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P–PERIODIC SOLUTIONS OF A q–INTEGRAL EQUATION WITH FINITE DELAY

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Dedicated to our friend and colleague Paul Eloe on the occasion of his retirement

(*Communicated by F. Atici*)

Abstract. A Volterra type integral equation with a finite delay is considered on a discrete non-additive time scale domain $q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, where $k \in \mathbb{N}$, $q > 1$. The existence of periodic solutions of this equation, which we call a q-integral equation, are shown employing the contraction mapping principle and a fixed point theorem due to Krasnosel'skii.

1. Introduction

Functional differential equations with finite as well as infinite delays arise in many applications. For the last fifty years or so, researchers have been studying various qualitative properties, such as the existence of solutions, of these equations. To study these equations, researchers normally convert them into integral equations and then apply suitable mathematical tools such as fixed point theorems in the analysis. For example, the functional differential equation defined on $\mathbb{R}$ with finite delay $h > 0$ given by

$$x'(t) = ax(t) - b(x(t), x(t-h)) + r(t), \quad a \neq 0, \tag{1}$$

becomes a Volterra type integral equation

$$x(t) = x(t-h)e^{ah} - \int_{t-h}^{t} b(x(s), x(s-h))e^{a(t-s)}ds + p(t), \tag{2}$$

where the integration is carried out from $t-h$ to t. We refer to [6] and the references therein for examples of delay functional differential equations and their applications.

Various generalizations of equation (2) are considered by the researches for studying the existence of continuous periodic solutions on the real line $\mathbb{R}$, which is an additively periodic time scale (cf. [5, 7]). Definition and examples of additively periodic time scales can be found in, for example, [9]. Kaufmann and Raffoul [10] were the first to study periodic solutions on additively periodic time scales when they studied a

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neutral nonlinear equation. It has been found that the notion of periodicity in so called non-additively periodic time scales is important. Non-additively periodic time scales are used in the study of the periodicity of q-difference equations, which occur in the field of quantum calculus. We refer the interested readers to [8], in which the authors studied the existence of periodic solutions of a quantum Volterra integral equation on non-additively periodic time scales. We also refer to [1] in which the authors provide a comprehensive study of periodicity on time scales.

In the present paper we study the periodicity of a generalized form of (2) on a non-additively periodic time scale $q^{\mathbb{N}_0}$ defined below. In particular, we consider the integral equation

$$x(t) = f(t,x(t),x(q^{-k}t)) - \int_{q^{-k}t}^{t} C(t,s)g(s,x(s),x(q^{-k}s))d_q s, \quad t \in q^{\mathbb{N}_0}, \tag{3}$$

where $k \in \mathbb{N}$, $q > 1$, $\mathbb{T} = \{q^n : n \in \{-k,-k+1,\dots,0,1,\dots\}\}$, $q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, $C : q^{\mathbb{N}_0} \times \mathbb{T}$, $f : q^{\mathbb{N}_0} \times \mathbb{R} \times \mathbb{R}$, $g : \mathbb{T} \times \mathbb{R} \times \mathbb{R}$, and q^{-k} is the finite delay. We assume f and g are continuous in their second and third variables. Here

$$\int_{q^m}^{q^n} f(s)d_q s := (q-1)\sum_{k=m}^{n-1} q^k f(q^k),$$

where $m = -\infty$ if the lower limit of integration is 0, and $n = \infty$ if the upper limit of integration is ∞.

2. P-periodicity

The first periodicity notion on $q^{\mathbb{N}_0}$ was given by Bohner and Chieochan in [4].

DEFINITION 1. ([4]) Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be P-periodic if

$$f(t) = q^P f(q^P t) \text{ for all } t \in q^{\mathbb{N}_0}. \tag{4}$$

Afterwards, Adivar [2] (see also [3]) introduced a more general periodicity notion on time scales that are not necessarily additively periodic. On $q^{\mathbb{N}_0}$, this is defined as follows.

DEFINITION 2. ([2]) Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be P-periodic if

$$f(q^P t) = f(t) \text{ for all } t \in q^{\mathbb{N}_0}.$$

Notice each of these definitions can easily be extended to the time scale $\mathbb{T}$.

In [8], it was shown that f is periodic with respect to Definition 1 if and only if $\tilde{f}(t) = tf(t)$ is periodic with respect to Definition 2. Other relationships between these definitions involving periodic solutions of q-difference and q-integral equations are also established in [8]. Therefore, the results we obtain in this paper with respect to Definition 2 can easily be extended to an appropriate integral equation with respect to Definition 1.

3. Existence of P-periodic solutions

In this section, we show the existence of a P-periodic solution of (3) in Theorem 2 by employing the fixed point theorem of Krasnosel'skii. We then employ the contraction principle in Theorem 3 to show the existence of a unique P-periodic solution of (3). We also present examples where the assumptions of Theorems 2 and 3 hold.

We make use of the following assumptions.

(A1) There exists a $P \in \mathbb{N}$ such that for all $t \in q^{\mathbb{N}_0}$, $s \in \mathbb{T}$ and $x, y \in \mathbb{R}$,

$$f(q^P t, x, y) = f(t, x, y),$$
$$g(q^P s, x, y) = g(s, x, y),$$

and

$$q^P C(q^P t, q^P s) = C(t, s).$$

(A2) There exists $a, b > 0$ such that for all $t \in q^{\mathbb{N}_0}$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leqslant a|x_1 - x_2| + b|y_1 - y_2|,$$

where $a + b < 1$.

(A3) There exists $c, d > 0$ such that for all $t \in \mathbb{T}$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \leqslant c|x_1 - x_2| + d|y_1 - y_2|.$$

(A4) There exists an $L > 0$ such that

$$\sup_{t \in q^{\mathbb{N}_0}} \int_{q^{-k}}^{t} |C(t, s)| d_q s < L.$$

Define the set $\mathcal{Q} = \{q^0, q^1, \ldots, q^P\}$. Define the Banach space

$$\mathscr{B} = \{x : \mathbb{T} \to \mathbb{R} : x(q^P t) = x(t)\},$$

with the norm

$$\|x\| = \max_{t \in \mathcal{Q}} |x(t)|.$$

Define the operator

$$Tx(t) = f(t, x(t), x(q^{-k} t)) - \int_{q^{-k} t}^{t} C(t, s) g(s, x(s), x(q^{-k} s)) d_q s, \quad t \in q^{\mathbb{N}_0},$$

and for $t \in \mathbb{T} \setminus q^{\mathbb{N}_0}$ with, for $m \in \mathbb{N}$, $q^{mP} t \in q^{\mathbb{N}_0}$ and $q^{(m-1)P} t \notin q^{\mathbb{N}_0}$, define

$$\begin{aligned} Tx(t) &= f(q^{mP} t, x(q^{mP} t), x(q^{-k} q^{mP} t)) - \int_{q^{-k} q^{mP} t}^{q^{mP} t} C(q^{mP} t, s) g(s, x(s), x(q^{-k} s)) d_q s \\ &= Tx(q^{mP} t). \end{aligned}$$

Defining T in this manner allows us to define T on all of $\mathscr{B}$ while not extending the domains of f or C. Notice if x is a fixed point of T, then for $t \in q^{\mathbb{N}_0}$,

$$Tx(t) = x(t) = f(t,x(t),x(q^{-k}t)) - \int_{q^{-k}t}^{t} C(t,s)g(s,x(s),x(q^{-k}s))d_qs,$$

and so x is a solution of (3).

EXAMPLE 1. If $q = 2$, $P = 3$, and $k = 4$, then $Tx(2^{-1}) := Tx(2^2)$, $Tx(2^{-2}) := Tx(2^1)$, $Tx(2^{-3}) := Tx(2^0)$, and $Tx(2^{-4}) := Tx(2^2)$.

LEMMA 1. *Assume (A1) is satisfied. Then the operator $T : \mathscr{B} \to \mathscr{B}$.*

Proof. Let $x \in \mathscr{B}$. For $t \in q^{\mathbb{N}_0}$, assumption (A1) gives that

$$\begin{aligned}
Tx(q^Pt) &= f(q^Pt,x(q^Pt),x(q^{-k}q^Pt)) - \int_{q^{-k}q^Pt}^{q^Pt} C(q^Pt,s)g(s,x(s),x(q^{-k}s))d_qs \\
&= f(q^Pt,x(q^Pt),x(q^{-k}q^Pt)) - \int_{q^{-k}t}^{t} q^PC(q^Pt,q^Ps)g(q^Ps,x(q^Ps),x(q^{-k}q^Ps))d_qs \\
&= f(t,x(t),x(q^{-k}t)) - \int_{q^{-k}t}^{t} C(t,s)g(s,x(s),x(q^{-k}s))d_qs \\
&= Tx(t).
\end{aligned}$$

For $t \in \mathbb{T} \setminus q^{\mathbb{N}_0}$ with, for $m \in \mathbb{N}$, $q^{mP}t \in q^{\mathbb{N}_0}$ and $q^{(m-1)P}t \notin q^{\mathbb{N}_0}$, notice

$$\begin{aligned}
Tx(q^Pt) &= Tx(q^{(m-1)P}q^Pt) \\
&= Tx(q^{mP}t) \\
&= Tx(t).
\end{aligned}$$

Therefore $Tx \in \mathscr{B}$. $\square$

THEOREM 1. (Krasnosel'skii, [11]) *Let $\mathscr{M}$ be a closed convex nonempty subset of a Banach space $\mathscr{B}$. Suppose that A and B map $\mathscr{M}$ into $\mathscr{B}$ such that*

(i) $x,y \in \mathscr{M}$ implies $Ax + By \in \mathscr{M}$,

(ii) A is a contraction mapping, and

(iii) B is a compact and continuous mapping.

Then there exists a $z \in \mathscr{M}$ with $z = Az + Bz$.

Let $m > 0$ and define

$$\mathscr{M}(m) = \{x \in \mathscr{B} : \|x\| \leqslant m\}.$$

THEOREM 2. *Assume (A1) and (A2) are satisfied. If there exists a positive constant m_0 such that*

$$\frac{\bar{F}+\bar{C}\bar{G}(m_0)(q^P-q^{-k})}{1-(a+b)} \leqslant m_0, \tag{5}$$

where

$$\bar{C} = \max_{(t,s)\in\mathscr{Q}\times\mathbb{T}} |C(t,s)|,$$

$$\bar{F} = \max_{t\in\mathscr{Q}} |f(t,0,0)|,$$

and

$$\bar{G}(m) = \max_{(t,x,y)\in\mathscr{Q}\times[-m,m]\times[-m,m]} |g(t,x,y)|.$$

then equation (3) has a P-periodic solution $x\in\mathscr{B}$ in the sense that $x(t)=x(q^Pt)$.

Proof. Define the mapping $A:\mathscr{B}\to\mathscr{B}$ by

$$Ax(t) = f(t,x(t),x(q^{-k}t)), \quad t\in q^{\mathbb{N}_0},$$

and for $t\in\mathbb{T}\setminus q^{\mathbb{N}_0}$ with, for $m\in\mathbb{N}$, $q^{mP}t\in q^{\mathbb{N}_0}$ and $q^{(m-1)P}t\notin q^{\mathbb{N}_0}$,

$$Ax(t) = f(q^{mP}t,x(q^{mP}t),x(q^{-k}q^{mP}t)) = Ax(q^{mP}t).$$

Define the mapping $B:\mathscr{B}\to\mathscr{B}$ by

$$Bx(t) = -\int_{q^{-k}t}^{t} C(t,s)g(s,x(s),x(q^{-k}s))d_qs, \quad t\in q_0^{\mathbb{N}},$$

and for $t\in\mathbb{T}\setminus q^{\mathbb{N}_0}$ with, for $m\in\mathbb{N}$, $q^{mP}t\in q^{\mathbb{N}_0}$ and $q^{(m-1)P}t\notin q^{\mathbb{N}_0}$,

$$Bx(t) = -\int_{q^{-k}q^{mP}t}^{q^{mP}t} C(q^{mP}t,s)g(s,x(s),x(q^{-k}s))d_qs = Bx(q^{mP}t).$$

Notice $Tx(t) = (Ax+Bx)(t)$ for all $t\in\mathbb{T}$.

First, we show A is a contraction mapping. For $x,y\in\mathscr{B}$ with $t\in q^{\mathbb{N}_0}$, by (A1),

$$\begin{aligned}|Ax-Ay|(t) &= |f(t,x(t),x(q^{-k}t)) - f(t,y(t),y(q^{-k}t))|\\ &\leqslant a|x(t)-y(t)| + b|x(q^{-k}t)-y(q^{-k}t)|\\ &\leqslant (a+b)\|x-y\|.\end{aligned}$$

For $t\in\mathbb{T}\setminus q^{\mathbb{N}_0}$ with, for $m\in\mathbb{N}$, $q^{mP}t\in q^{\mathbb{N}_0}$ and $q^{(m-1)P}t\notin q^{\mathbb{N}_0}$,

$$\begin{aligned}|Ax-Ay|(t) &= |f(q^{mP}t,x(q^{mP}t),x(q^{-k}q^{mP}t)) - f(q^{mP}t,y(q^{mP}t),y(q^{-k}q^{mP}t))|\\ &\leqslant a|x(q^{mP}t)-y(q^{mP}t)| + b|x(q^{-k}q^{mP}t)-y(q^{-k}q^{mP}t)|\\ &\leqslant (a+b)\|x-y\|.\end{aligned}$$

Thus

$$\|Ax - Ay\| \leqslant (a+b)\|x+y\|,$$

and since $a+b<1$, A is a contraction mapping.

Next, we show B is continuous and compact. Since g is continuous in its second and third variables, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|x-y\| < \delta$ implies

$$|g(t,x(t),x(q^{-k}t)) - g(t,y(t),y(q^{-k}t)| < \frac{\varepsilon}{\bar{C}(q^P - q^{-k})}$$

for all $t \in \mathscr{Q}$. Thus for $\|x-y\| < \delta$ and for $t \in \mathscr{Q}$,

$$\begin{aligned}
|Bx - By|(t) &\leqslant \int_{q^{-k}t}^{t} |C(t,s)||g(t,x(s),x(q^{-k}s)) - g(t,y(s),y(q^{-k}s))|d_q s \\
&< \frac{\varepsilon}{\bar{C}(q^P - q^{-k})}\bar{C}\int_{q^{-k}t}^{t} d_q s \\
&< \varepsilon.
\end{aligned}$$

Thus for $\|x-y\| < \delta$, $\|Bx - By\| < \varepsilon$. Therefore B is continuous. Since B is defined on a discrete domain, a diagonalization argument similar to the one found in [8] can be used to show B is compact.

Finally, we show for $x,y \in \mathscr{M}(m_0)$, $Ax + By \in \mathscr{M}(m_0)$. First, notice that for $t \in \mathscr{Q}$,

$$\begin{aligned}
&|f(t,x(t),x(q^{-k}t))| \leqslant |f(t,x(t),x(q^{-k}t)) - f(t,0,0)| + |f(t,0,0)| \\
&\leqslant (a+b)\|x\| + \bar{F}.
\end{aligned}$$

Then for $x,y \in \mathscr{M}(m_0)$ and $t \in \mathscr{Q}$,

$$\begin{aligned}
|Ax + By|(t) &= \left| f(t,x(t),x(q^{-k}t)) - \int_{q^{-k}t}^{t} C(t,s)g(s,y(s),y(q^{-k}s))d_q s \right| \\
&\leqslant am_0 + \bar{F} + \bar{C}\bar{G}(m_0)(q^P - q^{-k}) \leqslant m_0.
\end{aligned}$$

So $Ax + By \in \mathscr{M}(m_0)$. Therefore, by Theorem 1, there exists a $z \in \mathscr{M}(m_0)$ with $Az + Bz = z$. This z is a P-periodic solution of (3). □

EXAMPLE 2. Let $q = 2$ and $k = 3$. Define

$$f(2^n,x,y) = (-1)^n\left(\frac{1}{8}\cos x + \frac{1}{16}\cos y\right),$$

$$g(2^n,x,y) = (-1)^n(x+y),$$

and

$$C(t,s) = \frac{9}{80(t+s)}.$$

Then (A1) is satisfied with $P = 2$ and (A2) is satisfied with $a = \frac{1}{8}$ and $b = \frac{1}{16}$. Choose $m_0 = 100$. Then,

$$\bar{F} = f(1,0,0) = \frac{3}{16},$$

$$\bar{C} = C(1,2^{-3}) = \frac{1}{10},$$

and

$$\bar{G}(100) = g(1,100,100) = 200.$$

Thus

$$\frac{\bar{F} + \bar{C}\bar{G}(m_0)(2^2 - 2^{-3})}{1-(a+b)} = \frac{1243}{13} < 100.$$

Therefore by Theorem 2, (3) has a 2-periodic solution $x \in \mathscr{B}$.

THEOREM 3. *Assume (A1)-(A4) hold. If $(a+b)+L(c+d) < 1$, then (3) has a unique P-periodic solution.*

Proof. By Theorem 1, we know $T : \mathscr{B} \to \mathscr{B}$. Let $x, y \in \mathscr{B}$. Assumptions (A2)-(A4) give that if $t \in q^{\mathbb{N}_0}$,

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leqslant |f(t,x(t),x(q^{-k}t)) - f(t,y(t),y(q^{-k}t))| \\
&\quad + \int_{q^{-k}t}^{t} |C(t,s)||g(s,x(s),x(q^{-k}s)) - g(s,y(s),y(q^{-k}s))| d_q s \\
&\leqslant (a+b)\|x-y\| + (c+d)\|x-y\| \sup_{t\in q^{\mathbb{N}_0}} \int_{q^{-k}}^{t} |C(t,s)| d_q s \\
&\leqslant [(a+b)+L(c+d)]\,\|x-y\|.
\end{aligned}$$

If $t \in \mathbb{T} \setminus q^{\mathbb{N}_0}$ with, for $m \in \mathbb{N}$, $q^{mP}t \in q^{\mathbb{N}_0}$ and $q^{(m-1)P}t \notin q^{\mathbb{N}_0}$, then

$$|Tx(t) - Ty(t)| = |Tx(q^{mP}t) - Ty(q^{mP}t)|.$$

A similar argument can then be used to show

$$|Tx(t) - Ty(t)| \leqslant [(a+b)+L(c+d)]\,\|x-y\|.$$

Since $(a+b)+L(c+d) < 1$, T is a contraction mapping. Therefore, T has a unique fixed point x^* which is a unique solution of (3). □

EXAMPLE 3. Let $q = 2$ and $k = 3$. Define

$$f(2^n,x,y) = (-1)^n \left(\frac{1}{8}\cos x + \frac{1}{16}\cos y\right),$$

$$g(2^n,x,y) = (-1)^n \left(\frac{1}{8}\cos x + \frac{1}{16}\cos y\right),$$

and

$$C(t,s) = \frac{t}{(t+s)^2}.$$

Then (A1) is satisfied with $P = 2$, and (A2) and (A3) are satisfied with $a = c = \frac{1}{8}$ and $b = d = \frac{1}{16}$. Notice that if f is a positive deceasing function, then

$$\begin{aligned}
\int_{q^{m+1}}^{q^n} f(s) d_q s &= (q-1) \sum_{k=m+1}^{n-1} q^k f(q^k) \\
&< (q-1) \int_m^{n-1} q^t f(q^t) dt \\
&= \frac{q-1}{\ln q} \int_{q^m}^{q^{n-1}} f(s) ds \\
&< \frac{q-1}{\ln q} \int_{q^m}^{q^n} f(s) ds.
\end{aligned}$$

So

$$\begin{aligned}
\int_{1/8}^{t} \frac{t}{(t+s)^2} d_q s &< \frac{t}{8(t+1/8)^2} + \frac{1}{\ln 2} \int_{1/8}^{t} \frac{t}{(t+s)^2} ds \\
&< \frac{1}{\ln 2} \frac{64t^2 + 16t - 1}{2(8t+1)^2}.
\end{aligned}$$

So

$$\begin{aligned}
\sup_{t \in q^{\mathbb{N}_0}} \int_{1/8}^{t} |C(t,s)| d_q s &< \sup_{t \in [1/8, \infty)} \frac{1}{\ln 2} \frac{64t^2 + 16t - 1}{2(8t+1)^2} \\
&< 0.8.
\end{aligned}$$

Therefore,

$$(a+b) + L(c+d) < 0.3375 < 1.$$

So by Theorem 3, (3) has a unique 2-periodic solution.

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