Differential
Equations
& Applications
Volume 14, Number 2 (2022), 305–312
doi:10.7153/dea-2022-14-21

A FOURTH–ORDER ITERATIVE BOUNDARY VALUE PROBLEM WITH LIDSTONE BOUNDARY CONDITIONS

ERIC R. KAUFMANN

(*Communicated by J. Lyons*)

Abstract. Let $m \geqslant 2$ and $a > 0$. We consider the existence and uniqueness of solutions to the fourth-order iterative boundary value problem

$$x^{(4)}(t) = f(t, x(t), x^{[2]}(t), \ldots, x^{[m]}(t)), \quad -a \leqslant t \leqslant a,$$

with solutions satisfying Lidstone boundary conditions $x(-a) = x(a) = 0$, $x''(-a) = x''(a) = 0$. Here the iterative functions are defined by $x^{[2]}(t) = x(x(t))$ and for $j = 3, \ldots m, x^{[j]}(t) = x(x^{[j-1]}(t))$. The main tool employed to establish our results is the Schauder fixed point theorem.

1. Introduction

Let $a > 0$. In this paper we consider the existence and uniqueness of solutions of the fourth-order boundary value problem,

$$x^{(4)}(t) = f(t, x(t), x^{[2]}(t), \ldots, x^{[m]}(t)), \quad -a < t < a, \tag{1}$$

$$x(-a) = x(a) = 0, \tag{2}$$

$$x''(-a) = x''(a) = 0, \tag{3}$$

where $m \geqslant 2$. The iterative function values are defined by $x^{[2]}(t) = x(x(t))$ and for $j = 3, \ldots, m$, by $x^{[j]}(t) = x(x^{[j-1]}(t))$.

Throughout the paper we assume that $f : [-a, a] \times \mathbb{R}^{m+1} \to \mathbb{R}$ is continuous. We will propose further growth conditions on the function f later in the paper as needed. To the best of our knowledge, this is the first paper to consider fourth-order iterative boundary value problems.

The study of iterative differential equations can be traced back to the works of Eder [6] and Petuhov [13]. In 1965 Petuhov [13] considered the existence of solutions to the functional differential equation $x''(t) = \lambda x(x(t))$ under the conditions that x maps the interval $[-T, T]$ onto itself and that $x(0) = x(T) = \alpha$. Eder [6] considered the existence, uniqueness, and analyticity of solutions of the first order problem $x'(t) = x(x(t))$. Wang [15], in 1990, used Schauder's fixed point theorem to obtain a

Mathematics subject classification (2020): 34B15, 34K10, 39B05.

Keywords and phrases: Iterative differential equation, Schauder fixed point theorem, Lidstone boundary conditions.

solution of $x' = f(x(x(t)))$, $x(a) = a$. In 1993, Fečkan [10] used the Contraction Mapping Principle to establish the existence of solutions for the initial value problem for the iterative differential equation $x'(t) = f(x(x(t)))$, $x(0) = 0$. More recently, Kaufmann [11] used Schauder's fixed point theorem and the Contraction Mapping Principle to show the existence and uniqueness of solutions of the second order boundary value problems $x''(t) = f(t,x(t),\ x^{[2]}(t))$, $x(a) = a$, $x(b) = b$ and $x''(t) = f(t,x(t),x^{[2]}(t))$, $x(a) = b$, $x(b) = a$. Cheraiet, et. al. [2] in 2021 studied the nonlocal third-order differential iterative differential equation $x'''(t) + f(x^{[0]}(t), x^{[1]}(t), \ldots, x^{[n]}(t)) = 0$, $x(0) = 0$, $x''(0) = 0$, $\alpha \int_0^{\eta} x(t)\,dt = x(T)$. For more on iterative differential equations see the papers [12], [14], [15], [17], [18] and references therein.

The study of boundary value problems with Lidstone boundary conditions also has a rich history. Davis and Henderson [5] used shooting methods to obtain the existence of solutions for $y^{(4)} = f(x,y,y',y'',y''')$ with solutions satisfying Lidstone boundary conditions of the form, $y(x_1) = y_1$, $y''(x_2) = y_2$, $y''(x_3) = y_3$, $y(x_4) = y_4$, where $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4$. In [3], Davis, Eloe, and Henderson considered the existence of triple positive solutions of $y^{(2m)}(t) = f(y(t), \ldots, y^{(2j)}(t), \ldots, y^{(2(m-1))}(t))$, $0 \leqslant t \leqslant 1$, $y^{(2i)}(0) = 0 = y^{(2i)}(1)$, $0 \leqslant i \leqslant m-1$. In [9], Eloe used cone theoretic techniques to establish the existence of positive solutions of the Lidstone boundary value problem $y^{(2m)}(t) = \lambda a(t) f(y(t), \ldots, y^{(2j)}(t), \ldots, y^{2(m-1)}(t))$, $0 < t < 1$, $y^{(2i)}(0) = 0 = y^{(2i)}(1)$. See the papers [4], [7], [8], and references therein for more information about Lidstone boundary value problems.

The remainder of the paper is organized as follows. In section 2 we present preliminary material needed to prove our results. In particular, we transform the boundary value problem into a an integral equation. We also state a necessary condition concerning the norm of the difference of two iterative functions as well as Schauder's fixed point theorem. In section 3 we present our main theorems. The first result gives sufficient conditions for the existence of solutions of (1), (2), (3). Our second result gives an alternative inequality under which there exists a solution to the boundary value problem. By sharpening the inequality, solutions are shown to be unique.

2. Preliminaries

We begin this section by converting the boundary value problem (1), (2), (3), into an integral equation and stating properties of the kernel that will be used in the sequel.

The Green's function $G_1(t,s)$ associated with the second-order boundary value problem

$$\begin{aligned} -y'' &= 0, \\ y(-a) &= y(a) = 0, \end{aligned}$$

is

$$G_1(t,s) = \frac{1}{2a}\begin{cases} (a-s)(t+a), & -a \leqslant t \leqslant s \leqslant a, \\ (s+a)(a-t), & -a \leqslant s \leqslant t \leqslant a. \end{cases}$$

Note that $(a-s)(t+a), -a \leqslant t \leqslant s \leqslant a,$ is positive and increasing as a function of t. Hence, $0 \leqslant (a-s)(t+a) \leqslant (a-s)(s+a)$ for all $-a \leqslant t \leqslant s$. Similarly $0 \leqslant (s+a)(a-$

$t) \leqslant (s+a)(a-s)$, for all $s \leqslant t \leqslant a$. Thus,

$$0 \leqslant G_1(t,s) \leqslant G(s,s) = \frac{1}{2a}(a^2 - s^2).$$

Since $\max_{s\in[-a,a]} \frac{1}{2a}(a^2 - s^2) = \frac{a}{2}$, then $0 \leqslant G_1(t,s) \leqslant \frac{a}{2}$, for all $t,s \in [-a,a]$.

The Green's function $G(t,s)$ associated with

$$\begin{aligned} &x^{(4)} = 0,\\ &x(-a) = x(a) = 0,\\ &x''(-a) = x''(a) = 0, \end{aligned}$$

is

$$G(t,s) = \int_{-a}^{a} G_1(t,r)G_1(r,s)\,dr.$$

See [7] and [16] for details. From $G_1(t,s) \leqslant a/2$ we see that,

$$G(t,s) \leqslant \int_{-a}^{a} \left(\frac{a}{2}\right)^2 ds = \frac{a^3}{2}.$$

So, if x is a solution of (1), (2), (3), then x satisfies the integral equation

$$x(t) = \int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s))\,ds. \tag{4}$$

It can be shown that if $x \in C[-a,a]$ satisfies $-a \leqslant x(t) \leqslant a$ for all $t \in [-a,a]$ and the integral equation (4), then x is a solution of the boundary value problem (1), (2), (3). The following lemma holds.

LEMMA 2.1. *The function x is a solution of* (1)*,* (2)*,* (3) *if and only if $-a \leqslant x(t) \leqslant a$, $t \in [-a,a]$, and x is a fixed point of* (4)*.*

For our Banach space, we let $\Phi = (C[-a,a], \|\cdot\|)$, where the norm is given by $\|x\| = \max_{t\in[-a,a]} |x(t)|$. In view of Lemma 2.1, we seek a fixed point of the operator $T : \Phi \to \Phi$ defined by

$$(Tx)(t) = \int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s))\,ds. \tag{5}$$

When we consider the uniqueness of solutions, we will have need of the set

$$\Phi(K,M) = \{x \in \Phi : \|x\| \leqslant K \text{ and } |x(t_2) - x(t_1)| \leqslant M|t_2 - t_1|, t_1, t_2 \in [-a,a]\}, \tag{6}$$

as well as the following lemma (see [14] and [17]).

LEMMA 2.2. *If $x, y \in \Phi(K,M)$ then*

$$\left| x^{[m]}(t_1) - x^{[m]}(t_2) \right| \leqslant M^m |t_1 - t_2|,\ m = 0, 1, 2, \ldots \tag{7}$$

for all $t_1, t_2 \in [-a,a]$ *and*

$$\|x^{[m]} - y^{[m]}\| \leqslant \sum_{j=0}^{m-1} M^j \|x-y\|, \; m = 1,2,3,\ldots. \tag{8}$$

Proof. The inequality (7) follows by first noting that

$$\begin{aligned}
\left|x^{[2]}(t_1) - x^{[2]}(t_2)\right| &= |x(x(t_1)) - x(x(t_2))| \\
&= |x(s_1) - x(s_2)| \\
&\leqslant M|s_1 - s_2| \leqslant M^2|t_1 - t_2|,
\end{aligned}$$

and then proceeding by induction.

To show that the inequality (8) is valid, we first note that if $x, y \in \Phi(K,M)$, then $\|x-y\| \leqslant M^0\|x-y\|$. Now, suppose that (8) holds for some integer m. That is, for this integer m,

$$\|x^{[m]} - y^{[m]}\| \leqslant \sum_{j=0}^{m-1} M^j \|x-y\|.$$

For all $t \in [-a,a]$ we have,

$$\begin{aligned}
\left|x^{[m+1]}(t) - y^{[m+1]}(t)\right| &\leqslant \left|x(x^{[m]}(t)) - x(y^{[m]}(t))\right| + \left|x(y^{[m]}(t)) - y(y^{[m]}(t))\right| \\
&\leqslant |x(s_1) - x(s_2)| + \|x-y\| \\
&\leqslant M|s_1 - s_2| + \|x-y\| \\
&= M|x^{[m]}(t) - y^{[m]}(t)| + \|x-y\| \\
&\leqslant M\|x^{[m]} - y^{[m]}\| + \|x-y\| \\
&\leqslant M\sum_{j=0}^{m-1} M^j\|x-y\| + \|x-y\| = \sum_{j=0}^{m} M^j \|x-y\|.
\end{aligned}$$

Hence $\|x^{[m]} - y^{[m]}\| \leqslant \sum_{j=0}^{m-1} M^j\|x-y\|$ implies $\|x^{[m+1]} - y^{[m+1]}\| \leqslant \sum_{j=0}^{m} M^j\|x-y\|$ and (8) follows. □

We end this section by stating Schauder's fixed point theorem [1].

THEOREM 2.3. (Schauder) *Let A be a nonempty compact convex subset of a Banach space and let $T : A \to A$ be continuous. Then T has a fixed point in A.*

3. Existence and uniqueness of solutions

We state and prove our main results in this section as well as provide examples. Let $T : C[-a,a] \to C[-a,a]$ be defined by (5). In order for the solution of the boundary value problem to be well-defined we need the range of Tx to be bounded. In particular, we need $-a \leqslant (Tx)(t) \leqslant a$ for all $t \in [-a,a]$. For our first result we will assume the following condition.

(H1) There exists $\alpha \in L[-a,a]$ such that $|f(t,y_1,y_2,\ldots,y_{m+1})| \leqslant \alpha(t)$ for all $t \in [-a,a]$ and $y_i \in \mathbb{R}$, $i = 1,2,\ldots,m+1$.

THEOREM 3.1. *Suppose that condition (H1) holds. Assume that*

$$\frac{a^2}{2}\int_{-a}^{a}\alpha(s)\,ds \leqslant 1.$$

Then there exists a solution of the boundary value problem (1)*,* (2)*,* (3)*.*

Proof. Consider the convex set $\Phi_a = \{x \in \Phi : \|x\| \leqslant a\}$. Since $|G(t,s)| \leqslant \frac{a^3}{2}$, we have

$$\begin{aligned}|(Tx)(t)| &\leqslant \int_{-a}^{a}|G(t,s)||f(s,x(s),x^{[2]}(s),\ldots,x^{[m]}(s))|\,ds\\ &\leqslant \frac{a^3}{2}\int_{-a}^{a}\alpha(s)\,ds\\ &\leqslant a,\end{aligned}$$

for all $t \in [-a,a]$. Thus, $-a \leqslant (Tx)(t) \leqslant a$, $t \in [-a,a]$. Since T is a bounded linear operator, then T is continuous. Hence by Schauder's Theorem, there exists a fixed point $x \in \Phi_a$. By Lemma 2.1 the function x is a solution of (1), (2), (3). $\square$

As an example of Theorem 3.1 we consider the boundary value problem,

$$x^{(4)}(t) = ct^2\sin(x^{[2]}), \tag{9}$$

$$x(-\pi/2) = x(\pi/2) = 0, \tag{10}$$

$$x''(-\pi/2) = x''(\pi/2) = 0. \tag{11}$$

Here $m = 2$ and $f(t,x,x^{[2]}) = ct^2\sin(x^{[2]})$. Let $\alpha(t) = ct^2$. Then $|f(t,x_1,x_2)| \leqslant \alpha(t)$ for all $t \in [-\pi/2,\pi/2]$ and

$$\frac{1}{2}\left(\frac{\pi}{2}\right)^2\int_{-\pi/2}^{\pi/2}cs^2\,ds = \frac{\pi^5}{192}c.$$

Thus, if $c < \frac{192}{\pi^5} \approx 0.6274$, then there exists a solution of (9), (10), (11).

For our next results, we assume that f satisfies the following growth condition.

(H2) There exists $\alpha_\ell \in L[-a,a]$, $\ell = 1,2,\ldots,m+1$, such that

$$|f(t,x_1,x_2,\ldots,x_{m+1}) - f(t,y_1,y_2,\ldots,y_{m+1})| \leqslant \sum_{\ell=1}^{m+1}\alpha_\ell(t)\|x_\ell - y_\ell\|$$

for all $t \in [-a,a]$, and $x_i, y_i \in \mathbb{R}$, $i = 1,2,\ldots,m+1$.

THEOREM 3.2. *Let* $\rho = |f(s,0,0,\dots 0)|$. *Assume that condition (H2) holds and suppose that*

$$\rho a^3 + \frac{a^3}{2}\sum_{\ell=1}^{m+1}\int_{-a}^{a}\alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}a^k \leqslant 1. \tag{12}$$

Then there exists a solution of the boundary value problem (1), (2), (3).

Proof. We show that $T:\Phi(a,a)\to\Phi(a,a)$ where $K=M=a$ in (6). Let $x\in\Phi(a,a)$. Then,

$$\begin{aligned}
|(Tx)(t)| &\leqslant \int_{-a}^{a}|G(t,s)||f(s,x(s),x^{[2]}(s),\dots,x^{[m]}(s))|\,ds\\
&\leqslant \frac{a^3}{2}\int_{-a}^{a}|f(s,0,\dots,0)|\\
&\qquad\quad +|f(s,x(s),x^{[2]}(s),\dots,x^{[m]}(s))-f(s,0,\dots,0)|\,ds\\
&\leqslant \frac{a^3}{2}\int_{-a}^{a}\rho+\sum_{\ell=1}^{m+1}\alpha_\ell(s)\|x^{[\ell]}\|\,ds\\
&\leqslant \frac{a^3}{2}\left[2a\rho+\sum_{\ell=1}^{m+1}\int_{-a}^{a}\alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}a^k\|x\|\right]\\
&\leqslant a\left[a^3\rho+\frac{a^3}{2}\sum_{\ell=1}^{m+1}\int_{-a}^{a}\alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}a^k\right]\\
&\leqslant a.
\end{aligned}$$

Thus, $-a\leqslant Tx(t)\leqslant a$ for all $t\in[-a,a]$. By Schauder's Theorem, there exists a fixed point x of T in $\Phi(a,a)$. By Lemma 2.1, x is a solution of (1), (2), (3), and the proof is complete. □

COROLLARY 3.3. *Suppose either* $\rho>0$ *or* $\rho=0$ *and the inequality* (12) *is strict. Then there exists a unique solution of* (1), (2), (3).

Proof. If either $\rho>0$ or $\rho=0$ and the inequality (12) is strict, then

$$\frac{a^3}{2}\sum_{\ell=1}^{m+1}\int_{-a}^{a}\alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}a^k<1. \tag{13}$$

Assume x and y are distinct fixed points of (5). That is $x=Tx$ and $y=Ty$. From (H2), (8), and (13), we have,

$$\begin{aligned}
|x(t)-y(t)| &= |(Tx)(t)-(Ty)(t)|\\
&\leqslant \int_{-a}^{a}|G(t,s)|\big|f(s,x(s),x^{[2]}(s),\dots,x^{[m]})\\
&\qquad\qquad -f(s,y(s),y^{[2]}(s),\dots,y^{[m]})\big|\,ds\\
&\leqslant \frac{a^3}{2}\int_{-a}^{a}\sum_{\ell=1}^{m+1}\alpha_\ell(s)\|x^{[\ell]}-y^{[\ell]}\|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{a^3}{2}\int_{-a}^{a}\sum_{\ell=1}^{m+1}\alpha_\ell(s)\sum_{k=0}^{\ell-1}a^k\|x-y\|\,ds\\
&\leqslant \left[\frac{a^3}{2}\int_{-a}^{a}\sum_{\ell=1}^{m+1}\alpha_\ell(s)\sum_{k=0}^{\ell-1}a^k\,ds\right]\|x-y\|\\
&< \|x-y\|.
\end{aligned}
$$

Hence we have the contradiction $\|x-y\|<\|x-y\|$. Thus, the solution is unique and the proof is complete. □

As an example of Corollary 3.3, we again consider the boundary value problem (9), (10), (11). Note that $f(t,x_1,x_2)=ct^2\sin(x_2)$ satisfies

$$|f(t,x_1,x_2)-f(t,y_1,y_2)|\leqslant ct^2|x_2-y_2|$$

for all $t,x_2,y_2\in[-\pi/2,\pi/2]$. So, $\rho=0,\alpha_0(t)=\alpha_1(t)=0$, and $\alpha_2(t)=ct^2$. The left hand side of (12) becomes

$$
\begin{aligned}
\rho a^3+\frac{a^3}{2}\sum_{\ell=1}^{3}\int_{-a}^{a}\alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}a^k &= \frac{a^3}{2}\int_{-a}^{a}\alpha_2(s)\,ds(1+a+a^2)\\
&= \frac{\pi^3}{16}\frac{c\pi^3}{12}\left(1+\frac{\pi}{2}+\frac{\pi^2}{4}\right)\\
&= \frac{\pi^6}{192}\left(1+\frac{\pi}{2}+\frac{\pi^2}{4}\right)c.
\end{aligned}
$$

Consequently, if $c<\frac{768}{\pi^6(4+2\pi+\pi^2)}\approx 0.0386$, then by Corollary 3.3 there exists a unique solution to (9), (10), (11).

REMARK 1. We can generalize the boundary value problem to

$$
\begin{aligned}
&x^{(4)}(t)=f(t,x(t),x^{[2]}(t),\ldots,x^{[m]}(t)),\quad a<t<b,\\
&x(-a)=x(b)=0,\\
&x''(-a)=x''(b)=0,
\end{aligned}
$$

where $a\geqslant 0$ and $b>0$, and obtain equivalent results to Theorem 3.1, Theorem 3.2, and Corollary 3.3. Furthermore, if we let $a=0$, we can also obtain results on the positivity of solutions.

REMARK 2. The technique can easily be extended to the $(2n)^{th},n\geqslant 2$, order boundary value problem,

$$
\begin{aligned}
&(-1)^n x^{(2n)}(t)=f(t,x(t),x^{[2]}(t),\ldots,x^{[m]}(t)),\quad a<t<b,\\
&x(a)=x''(a)=\cdots=x^{(2(n-1))}(a)=0,\\
&x(b)=x''(b)=\cdots=x^{(2(n-1))}(b)=0,
\end{aligned}
$$

and similar results to our main theorems may be obtained.

REFERENCES

[1] T. A. BURTON, *Stability by fixed point theory for functional differential equations*, Dover Publications, Inc., Mineola, NY, 2006.

[2] S. CHERAIET, A. BOUAKKAZ, AND R. KHEMIS, *Bounded positive solutions of an iterative three-point boundary-value problem with integral boundary condtions*, J. Appl. Math. Comput., **65**, 1–2 2021, 597–610.

[3] J. M. DAVIS, P. W. ELOE, AND J. HENDERSON, *Triple positive solutions and dependence on higher order derivatives*, J. Math. Anal. Appl., **237**, 2 1999, 710–720.

[4] J. M. DAVIS, P. W. ELOE, AND J. HENDERSON, *Comparison of eigenvalues for discrete Lidstone boundary value problems*, Dynam. Systems Appl., **8**, 3–4 1999, 381–388.

[5] J. M. DAVIS AND J. HENDERSON, *Uniqueness implies existence for fourth-order Lidstone boundary value problems*, PanAmer. Math. J., **8**, 4 1998, 23–35.

[6] E. EDER, *The functional-differential equation* $x'(t) = x(x(t))$, J. Differential Equations, **54**, 3 1984, 390–400.

[7] P. W. ELOE, J. HENDERSON, AND H. B. THOMPSON, *Extremal points for impulsive Lidstone boundary value problems*, Mathematical and Computer Modelling, **32**, 2000, pages 687–698.

[8] P. W. ELOE AND M. N. ISLAM, *Monotone methods and fourth order Lidstone boundary value problems with impulse effects*, Commun. Appl. Anal., **5**, 1 2001, 113–120.

[9] P. W. ELOE, *Nonlinear eigenvalue problems for higher order Lidstone boundary value problems*, Electron. J. Qual. Theory Differ. Equ., **2000**, no. 2, 1–8.

[10] M. FEČKAN, *On a certain type of functional-differential equations*, Math. Slovaca, **43**, 1 (1993), 39–43.

[11] E. R. KAUFMANN, *Existence and uniqueness of solutions for a second-order iterative boundary-value problem*, Electron. J. Differential Equations, **2018**, no. 150, 1–6.

[12] X. P. LIU AND M. JIAN, *Initial value problem for a second order non-autonomous functional-differential iterative equation*, Acta Math. Sinica (Chin. Ser.), **45**, 4 2002, 711–718.

[13] V. R. PETUHOV, *On a boundary value problem*, Trudy Sem. Teor. Differencial. Uravneniĭ s Otklon. Argumentom Univ. Družby Narodov Patrisa Lumumby, **3**, 1965, 252–255.

[14] J. SI AND X. WANG, *Smooth solutions of a nonhomogeneous iterative functional-differential equation with variable coefficients*, J. Math. Anal. Appl., **226**, 2 1998, 377–392.

[15] K. WANG, *On the equation* $x'(t) = f(x(x(t)))$, Funkcial. Ekvac., **33**, 3 1990, 405–425.

[16] P. J. Y. WONG, *Triple solutions of complementary Lidstone boundary value problems via fixed point theorems*, Bound. Value Probl., **2014**, 2014:125, 21 pp.

[17] H. Y. ZHAO AND J. LIU, *Periodic solutions of an iterative functional differential equation with variable coefficients*, Math. Methods Appl. Sci., **40**, 1 2017, 286–292.

[18] H. ZHAO, *Smooth solutions of a class of iterative functional differential equations*, Abstr. Appl. Anal., **2012**, Art. ID 954352, 13 pp.

(Received February 25, 2022)

Eric R. Kaufmann
Department of Mathematics & Statistics
University of Arkansas at Little Rock
Little Rock, AR 72204, USA
e-mail: erkaufmann@ualr.edu