quad t \in [0, T]. \tag{2.7}$$

By Theorem 1, there exists a positive constant M with

$$|L(t, I(t), c)| < M$$

for all $c(t) \in [0,1]$ and $I(t) \in [0,1]$, $t \in [0,T]$.

Now we define the auxiliary system

$$\begin{cases} \frac{dz}{dt} = c_0(t)M + (1 - c_0(t))L(t, I, c(t)), \\ \frac{dI}{dt} = \delta I(1 - I - R) - \eta IR - \nu I - \Lambda I + c(t)R, \\ \frac{dR}{dt} = \eta IR + \nu I - \Lambda R - c(t)R, \end{cases} \tag{2.8}$$

with the control $(c_0, c) \in \mathcal{U} \times \mathcal{U}$ and the initial condition $(0, I_0, R_0)$, and consider the minimization problem:

$$\min_{(c_0,c)\in\mathcal{U}\times\mathcal{U}} z(T; c_0, c) = \min_{(c_0,c)\in\mathcal{U}\times\mathcal{U}} \int_0^T [c_0(s)M + (1 - c_0(s))L(s, I(s), c(s))]ds. \tag{2.9}$$

By the convexity of L in c and [2, Theorem 5.1.1], there exists an optimal control (c_0^*, c^*) for the auxiliary minimization problem (2.8), (2.9). We claim that $c_0^*(t) = 0$ for almost all $t \in [0,T]$ as otherwise we will have $z^*(T; 0, c^*) < z^*(T; c_0^*, c^*)$ since $L(t, I^*(t), c^*(t)) < M$ on $[0,T]$, where (z^*, I^*, R^*) denotes the solution corresponding to (c_0^*, c^*).

Therefore, c^* is an optimal control of the original minimization problem (2.2), (2.5) with (I^*, R^*) the optimal solution. $\square$

By the Pontryagin Maximum Principle [18, Theorem III.3.1], we also obtain the necessary conditions for the optimal solution. Let x and f be defined by (2.3), L be defined by (2.7),

$$f_x(x,c) = \begin{bmatrix} \delta - 2\delta I - \delta R - \eta R - \nu - \Lambda & -\delta I - \eta I + c(t) \\ \eta I + \nu & \eta I - \Lambda - c(t) \end{bmatrix}, \quad \text{and} \quad L_x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

THEOREM 3. *Let $c^* \in \mathcal{U}$ and $x^* = (I^*, R^*)$ be a solution of the minimization problem (2.5) for the model (2.2) with an initial condition $(I_0, R_0) \in \Delta_2$. Let $p = (p_1(t), p_2(t))$ be the solution of the adjoint system*

$$\begin{cases} \frac{dp}{dt} = -f_x(x^*, c^*)p + L_x, \quad t \in [0,T], \\ p(T) = (0,0). \end{cases}$$

Then for any $t \in (0,T)$,

$$p(t)f(x^*(t), c^*(t)) + L(t, I^*(t), c^*(t)) = \max_{\alpha\in[0,1]} \{p(t)f(x^*(t), \alpha) - L(t, I^*(t), \alpha)\}.$$

3. Numerical simulations

In this section, we numerically solve the minimization problem (2.5) subject to model (2.2) with different weight functions. All the simulations use the same model parameters given in Table 2 and the initial value $(I_0, R_0) = (0.23797239778150392, 0)$, which are taken from [9].

Table 2: *ODE model parameters.*

Parameter	Λ	δ	μ	η	ν
Value	0.17978378	0.27074364	0.06651327	0.01184467	0.01254274

We consider four weight functions w_i, $i = 1, \dots, 4$ defined by

$$w_1(t) \equiv 0.01,\ t \in [0,4],$$
$$w_2(t) \equiv 0.03,\ t \in [0,4],$$
$$w_3(t) \in C[0,4] \text{ with } w_3(t) = \begin{cases} 0.1,\ t \in \cup_{k=0}^{3} [k+0.25+1/\varepsilon, k+1-1/\varepsilon], \\ 0.005,\ t \in \cup_{k=0}^{3} [k+1/\varepsilon, k+0.25-1/\varepsilon], \end{cases}$$
$$w_4(t) \in C[0,4] \text{ with } w_4(t) = \begin{cases} 0.005,\ t \in [0, 1.5-1/\varepsilon], \\ 0.055,\ t \in [1.5+1/\varepsilon, 4], \end{cases}$$

where $\varepsilon \gg 0$. The graphs of w_i, $i = 1, \dots, 4$, are given in Figure 2. It is clear that the goal of w_1 and w_2 is to minimize the cost over the entire interval $[0,4]$ with different weights; the goal of w_3 is to emphasize the minimization of the cost on $(0.25,1) \cup (1.25,2) \cup (2.25,3) \cup (3.25,4)$; and the goal of w_4 is to emphasize the minimization of the cost on $(1.5,4]$.

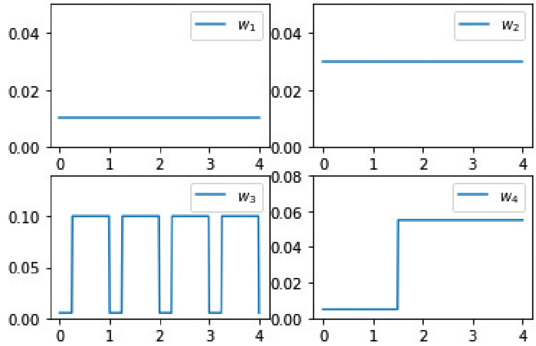

Figure 2: *Graphs of the weight functions.*

The optimization problems are solved by Python. The solutions (I only) and the corresponding controls are plotted in Figure 3 and Figure 4 respectively. The solution without control ($c(t) \equiv 0$ on $[0,4]$) is also plotted in Figure 3 as a benchmark. It is clear that all the controls increase the number of OSN users I as expected. Note that $w_1(t) < w_2(t)$ on $[0,4]$ implies that w_1 has less concern than w_2 on the total cost, i.e., the policy with respect to w_1 can allocate more resources than the policy with respect to w_2. As a result, the I with respect to w_1 is larger than the I with respect to w_2.

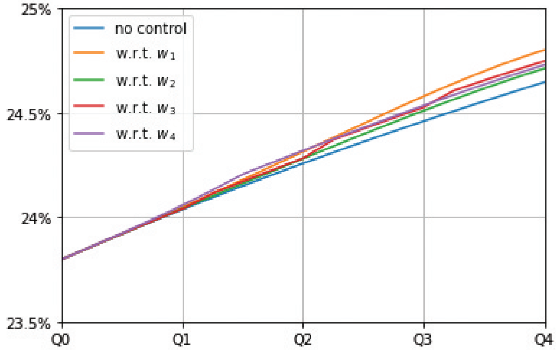

Figure 3: *Numerical solutions (I) with respect to (w.r.t.) various weight functions.*

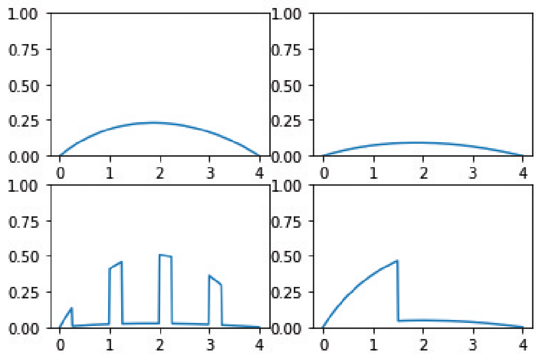

Figure 4: *Upper left: the control w.r.t. w_1. Upper right: the control w.r.t. w_2. Lower left: the control w.r.t. w_3. Lower right: the control w.r.t. w_4.*

4. Conclusions and discussions

In this paper, we extended models (1.2) and (1.3) to (2.1) and (2.2) by introducing a control term that represents the effort to attract the people who quit the OSN due to various reasons. The optimal control problem subject to a type of objective functions was investigated and numerical simulations were used to demonstrate the applications.

The results show that the inclusion of a control term does not impact the uniqueness and positivity of the solutions for the OSN models, and the control problem has an optimal solution for any initial conditions. The practical meaning of the control term further demonstrates that it is feasible to increase the OSN users in a short period by attracting some previous users who quit the OSN. Moreover, the involvement of weighted functions in objective functions reflects the level of emphasis on the cost and allows us to derive various policies (controls) to balance the maximization of OSN users and the minimization of the cost. Therefore, this paper supplements the work in [9] and the approach presented in this paper will be a practical reference for business planning.

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