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QUALITATIVE ANALYSIS OF DYNAMIC EQUATIONS ON TIME SCALES USING LYAPUNOV FUNCTIONS

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Abstract. We employ Lyapunov functions to study boundedness and stability of dynamic equations on time scales. Most of our Lyapunov functions involve the term $|x|$ and its Δ-derivative. In particular, we prove general theorems regarding qualitative analysis of solutions of delay dynamical systems and then use Lyapunov functionals that partially include $|x|$ to provide examples.

1. Introduction

We assume the reader is familiar with the concept on time scales and for introductory and basic materials we refer the reader to [5]. On the other hand when the time scale is the set of discrete numbers, we refer the reader to [8], [9], [14], and [19].

We are interested in the study of certain dynamic equations on time scales and to analyze the asymptotic properties of their solutions. There has been some recent interest in the topic and we refer to [4, 11, 12], and the bibliography therein. We observe here that in the study of stability, boundedness and the existence of solutions of a given dynamical equation on time scales, there is a good chance that Lyapunov functions/functionals will be used, and most likely will involve the term $|x|$. As consequence, the delta derivative of $|x|$. Let $\mathbb{T}$ be an arbitrary time scale and let $x:\mathbb{T}\to\mathbb{R}\setminus\{0\}$. If $\mathbb{T}=\mathbb{R}$, then one can easily find $\frac{d}{dt}|x(t)| = \frac{x(t)}{|x(t)|}x'(t)$ by using the equation $x^2(t)=|x(t)|^2$ and the product rule in real case. However, due to the product rule on time scales ($(fg)^\Delta = f^\Delta g^\sigma + fg^\Delta$), such straightforward calculations is not possible.

Let $\sigma(t)$ and $\mu(t)$ be the forward jump operator and the graininess function, respectively of the time scale $\mathbb{T}$. We have the following lemmas.

LEMMA 1. *For any $t\in\mathbb{T}$, we have*

$$|x|^\Delta = \frac{x+x^\sigma}{|x|+|x^\sigma|}x^\Delta \text{ for } x\neq 0. \tag{1}$$

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Proof. For the proof see [4, p.13]. □

We observe that the factor on the right side of (1) that multiplies x^{Δ} depends not only on the sign of $x(t)$ but also on that of $x^{\sigma}(t)$. Hence, the expression $|x|^{\Delta} = \frac{x}{|x|}x^{\Delta}$ holds only if $xx^{\sigma} \geqslant 0$ and $x \neq 0$. We will try to separate this case from the case $xx^{\sigma} < 0$ by separating the time scale $\mathbb{T}$ into two parts as follows (see also ([3]))

$$\begin{aligned} \mathbb{T}_x^- &:= \{s \in \mathbb{T} : x(s)x^{\sigma}(s) < 0\}, \\ \mathbb{T}_x^+ &:= \{s \in \mathbb{T} : x(s)x^{\sigma}(s) \geqslant 0\}. \end{aligned} \tag{2}$$

Observe that the set $\mathbb{T}_x^-$ consists only of right scattered points of $\mathbb{T}$. The next result gives a clear distinction between the relation of $|x|^{\Delta}$ and $\frac{x}{|x|}x^{\Delta}$.

LEMMA 2. [3, Lemma 5] *Let $x \neq 0$ be Δ-differentiable, then*

$$|x(t)|^{\Delta} = \begin{cases} \frac{x(t)}{|x(t)|}x^{\Delta}(t) & \textit{if } t \in \mathbb{T}_x^+ \\ -\frac{2}{\mu(t)}|x(t)| - \frac{x(t)}{|x(t)|}x^{\Delta}(t) & \textit{for } t \in \mathbb{T}_x^- \end{cases}.$$

In particular,

$$\frac{x}{|x|}x^{\Delta} \leqslant |x|^{\Delta} \leqslant -\frac{x}{|x|}x^{\Delta}(t) \textit{ for all } t \in \mathbb{T}_x^-. \tag{3}$$

Proof. We refer to [3, Lemma 5]. □

Lemma 2 implies the following.

REMARK 1. If $xx^{\sigma} \neq 0$, then

$$\frac{x}{|x|}x^{\Delta} \leqslant |x|^{\Delta} \leqslant \frac{x^{\sigma}}{|x^{\sigma}|}x^{\Delta} \text{ for } t \in \mathbb{T}. \tag{4}$$

Note that if $xx^{\sigma} \neq 0$ and $t \in \mathbb{T}_x^+$, then $\frac{x}{|x|} = \frac{x^{\sigma}}{|x^{\sigma}|}$, and therefore, (4) gives $\frac{x}{|x|}x^{\Delta} = |x|^{\Delta}$. Moreover, if $t \in \mathbb{T}_x^-$, then $-\frac{x}{|x|} = \frac{x^{\sigma}}{|x^{\sigma}|}$ and the inequality (4) is equivalent to (3).

In Sections 2 and 3, dynamic equations and delay dynamic equations are considered, respectively. In addition, we state some stability definitions and make use of Lyapunov functionals to obtain results concerning boundedness of solutions and stability of the zero solution. In Section 2, we recall few results from [1].

2. Boundedness and stability

In this section we define what Peterson and Tisdell [13] call a type I Lyapunov function and summarize a few of the results. Through this paper we use the notation

$$[t_0, \infty)_{\mathbb{T}} =: [t_0, \infty) \cap \mathbb{T}.$$

We begin by considering the boundedness and uniqueness of solutions to the first-order dynamic equation

$$x^{\Delta} = f(t,x), \quad t \geqslant 0, \tag{5}$$

subject to the initial condition

$$x(t_0) = x_0, \quad t_0 \geqslant 0, \quad x_0 \in \mathbb{R}, \tag{6}$$

where here $x(t) \in \mathbb{R}^n$, $f : [0,\infty)_{\mathbb{T}} \times D \to \mathbb{R}^n$ where $D \subset \mathbb{R}^n$ and open and f is a continuous, nonlinear function and t is from a so-called time scale" $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$). Throughout this section we assume $0 \in \mathbb{T}$ (for convenience) and $f(t,0) = 0$ when discussing stability, for all t in the time scale interval $[0,\infty)_{\mathbb{T}}$, and call the zero solution the trivial solution of (5). Equation (5) subject to (6) is known as an initial value problem (IVP) on time scales.

DEFINITION 1. We say $V : \mathbb{R}^n \to \mathbb{R}^+$ is a *"type I" Lyapunov function* and say "type I" on $\mathbb{R}^n$ provided

$$V(x) = \sum_{i=1}^{n} V_i(x_i) = V_1(x_1) + \ldots + V_n(x_n),$$

where each $V_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable and $V_i(0) = 0$.

The following Chain Rule shall be very useful throughout the remainder of this paper. Its proof can be found in Potzsche [15] and Bohner and Peterson [5], Theorem 1.90.

THEOREM 1. *Let $V : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $x : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $V \circ x$ is delta differentiable and*

$$[V(x(t))]^{\Delta} = \left\{ \int_0^1 V'\left(x(t) + h\mu(t)x^{\Delta}(t)\right) dh \right\} x^{\Delta}(t).$$

Now assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a "type I" function and x is a solution to (5). Then by the results of [1] we are motivated to define $\dot{V} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ by either of the following identities

$$\begin{aligned} \dot{V}(t,x) &= \left\{ \int_0^1 \nabla V(x + h\mu(t)f(t,x))dh \right\} f(t,x) \\ &= \sum_{i=1}^{n} \left\{ \int_0^1 V_i'(x_i + h\mu(t)f_i(t,x))\, dh \right\} f_i(t,x) \\ &= \begin{cases} \sum_{i=1}^{n} \{V_i(x_i + \mu(t)f_i(t,x)) - V_i(x_i)\} / \mu(t), & \text{when } \mu(t) \neq 0, \\ \sum_{i=1}^{n} V_i'(x_i) f_i(t,x), & \text{when } \mu(t) = 0, \end{cases} \end{aligned} \tag{7}$$

For $x \in \mathbb{R}^n$, $||x||$ denotes the Euclidean norm of x. For any $n \times n$ matrix A, define the norm of A by $|A| = \sup\{|Ax| : ||x|| \leqslant 1\}$. If x is scalar then we use the supremum norm.

For the purpose of studying the stability of the zero solution we ask that $f(t,0)=0$. Below we state some stability definitions on time scales. Similar ones can be found for the continuous case by referring to [6].

DEFINITION 2. We say a solution $x(t)$ of (5) is bounded if for any $t_0 \in [0,\infty)$ and number r there exists a number $\alpha(t_0,r)$ depending on t_0 and r such that $||x(t,t_0,x_0)|| \leqslant \alpha(t_0,r)$ for all $t \geqslant t_0$ and x_0, $|x_0|<r$. It is uniformly bounded if α is independent of t_0.

DEFINITION 3. The zero solution of (5) is stable (S) if for each $\varepsilon>0$, there is a $\delta=\delta(\varepsilon)>0$ such that $|x_0|<\delta$ implies $|x(t,t_0,x_0)|<\varepsilon$. It is uniformly stable (US) if δ is independent of t_0.

DEFINITION 4. The zero solution of (5) is uniformly asymptotically stable (UAS) if it is (US) and there exists a $\gamma>0$ with the property that for each $\mu>0$ there exists $T=T(\gamma)>0$ such that $|x_0|<\gamma, t \geqslant t_0+T$ implies $|x(t,t_0,x_0)|<\mu$.

Throughout this paper we denote wedges by Z_i, $i=1,2,3,....$ such that $Z_i[0,\infty)_{\mathbb{T}} \to [0,\infty)$ be continuous with $Z_i(0)=0$, $Z_i(r)$ strictly increasing, and $Z_i(r)\to\infty$ as $r\to\infty$.

THEOREM 2. *Suppose there exists a "type I" function $V(t,x)$ and Z_1 such that*

$$Z_1(||x||) \leqslant V(t,x),\ t \geqslant t_0 \tag{8}$$

$$\dot{V}(t,x) \leqslant 0, \tag{9}$$

where $\dot{V}(t,x)$ is given by (7) and

$$Z_1(||x||) \to \infty,\ as\ ||x|| \to \infty. \tag{10}$$

Assume for any initial time t_0 with $x(t_0)=x_0$, and $V(t_0,x_0)$ is bounded, then solutions of (5) are bounded. If the bound on $V(t_0,x_0)$ is uniform regardless of t_0, then the solutions are said to be uniformly bounded (UB).

Proof. We refer to [1, Lemma 5]. □

The next example is to illustrate the use of such Lyapunov "type I" functions.

EXAMPLE 1. Consider the scalar nonlinear Volterra integro-dynamic equation

$$x^{\Delta}(t)=a_1(t)x(t)+a_2(t)\frac{x(t)}{1+x^2(t)},\ t\in[0,\infty)_{\mathbb{T}}. \tag{11}$$

Let the function $\alpha:\mathbb{T}\to\mathbb{R}$ be defined by

$$\alpha(t):=\begin{cases}-\frac{2}{\mu(t)}-a_1(t) & \text{for } t\in[0,\infty)_{\mathbb{T}_x^-}\\ a_1(t) & \text{for } t\in[0,\infty)_{\mathbb{T}_x^+}\end{cases}. \tag{12}$$

If there exists a $\beta > 0$ such that

$$\alpha(t) + |a_2(t)| \leqslant -\beta \text{ for all } t \in [0,\infty)_{\mathbb{T}}$$

then all solutions of (11) are bounded. To see this, consider the Lyapunov function

$$V(t,x) = |x(t)|.$$

Then along the solutions of (11) we have by using Lemma 2

$$\begin{aligned}
\dot{V}(t,x) &= \frac{x}{|x|} x^{\Delta}(t) \\
&= \frac{x}{|x|}\left(a_1(t)x(t) + a_2(t)\frac{x(t)}{1+x^2(t)}\right) \\
&= \frac{x^2}{|x|}\left(a_1(t) + a_2(t)\frac{1}{1+x^2(t)}\right) \\
&\leqslant (a_1(t) + |a_2(t)|)\,|x(t)| \\
&= (\alpha(t) + |a_2(t)|)\,|x(t)| \\
&\leqslant -\beta|x(t)|
\end{aligned}$$

for $t \in [0,\infty)_{\mathbb{T}_x^+}$. On the other hand, for $t \in [0,\infty)_{\mathbb{T}_x^-}$, we have

$$\begin{aligned}
\dot{V}(t,x) &= -\frac{2}{\mu(t)}|x(t)| - \frac{x(t)}{|x(t)|}x^{\Delta}(t) \\
&= -\frac{2}{\mu(t)}|x(t)| - \frac{x}{|x|}\left(a_1(t)x(t) + a_2(t)\frac{x(t)}{1+x^2(t)}\right) \\
&= -\frac{2}{\mu(t)}|x(t)| - \frac{x^2}{|x|}\left(a_1(t) + a_2(t)\frac{1}{1+x^2(t)}\right) \\
&= |x(t)|\left(-\frac{2}{\mu(t)} - a_1(t) - a_2(t)\frac{1}{1+x^2(t)}\right) \\
&\leqslant \left(-\frac{2}{\mu(t)} - a_1(t) + |a_2(t)|\right)|x(t)| \\
&= (\alpha(t) + |a_2(t)|)\,|x(t)| \\
&\leqslant -\beta|x(t)|.
\end{aligned}$$

Clearly,

$$Z_1(||x||) = ||x|| \to \infty, \text{ as } ||x|| \to \infty.$$

Therefore, the results follow from Theorem 2.

THEOREM 3. *Let V be a "type I" and $Z_i, i = 1,2,$ be defined as before. Assume that for $t_0 \geqslant 0$ every solution $x(t) = x(t,t_0,x_0)$ of* (5) *satisfies*

$$Z_1(|x(t)|) \leqslant V(t,x(\cdot)) \leqslant \alpha Z_2(|x(t)|) \tag{13}$$

and

$$\dot{V}_{(5)}(t,x(\cdot)) \leqslant -\rho Z_2(|x(t)|) \tag{14}$$

for some constants ρ *and* $\alpha > 0$.

Then solutions of (5) *are uniformly bounded and the zero solution of* (5) *is (UAS)*

Proof. Let $H > 0$ such that $|x(t_0)| < H$, and set $V(t) = V(t,x(\cdot))$. By (14), $V(t)$ is monotonically decreasing and hence, by (13), we have

$$\begin{aligned} Z_1(|x(t)|) \leqslant V(t) &\leqslant V(t_0) \\ &\leqslant \alpha Z_2(H). \end{aligned} \tag{15}$$

Let $\varepsilon > 0$ be given. Choose H such that $H < \varepsilon$ and

$$\alpha Z_2(H) < Z_1(\varepsilon).$$

Hence from (15), we have $|x(t)| < \varepsilon$, for $t \geqslant t_0$. Consequently, the zero solution of (5) is (US). Also, it follows from (15) that

$$|x(t)| < Z_1^{-1}\big(\alpha Z_2(H)\big),$$

which implies solutions of (5) are (UB).

Integrate (14) from t_0 to t to obtain

$$-V(t_0) \leqslant V(t) - V(t_0) \leqslant -\rho \int_{t_0}^{t} Z_2(|x(s)|)\Delta s$$

and hence

$$\int_{t_0}^{t} Z_2(|x(s)|)\Delta s \leqslant \frac{V(t_0)}{\rho} \leqslant \frac{\alpha Z_2(H)}{\rho}.$$

On the other hand, if we integrate (13) from t_0 to t we arrive at

$$\begin{aligned} \int_{t_0}^{t} V(s)\Delta s &\leqslant \alpha \frac{\alpha Z_2(H)}{\rho} \\ &= \frac{\alpha^2}{\rho} Z_2(H) \stackrel{def}{=} a Z_2(H). \end{aligned} \tag{16}$$

Since $V(t)$ is positive and decreasing for all $t \in [t_0,\infty)_{\mathbb{T}}$, we have

$$\int_{t_0}^{t} V(s)\Delta s \geqslant V(t)(t-t_0).$$

Let $\varepsilon > 0$ be given. Then, for $t \geqslant t_0 + \frac{aZ_2(H)}{Z_1(\varepsilon)}$ we have form (13) and (16) that

$$Z_1(|x(t)|) \leqslant V(t) \leqslant \frac{aZ_2(H)}{t-t_0} < Z_1(\varepsilon). \tag{17}$$

Hence, inequality (17) implies that

$$|x(t)| \leqslant Z_1^{-1}\Big(\frac{aZ_2(H)}{t-t_0}\Big) < \varepsilon.$$

From this we have the (UAS). □

REMARK 2. According to Theorem 3, the zero solution of (11) is (UAS).

We have the following simple example.

EXAMPLE 2. For all $t \in [0,\infty)_{\mathbb{T}}$, we assume the functions $a(t)$ and $b(t)$ are right-dense continuous with $a(t) > 0$. Consider the two dimensional system

$$\begin{cases} x^{\Delta} = -a(t)x + b(t)y, \\ y^{\Delta} = -b(t)x - a(t)y \end{cases}. \tag{18}$$

If

$$-2a(t) + \mu(t)(a^2(t) + b^2(t) \leqslant -\alpha$$

for positive constant α, then $(0,0)$ of (18) is (UAS). To see this, we let

$$V(x,y) = x^2 + y^2.$$

Then, we have along the solutions of (18) that

$$\begin{aligned} \dot{V}(t,x) &= 2x \cdot f(t,x) + \mu(t)\|f(t,x)\|^2 \\ &= 2x(-a(t)x + b(t)y) + 2y(-b(t)x - a(t)y) \\ &\quad + \mu(t)\big(-a(t)x + b(t)y\big)^2 + \mu(t)\big(-b(t)x - a(t)y\big)^2 \\ &= \big[-2a(t) + \mu(t)(a^2(t) + b^2(t)\big](x^2 + y^2). \end{aligned}$$

Thus, by Theorem 3 , $(0,0)$ of (18) is (UAS) and all its solutions are uniformly bounded.

3. Delay dynamic equations

Next we switch to delay dynamic equations and prove that if there exists a Lyapunov functional with certain criterion, then the zero solution is (US). We consider the general dynamic system with delay

$$x^{\Delta}(t) = f(t, x(\delta(t))),\ t \in [t_0,\infty)_{\mathbb{T}} \tag{19}$$

on an arbitrary time scale $\mathbb{T}$ which is unbounded above and $0 \in \mathbb{T}$. We assume the function f is rd-continuous, where $x \in \mathbb{R}^n$ and $f : [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ with $f(t,0) = 0$. The delay term $\delta : [t_0,\infty)_{\mathbb{T}} \to [\delta(t_0),\infty)_{\mathbb{T}}$ is strictly increasing, invertible and delta differentiable such that $\delta(t) < t$, $\left|\delta^{\Delta}(t)\right| < \infty$ for $t \in \mathbb{T}$, and $\delta(t_0) \in \mathbb{T}$.

For more on delay dynamic equations on time scales we refer to [4] and [18]. Let $t_0 \in \mathbb{T}$ and let $\phi : [\delta(t_0), t_0]_{\mathbb{T}} \to \mathbb{R}^n$, be a given $rd-$continuous initial function. We say that $x(t) := x(t; t_0, \phi)$ is the solution of (19) if $x(t) = \phi(t)$ on $[\delta(t_0), t_0]_{\mathbb{T}}$ and satisfies (19) for all $t \geqslant t_0$. We set

$$E_{t_0} = [\delta(t_0), t_0]_{\mathbb{T}}$$

that we call the *the initial interval.*

For $x \in R^n$, $|x|$ denotes the Euclidean norm of x. In addition, for any $n \times n$ matrix A, $|A|$ will denote any compatible norm so that $|Ax| \leqslant |A||x|$. For positive constant H, we let $C_H(t)$ denote the set of rd-continuous functions $\psi : [\delta(t_0), t]_{\mathbb{T}} \to R^n$ and $\|\psi\| = \sup\{|\psi(s)| : \delta(t_0) \leqslant s \leqslant t\} \leqslant H$. In addition ϕ_t denotes $\phi \in C_H(t)$.

DEFINITION 5. Let $x(t) = 0$ be a solution of (19).

(a) The zero solution of (19) is stable if for each $\varepsilon > 0$ and $t_1 \geqslant t_0$ there exists $\delta* > 0$ such that $[\phi \in C_H(t_1), \|\phi\| < \delta*, t \geqslant t_1]$ imply that $|x(t, t_1, \phi)| < \varepsilon$.

(b) The zero solution of (19) is uniformly stable if it is stable and if $\delta*$ is independent of $t_1 \geqslant t_0$.

(c) The zero solution of (19) is asymptotically stable if it is stable and if for each $t_1 \geqslant t_0$ there is an $\eta > 0$ such that $[\phi \in C_H(t_1), \|\phi\| < \eta]$ imply that $|x(t, t_1, \phi)| \to 0$ as $t \to \infty$.

THEOREM 4. *Assume the existence of a scalar functional $V(t, \psi_t)$ that is C_{rd} in t and locally Lipschitz in ψ_t when $t \geqslant t_0$ and $\psi_t \in E_t$ with $||\psi_t|| < D$ for positive constant D. Assume $V(t, 0) = 0$ and*

$$Z_1(|\psi(t)|) \leqslant V(t, \psi_t). \tag{20}$$

(a) If

$$\dot{V}(t, \psi_t) \leqslant 0 \text{ for } t_0 \leqslant t < \infty \text{ and } ||\psi_t|| \leqslant D, \tag{21}$$

then the zero solution of (19) is stable.

(b) If in addition to (a),

$$V(t, \psi_t) \leqslant Z_2(||\psi_t||), \tag{22}$$

then the zero solution of (19) is uniformly stable.

(c) If there is an $M > 0$ with $|f(t, \psi_t)| \leqslant M$ for $t_0 \leqslant t < \infty$ and $||\psi_t|| \leqslant D$, and if

$$\dot{V}(t, \psi_t) \leqslant -Z_2(|\psi(t)|), \tag{23}$$

then the zero solution of (19) is asymptotically stable.

Proof. Let $\varepsilon > 0$ be given such that $\varepsilon < D$. Let $t_1 \geqslant t_0$ and since V is continous and $V(t,0) = 0$ there exists $\delta* > 0$ such that $\phi \in E_{t_1}$ with $||\phi_{t_1}|| < \delta*$ implies that $V(t_1, \phi_{t_1}) < Z_1(\varepsilon)$. Due to condition (21) we have

$$\begin{aligned} Z_1(|x(t,t_1,\phi_{t_1}|) \leqslant V(t,x(t,t_1,\phi_{t_1})) &\leqslant V(t_1,\phi_{t_1}) \\ &\leqslant Z_1(\varepsilon), \end{aligned}$$

from which it follows that

$$|x(t,t_1,\phi_{t_1})| \leqslant W_1^{-1}\big(Z_1(\varepsilon)\big) = \varepsilon.$$

This concludes the proof of (a).

For part (b) we let $\varepsilon > 0$ be given such that $\varepsilon < D$. Let $\delta* > 0$ with $Z_2(\delta*) < Z_1(\varepsilon)$. Set $t_1 \geqslant t_0$ and $\phi_{t_1} \in E_{t_1}$ with $||\phi_{t_1}|| < \delta*$. Then

$$\begin{aligned} Z_1(|x(t,t_1,\phi_{t_1})|) \leqslant V(t,x_t(t_1,\phi_{t_1})) &\leqslant V(t_1,\phi_{t_1}) \\ &\leqslant Z_2(\delta*) < Z_1(\varepsilon). \end{aligned}$$

It follows that

$$|x(t,t_1,\phi_{t_1})| \leqslant W_1^{-1}\big(Z_1(\varepsilon)\big) = \varepsilon.$$

This concludes the proof of (b).

To prove (c), let $t_1 \geqslant t_0$ be given and let $0 < \varepsilon < D$. Find δ as in part (b) and take $\eta = \delta*$. Let $\phi_{t_1} \in C_{t_1}(H)$ with $||\phi_{t_1}|| < \delta*$. We write $x(t) = x(t,t_1,\phi_{t_1})$. Thew proof will be done vis contradiction. Thus we assume $x(t) \nrightarrow 0$ as $t \to \infty$. Then there is an $\varepsilon_1 > 0$ and a sequence $\{t_n\} \to \infty$ with $|x(t_n)| \geqslant \varepsilon_1$. Since $|f(t,\psi_t)| \leqslant M$, for $t_0 \leqslant t < \infty$ and $||\psi_t|| \leqslant D$, there is a $T > 0$ and an $\varepsilon_2 < \varepsilon_1$ with $|x(t_n)| \geqslant \varepsilon_2$, for $t_n \leqslant t \leqslant t_n + T$. A combination of this and condition (23) we have

$$\begin{aligned} 0 \leqslant V(t,x_t) &\leqslant V(t_1,\phi_{t_1}) - \int_{t_1}^{t} Z_2(|x(s|)\Delta s \\ &\leqslant V(t_1,\phi_{t_1}) - \sum_{i=2}^{n} \int_{t_i}^{t_i+T} Z_2(|x(s|)\Delta s \\ &\leqslant V(t_1,\phi_{t_1}) - \sum_{i=2}^{n} \int_{t_i}^{t_i+T} Z_2(\varepsilon_2)\Delta s \\ &= V(t_1,\phi_{t_1}) - nTZ_2(\varepsilon_2) \to -\infty, \ \text{ as } n \to \infty, \end{aligned}$$

a contradiction. This concludes the proof of (c) □

As an application of Theorem 4, we consider the scalar delay dynamical system

$$x^{\Delta}(t) = b(t)x(t) + a(t)x(\delta(t)), \ t \in [t_0,\infty)_{\mathbb{T}} \tag{24}$$

where $a,b : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ are continuous and the delay function $\delta(t)$ satisfies all the requirements as given in the beginning of this section. For more on stability results for equations that are similar to (24) we refer the reader to [2] and [4]. We have the following theorem.

THEOREM 5. *Let γ be a positive constant and define the function $\alpha : \mathbb{T} \to (-\infty, 0)$ by*

$$\alpha(t) := \begin{cases} \frac{|1+\mu(t)b(t)|-1+\gamma}{\mu(t)} & \text{for } t \in [0,\infty)_{\mathbb{T}_-} \\ b(t)+\gamma & \text{for } t \in [0,\infty)_{\mathbb{T}_+} \end{cases} \tag{25}$$

In addition we assume that

$$|a(t)| - \gamma\delta^{\Delta}(t) \leqslant 0.$$

(i) If

$$\alpha(t) \leqslant 0, \tag{26}$$

then the zero solution of (24) *is uniformly stable.*

(ii) If

$$|b(t)| + \delta < 1, \tag{27}$$

then the zero solution of (24) *is asymptotically stable.*

(iii) If inequality (27) *holds, then solutions of* (24) *are integrable.*

Proof. Let

$$V(t,x) = |x(t)| + \gamma \int_{\delta(t)}^{t} |x(s)|\Delta s. \tag{28}$$

Then along the solutions of (24) we have for $t \in [0,\infty)_{\mathbb{T}_+}$ that

$$\begin{aligned} \dot{V}(t,x) &= \frac{x}{|x|} x^{\Delta}(t) + \gamma|x(t)| - \gamma|x(\delta(t))\delta^{\Delta}(t) \\ &= \frac{x}{|x|}\big(b(t)x(t) + a(t)x(\delta(t))\big) + \gamma|x| - \gamma|x(\delta(t))|\delta^{\Delta}(t) \\ &\leqslant (b(t)+\gamma)|x| + \big(|a(t)| - \gamma\delta^{\Delta}(t)\big)|x(t)| \\ &\leqslant \alpha|x|. \end{aligned}$$

On the other hand, for $t \in [0,\infty)_{\mathbb{T}_-}$, then t is right scattered (i.e., $\mu(t) > 0$) and, we have that

$$\begin{aligned} &\dot{V}(t,x) \\ &= |x|^{\Delta} + \gamma|x(t)| - \gamma|x(\delta(t))\delta^{\Delta}(t) \\ &= \frac{|x+\mu(t)x^{\Delta}| - |x|}{\mu(t)} + \gamma|x(t)| - \gamma|x(\delta(t))\delta^{\Delta}(t) \\ &= \frac{|x+\mu(t)\big(b(t)x(t)+a(t)x(\delta(t))\big)| - |x|}{\mu(t)} + \gamma|x(t)| - \gamma|x(\delta(t))\delta^{\Delta}(t) \\ &\leqslant \frac{|x+\mu(t)b(t)x(t)| + \mu(t)|a(t)||x(\delta(t)| - |x|}{\mu(t)} + \gamma|x(t)| - \gamma|x(\delta(t))\delta^{\Delta}(t) \\ &\leqslant \frac{(|1+\mu(t)b(t)||x(t)| + \mu(t)|a(t)||x(\delta(t)| - |x| + \mu(t)\gamma|x(t)| - \mu(t)\gamma|x(\delta(t))\delta^{\Delta}(t)}{\mu(t)} \\ &= \frac{\big(|1+\mu(t)b(t)| - 1 + \gamma\big)}{\mu(t)}|x(t)| + \big(|a(t)| - \gamma\delta^{\Delta}(t)\big)|x(\delta(t))| \leqslant \alpha|x|. \end{aligned}$$

If $\alpha(t) = 0$, then we have $\dot{V}(t,x) \leqslant 0$. It is easy to see that (a) of Theorem 4 is satisfied with $Z_1(|\psi(t)|) = |\psi(t)|$. Left to verify (22). Due to condition (27) we have that

$$\dot{V}(t,x) \leqslant -\eta|x(t)| \text{ for some positive constant } \gamma. \tag{29}$$

Let $t_1 \in [t_0,\infty)_{\mathbb{T}}, \phi \in C_H(t_1), x(t) = x(t,t_1,\phi)$. Then by integrating (29) from $\delta(t_1)$ to t we arrive at

$$V(x_t) - V(\phi_{t_1}) \leqslant -\eta \int_{\delta(t_1)}^{t_1} |x(s)|\Delta s \leqslant 0.$$

Or,

$$V(x_t) - V(\phi_{t_1}) \leqslant 0.$$

Now using

$$|x(t)| \leqslant V(x_t),$$

and since V is decreasing we arrive at

$$\begin{aligned}|x(t)| &\leqslant V(x_t) \leqslant V(\phi_{t_1}) \\ &= \phi(t_1) + \eta \int_{\delta(t_1)}^{t_1} |\phi(s)|\Delta s \\ &\leqslant \left[1 + \eta(t_1 - \delta(t_1))||\phi_{t_1}||\right].\end{aligned}$$

Thus we may take $Z_2(||\psi_t||) = \left[1 + \eta(t_1 - \delta(t_1))||\phi_{t_1}||\right]$. This completes the proof of (i) and (ii). To prove the integrability of all solutions we integrate (ii) (29) from t_0 to t and get

$$V(t,x_t) - V(t_0,\phi_{t_0}) \leqslant \int_{t_0}^{t} |x(s)|\Delta,$$

from which we arrive at

$$\begin{aligned}\int_{t_0}^{t} |x(s)|\Delta s &\leqslant -V(t,x_t) + V(t_0,\phi_{t_0}) \\ &\leqslant V(t_0,\phi_{t_0}) \\ &\leqslant |\phi_{t_0}| + \eta \int_{\delta(t_0)}^{t_0} |\phi(s)|\Delta s < \infty.\end{aligned}$$

This completes the proof. $\square$

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