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p–DEFORMATION

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Dedicated to Paul Eloe on the Occasion of His Retirement

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Abstract. In this article we introduce the so called p-deformed algebra $\mathscr{V}$. The notion of p-deformation is connected to the well-known q-deformation by the simple relation $p = \frac{q^2+q^{-2}}{2}$. Thus the p-deformed algebra $\mathscr{V}$ will have representations in terms of q-difference operators. There are isomorphisms of $\mathscr{V}$ to the q-deformed Weyl algebra $\mathscr{W}$ and to the well known algebra $\mathscr{U} = \mathscr{U}_q$, the q-deformation of the universal enveloping algebra $U_q(\mathfrak{sl}(2))$, extended by an involution. It turns out that the presentation of the p-deformed algebra $\mathscr{V}$ is more symmetric than the ones of its q counterparts. Especially the limit $p \to \pm 1$ can be performed in a direct and quite consistent manner. For $p^2 = 1$ the p-deformed algebra contains copies of the classical Weyl algebra, the Lie superalgebra $\mathfrak{osp}(1|2)$ and the Lie algebra $\mathfrak{sl}(2)$. Finally we will see that the p-deformed algebra $\mathscr{V}$ contains a "squared copy" of itself.

1. Introduction

We start out with considering some q-difference operators, such as

$$
\begin{aligned}
X^+ f(x) &= x \cdot f(x) & \qquad G f(x) &= \frac{q^2 f(q^2x) + f(q^{-2}x)}{q^2+1} \\
X^- f(x) &= \frac{f(q^2x) - f(q^{-2}x)}{x(q^2-q^{-2})} & \qquad H f(x) &= \frac{q^2 f(q^2x) - f(q^{-2}x)}{q^2-1},
\end{aligned}
$$

acting on a space $\mathscr{F}$ of functions defined on a discrete q^2-grid. q is a formal variable, it may be interpreted as a fixed complex nonzero number.

For $q \to +1$ or $q \to -1$ some of the above operators become singular, so the questions about an appropriate limit for $q \to \pm 1$ arises. A suitable approach to studying this limit is passing to an algebraic context. Just consider the operators as elements of an associative noncommutative algebra that fulfill certain relations, such as

$$
\begin{aligned}
X^-X^+ - X^+X^- &= G \\
X^-X^+ + X^+X^- &= H \\
GH - HG &= 0.
\end{aligned}
$$

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Then the initial space $\mathscr{F}$ of functions becomes a representation of this algebra. It is now possible to perform the above mentioned limit within the relations, the “limiting algebra” has a corresponding representation on a space of continuous functions.

When further abstracting this approach one gets finally to an algebra $\mathscr{V}$ which is – the other way round – the point of departure for our study. We give a short overview of this study.

In Section 2 we introduce our main object, the so called p-deformed algebra $\mathscr{V}$. This algebra will prove to unfold a variety of special cases that are well known in mathematical physics, especially quantum theory. Moreover, singularities that usually appear in q-deformed versions of classical algebras will disappear by passing to the above p-deformation. After introducing the variable q, related to p by $\frac{q^2+q^{-2}}{2} = p$, and further elements $S, S', K, K' \in \mathscr{V}$ we will be able to establish isomorphisms of $\mathscr{V}$ to the so called q-deformed Weyl algebra $\mathscr{W}$ and to the universal enveloping algebra $\mathscr{U} = \mathscr{U}_q(\mathfrak{sl}(2))$, also called a quantum group.

In Section 3 we investigate the quotient algebra $\mathscr{V}_0$ of $\mathscr{V}$, defined by setting the so called Casimir element to zero. Depending on certain values of p we are going to present some special properties and then representations of this algebra $\mathscr{V}_0$. It will turn out that for the cases $p = \pm 1$, after a certain transformation the defining relations of $\mathscr{V}_0$ reduce to the ones of the Lie superalgebra $\mathfrak{osp}(1|2)$, extended by an involution, that is in the center (case $p = +1$) or in the so called supercenter (case $p = -1$) of $\mathscr{V}_0$.

In the final Section 4 we establish an Embedding Theorem. The mapping $\tilde{\eta} : \tilde{\mathscr{V}} \to \mathscr{V}_0$ embeds a “squared” copy $\tilde{\mathscr{V}}$ of $\mathscr{V}$ into the algebra $\mathscr{V}_0$. The other way round $\mathscr{V}_0$ can be considered as a “square root algebra” of $\tilde{\mathscr{V}}$.

2. The algebra $\mathscr{V}$

We call an associative algebra with a unit an algebra, for short. Recall [5, p. 3] that a mapping between two algebras $\mathscr{A} \to \widetilde{\mathscr{A}}$ is called an algebra morphism, if it is a morphism of the underlying unital rings. The unit 1 has to be preserved.

We will use the standard notations for commutators and anticommutators

$$[A,B] := AB - BA, \qquad \{A,B\} := AB + BA.$$

Let $\mathbb{C}[p]$ denote the commutative $\mathbb{C}$-algebra of polynomials in the variable p with complex coefficients. The unital associative $\mathbb{C}[p]$-algebra $\mathscr{V}$ is defined by a presentation with generating elements

$$X^+, X^-, C$$

and relations

$$\begin{array}{lrcl}
(V_0) & [C,X^+] = [C,X^-] & = & 0 \\
(V_1) & [X^-X^+, X^+X^-] & = & 0 \\
(V_2) & (X^+)^2X^- + X^-(X^+)^2 & = & 2pX^+X^-X^+ + X^+C \\
(V_3) & (X^-)^2X^+ + X^+(X^-)^2 & = & 2pX^-X^+X^- + X^-C.
\end{array} \tag{1}$$

Additionally we define

$$(V_4) \quad P \quad := \quad (X^-X^+)^2 + (X^+X^-)^2 - 2pX^-(X^+)^2X^- - C(X^-X^+ + X^+X^-).$$

2.1. Representation by q-difference operators

We make the abstract algebra $\mathscr{V}$ more concrete by stating a representation. Given the variable p, considered as a complex nonzero number, choose a complex number $q \neq 0$ such that

$$\tfrac{q^2+q^{-2}}{2} = p. \tag{2}$$

For $p^2 = 1$ there are two solutions for q, otherwise there are four.

In algebraic terms the relation (2) induces an injection of polynomial rings $\mathbb{C}[p] \hookrightarrow \mathbb{C}[q]$. In order to have rational expressions in q available in $\mathscr{V}$, we tacitly replace $\mathscr{V}$ by the tensor product $\mathscr{V} \otimes_{\mathbb{C}[p]} \mathbb{C}(q)$, where $\mathbb{C}(q)$ is the field of rational functions with variable q.

In the course of this paper the variable q will only appear within the following Theorem 1 and in Subsection 2.4. The main properties of the algebras $\mathscr{V}$ and $\mathscr{V}_0$ in Section 3 the embedding in Section 4 only involve the variable p, they are independent of the choice for q in (2).

THEOREM 1. *Let $p^2 \neq 1$, that is $q \notin \{1, i, -1, -i\}$. Then there is a representation of $\mathscr{V}$ on a space $\mathscr{F}$ of functions defined on a q^2-grid, given by*

$$\begin{array}{rclcrcl} X^+f(x) &=& x \cdot f(x), & \qquad & X^-f(x) &=& \frac{f(q^2x)-f(q^{-2}x)}{x(q^2-q^{-2})}, \\ Cf(x) &=& 0, & & Pf(x) &=& f(x). \end{array} \tag{3}$$

The operator X^- is the well known symmetric q^2 difference operator, sometimes called the Hahn operator or the Jackson operator. Note that it is invariant with respect to replacing q by any other solution $\pm q^{\pm 1}$ of equation (2). So the representation (3) only depends on p, the choice of q according to (2) is irrelevant.

Proof. We check some of the defining relations (V_1)–(V_4) by computing some of the composite operators appearing there.

$$\begin{aligned} X^-X^+f(x) &= X^-xf(x) = \tfrac{q^2f(q^2x)-q^{-1}f(q^{-2}x)}{q^2-q^{-2}} \\ X^+X^-f(x) &= X^+\tfrac{f(q^2x)-f(q^{-2}x)}{x(q^2-q^{-2})} = \tfrac{f(q^2x)-f(q^{-2}x)}{q^2-q^{-2}} \\ X^-(X^+)^2X^-f(x) &= X^+(X^-)^2X^+f(x) = \tfrac{q^2f(q^4x)-(q^2+q^{-2})f(x)+q^{-2}f(q^{-4}x)}{(q^2-q^{-2})^2} \end{aligned}$$

$$\begin{aligned} (X^+)^2X^-f(x) &= x\tfrac{f(q^2x)-f(q^{-2}x)}{q^2-q^{-2}} \\ X^-(X^+)^2f(x) &= X^-x^2f(x) = \tfrac{q^4xf(q^2x)-q^{-4}xf(q^{-2}x)}{q^2-q^{-2}} \end{aligned}$$

$$\begin{aligned}
[(X^+)^2X^- + X^-(X^+)^2]f(x) &= \tfrac{(q^4+1)xf(q^2x)-(q^{-4}+1)xf(q^{-2}x)}{q^2-q^{-2}} \\
X^+X^-X^+f(x) &= x\tfrac{q^2f(q^2x)-q^{-1}f(q^{-2}x)}{q^2-q^{-2}} \\
(q^2+q^{-2})X^+X^-X^+f(x) &= \tfrac{(q^4+1)xf(q^2x)-(q^{-4}+1)xf(q^{-2}x)}{q^2-q^{-2}}. \qquad \square
\end{aligned}$$

REMARK 1. If $p = \cos\frac{2\pi}{n}$ for some $n = 3,4,\ldots$, then one of the solutions of (2) fulfills $q^2 = e^{\frac{2\pi i}{n}}$, a primitive n-th root of unity. This means that the above representation (3) acts on functions defined on the finite multiplicative group $E_n = \{e^{\frac{2\pi ik}{n}} \in \mathbb{C} \mid k = 0,\ldots,n-1\}$ The representation space is finite dimensional with dimension n.

As an example we choose $n = 4$, that is $p = 0$ and then $q^2 = +i$ or $q^2 = -i$. Representation (3) now acts on functions with domain $\{1,i,-1,-i\}$. It is given by complex 4×4 matrices

$$X^+ \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \qquad G \mapsto \frac{1}{2}\begin{pmatrix} 0 & 1+i & 0 & 1-i \\ 1-i & 0 & 1+i & 0 \\ 0 & 1-i & 0 & 1+i \\ 1+i & 0 & 1-i & 0 \end{pmatrix},$$

$$X^- \mapsto \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \\ 1 & 0 & -1 & 0 \end{pmatrix}, \qquad H \mapsto \frac{1}{2}\begin{pmatrix} 0 & 1-i & 0 & 1+i \\ 1+i & 0 & 1-i & 0 \\ 0 & 1+i & 0 & 1-i \\ 1-i & 0 & 1+i & 0 \end{pmatrix}.$$

Note that $(X^-)^2 = 0$, $(X^+)^4 = 1$ and $P = G^2 = H^2 = 1$.

2.2. Properties of $\mathscr{V}$

We are going to examine some properties of $\mathscr{V}$ in a purely algebraic fashion.

1. (V_0) shows that the so called Casimir element C is in the center of $\mathscr{V}$. The three elements C, X^-X^+, X^+X^- generate a commutative subalgebra.

2. The element P defined in (V_4) is another element in the center, in fact we have

$$\begin{aligned}
&[P,X^+] \\
&= [(X^-X^+)^2 + (X^+X^-)^2 - 2pX^-(X^+)^2X^- - C(X^-X^+ + X^+X^-)]X^+ \\
&\quad - X^+[(X^-X^+)^2 + (X^+X^-)^2 - 2pX^-(X^+)^2X^- - C(X^-X^+ + X^+X^-)] \\
&= X^-X^+[X^-(X^+)^2 - 2pX^+X^-X^+ - CX^+] \\
&\quad - [(X^+)^2X^- - 2pX^+X^-X^+ - CX^+]X^+X^- \\
&= X^-X^+[-(X^+)^2X^-] - [-X^-(X^+)^2]X^+X^- = 0
\end{aligned}$$

and similarly $[P,X^-] = 0$. Then (V_4) implies $[P,C] = 0$.

3. The notation with the superscripts $+$ and $-$ is a reminiscence to ladder theory, see [4]. X^+ and X^- act as raising (or creation) and lowering (or annihilation) operators in a ladder. Ladder theory provides a mean to systematically compute finite dimensional representations of the algebra $\mathscr{V}$. We won't consider this direction in this article.

2.3. G-H transformation

In this section we specify the commutator G and anticommutator H of the ladder operators X^+ an X^-. They will later serve to identify further algebraic properties of the algebras $\mathscr{V}$ and $\mathscr{V}_0$.

THEOREM 2. (G-H transformation) *We additionally define the commutator and anticommutator of $X^-, X^+ \in \mathscr{V}$*

$$G \;:=\; X^-X^+ - X^+X^-, \qquad H \;:=\; X^-X^+ + X^+X^-.$$

Then we can rewrite and supplement the relations (V_0)–(V_4) in the following way.

$$\begin{array}{lrcl}
(T_0) & [C,X^+] & = & [C,X^-] \;=\; 0 \\
(T_1) & [G,H] & = & 0 \\
(T_2) & GX^+ & = & X^+[pG+(p-1)H+C] \\
 & HX^+ & = & X^+[pH+(p+1)G+C] \\
(T_3) & GX^- & = & X^-[pG-(p-1)H-C] \\
 & HX^- & = & X^-[pH-(p+1)G+C] \\
(T_4) & P & = & \frac{p+1}{2}G^2 - \frac{p-1}{2}H^2 - CH \\
(T_5) & G & = & [X^-,X^+] \\
(T_6) & H & = & \{X^-,X^+\}.
\end{array} \tag{4}$$

Within the representation (3) the elements G and H act as

$$Gf(x) \;=\; \frac{q^2 f(q^2x)+f(q^{-2}x)}{q^2+1}, \qquad Hf(x) \;=\; \frac{q^2 f(q^2x)-f(q^{-2}x)}{q^2-1}.$$

Only exemplarily we prove that the first relation in (T_2) follows from (V_2).

$$\begin{aligned}
GX^+ &= X^-(X^+)^2 - X^+X^-X^+ \\
&\overset{(V_2)}{=} -(X^+)^2X^- + 2pX^+X^-X^+ + X^+C - X^+X^-X^+ \\
&= X^+[p(X^-X^+ - X^+X^-) + (p-1)(X^-X^+ + X^+X^-) + C] \\
&= X^+[pG + (p-1)H + C].
\end{aligned}$$

When taking account of (V_2), the relation (T_4) is a reformulation of (V_4) as follows

$$\begin{aligned}
P &\overset{(V_4)}{=} (X^-X^+)^2 + (X^+X^-)^2 - 2pX^-(X^+)^2X^- - C(X^-X^+ + X^+X^-) \\
&= \tfrac{p+1}{2}(X^-X^+ - X^+X^-)^2 - \tfrac{p-1}{2}(X^-X^+ + X^+X^-)^2 - C(X^-X^+ + X^+X^-) \\
&= \tfrac{p+1}{2}G^2 - \tfrac{p-1}{2}H^2 - CH.
\end{aligned}$$

2.4. Connecting p-deformation to q-deformation

In order to establish the connection of the p-deformed algebra to the classical well-known q-deformed universal enveloping algebra $\mathscr{U}$ of $\mathfrak{sl}_2(\mathbb{C})$, we define the additional elements S, S^{-1} and K, K^{-1}. With respect to representations these are shift operators that are decisive in the literature about q-deformation, see especially [5, p. 122].

THEOREM 3. (S transformation) *If $q^2 \neq 1$ or $C = 0$ we are able to define the following two elements in $\mathscr{V}$*

$$\begin{aligned} S &:= qX^-X^+ - q^{-1}X^+X^- + \tfrac{C}{q-q^{-1}} \\ S' &:= q^{-1}X^-X^+ - qX^+X^- - \tfrac{C}{q-q^{-1}}. \end{aligned} \tag{5}$$

Then the relations (V_0)–(V_4) transform into

$$\begin{array}{lrclrcl} (W_0) & [C,X^+] & = & [C,X^-] \;=\; 0 & & & \\ (W_1) & [S,S'] & = & 0 & & & \\ (W_2) & SX^+ & = & q^2X^+S & S'X^+ & = & q^{-2}X^+S' \\ (W_3) & SX^- & = & q^{-2}X^-S & S'X^- & = & q^2X^-S' \\ (W_4) & (q-q^{-1})^2P & = & (q-q^{-1})^2SS' + C^2 & & & \\ (W_5) & (q+q^{-1})[X^-,X^+] & = & S+S' & & & \\ (W_6) & (q-q^{-1})^2\{X^-,X^+\} & = & (q-q^{-1})(S-S') - 2C. & & & \end{array} \tag{6}$$

Within the representation (3) the elements S and S' act as

$$Sf(x) \;=\; qf(q^2x), \qquad S'f(x) \;=\; q^{-1}f(q^{-2}x).$$

Proof. (W_5) and (W_6) only repeat the definitions (5). The computations

$$\begin{aligned} & SX^+ - q^2X^+S \\ =\; & [qX^-X^+ - q^{-1}X^+X^- + \tfrac{1}{q-q^{-1}}C]X^+ - q^2X^+[qX^-X^+ - q^{-1}X^+X^- + \tfrac{1}{q-q^{-1}}C] \\ =\; & q\left[(X^+)^2X^- + X^-(X^+)^2 - (q^2+q^{-2})X^+X^-X^+ + \tfrac{1-q^2}{q(q-q^{-1})}C\right] \end{aligned}$$

and similarly for the differences $S'X^+ - q^{-2}X^+S'$, $SX^- - q^{-2}X^-S$, $S'X^- - q^2X^-S'$, show that $(V_2),(V_3)$ are equivalent to $(W_2),(W_3)$. Also we check the equivalence of the definition (V_4) and the relation (W_4) as follows.

$$\begin{aligned} & (q-q^{-1})^2SS' + C^2 \\ &= [(q^2-1)X^-X^+ - (1-q^{-2})X^+X^- + C]\cdot[(1-q^{-2})X^-X^+ - (q^2-1)X^+X^- - C] + C^2 \\ &= (q-q^{-1})^2[(X^-X^+)^2 + (X^+X^-)^2] - (q^2-1)^2X^-(X^+)^2X^- - (1-q^{-2})X^+(X^-)^2X^+ \\ &\qquad + C[-(q^2-1)X^-X^+ + (1-q^{-2})X^+X^- + (1-q^{-2})X^-X^+ - (q^2-1)X^+X^-] \\ &= (q-q^{-1})^2[(X^-X^+)^2 + (X^+X^-)^2 - (q^2+q^{-2})X^-(X^+)^2X^- - C(X^-X^+ + X^+X^-)] \\ &= (q-q^{-1})^2P. \quad \square \end{aligned}$$

REMARK 2. When we add the relations $q = S = S' = 1$, then we end up with the classical Weyl algebra with generators X^+, X^- and the canonical commutation relation (W_5), that is $[X^-, X^+] = 1$. So we will denote the presentation (6) of the algebra $\mathscr{V}$ as the q-deformed Weyl algebra $\mathscr{W}$. This algebra was studied in the article [3].

REMARK 3. In [7, p. 134] the authors introduce the so called symmetric q-oscillator algebra $\mathscr{A}_q$. We remark that there is an isomorphism $\mathscr{W}/(SS' = 1) \to \mathscr{A}_q$ given by

$$q \mapsto q^{\frac{1}{2}}, \qquad X^+ \mapsto a^+, \qquad X^- \mapsto a,$$
$$S \mapsto q^{\frac{1}{2}} q^N, \qquad S' \mapsto q^{-\frac{1}{2}} q^{-N}.$$

THEOREM 4. (K transformation) *If $q^4 \neq 1$ we can introduce the elements*

$$\begin{aligned} K &:= \tfrac{q-q^{-1}}{q+q^{-1}} S = \tfrac{q^2-1}{q+q^{-1}} X^- X^+ + \tfrac{q^{-2}-1}{q+q^{-1}} X^+ X^- + \tfrac{C}{q+q^{-1}} \\ K' &:= \tfrac{q-q^{-1}}{q+q^{-1}} S' = \tfrac{q^{-2}-1}{q+q^{-1}} X^- X^+ + \tfrac{q^{2}-1}{q+q^{-1}} X^+ X^- + \tfrac{C}{q+q^{-1}}. \end{aligned} \tag{7}$$

Then the relations (V_0)–(V_4) transform into

$$\begin{array}{llll} (U_0) & [C, X^+] = [C, X^-] = 0 & & \\ (U_1) & [K, K'] = 0 & & \\ (U_2) & KX^+ = q^2 X^+ K & & K'X^+ = q^{-2} X^+ K' \\ (U_3) & KX^- = q^{-2} X^- K & & K'X^- = q^2 X^- K' \\ (U_4) & (q-q^{-1})^2 P = C^2 - (q+q^{-1})^2 KK' & & \\ (U_5) & (q-q^{-1})[X^-, X^+] = K - K' & & \\ (U_6) & (q-q^{-1})^2 \{X^-, X^+\} = (q+q^{-1})(K+K') - 2C. & & \end{array}$$

Within the representation (3) the elements K and K' act as

$$Kf(x) = \tfrac{q^2-1}{q+q^{-1}} f(q^2 x), \qquad K'f(x) = \tfrac{q^{-2}-1}{q+q^{-1}} f(q^{-2} x).$$

Proof. For the proof just replace $S = \frac{q+q^{-1}}{q-q^{-1}} K$ and $S' = \frac{q+q^{-1}}{q-q^{-1}} K'$ within the relations (W_1)–(W_6) in (6). $\square$

REMARK 4. When renaming the elements $E = -X^+$, $F = X^-$, $L = G = [X^-, X^+]$, it turns out that this algebra with the additional relation $KK' = 1$ coincides with the algebra $\mathscr{U} = \mathscr{U}_q$. This ist the q-deformation of the universal enveloping algebra $U(\mathfrak{sl}(2))$ of the Lie algebra $\mathfrak{sl}(2)$, extended by a central involution. $\mathscr{U}_q$ is also called a quantum group, see [5, p. 125] or [7, p. 53]. Thus we have established a morphism of algebras $\mathscr{V}/(KK' = 1) \to \mathscr{U}_q$. In this context the element $C_q = \frac{C}{(q-q^{-1})^2}$ in the center of $\mathscr{U}$ is called the quantum Casimir element.

3. The algebra $\mathscr{V}_0$

Within the algebra $\mathscr{V}$ we now set $C=0$. In other words we pass to the algebra $\mathscr{V}_0 := \mathscr{V}/(C=0)$ that is defined as the quotient algebra of $\mathscr{V}$ by the two sided ideal $(C=0)$.

We have already seen a representation of this algebra $\mathscr{V}_0$ for $p^2 \neq 1$ in Theorem 1. We will now investigate the algebra $\mathscr{V}_0$ and give representations for $p=1$ and $p=-1$.

3.1. The algebra $\mathscr{V}_0$, where $p=1$

Here the relations (T_1)–(T_6), see (4), have the following form

$$\begin{array}{lrcllcl}
(T_1) & [G,H] & = & 0 & & & \\
(T_2) & [G,X^+] & = & 0 & [H,X^+] & = & 2X^+G \\
(T_3) & [G,X^-] & = & 0 & [H,X^-] & = & -2X^-G \\
(T_4) & P & = & G^2 & & & \\
(T_5) & G & = & [X^-,X^+] & & & \\
(T_6) & H & = & \{X^-,X^+\}. & & &
\end{array}$$

We see that G commutes with all elements of $\mathscr{V}_0$, it is in the center.

THEOREM 5. *We add to $\mathscr{V}_0$ with $p=1$ the relation $G^2=1$ and perform a "G-transformation". That means we define additional elements in $\mathscr{V}$ according to*

$$\begin{array}{lllllllll}
F := GH, & Z^+ & := & X^+, & & Z^- & := & GX^-, & \\
 & \tilde{Z}^+ & := & (Z^+)^2 & = (X^+)^2, & \tilde{Z}^- & := & (Z^-)^2 & = (X^-)^2.
\end{array}$$

Then one can thoroughly derive from the above set of relations (T_1)–(T_6) the equivalent set (O_1)–(O_6) and additionally the "tilde" set $(\tilde{O}_2),(\tilde{O}_3),(\tilde{O}_5)$–$(\tilde{O}_9)$

$$\begin{array}{lrcllcl}
(O_1) & [G,F] & = & 0 & & & \\
(O_2) & [G,Z^+] & = & 0 & [F,Z^+] & = & 2Z^+ \\
(O_3) & [G,Z^-] & = & 0 & [F,Z^-] & = & -2Z^- \\
(O_4) & P=G^2 & = & 1 & & & \\
(O_5) & [Z^-,Z^+] & = & 1 & & & \\
(O_6) & & & & \{Z^-,Z^+\} & = & F \\
 & & & & & & \\
(\tilde{O}_2) & [G,\tilde{Z}^+] & = & 0 & [F,\tilde{Z}^+] & = & 4\tilde{Z}^+ \\
(\tilde{O}_3) & [G,\tilde{Z}^-] & = & 0 & [F,\tilde{Z}^-] & = & -4\tilde{Z}^- \\
(\tilde{O}_5) & & & & [\tilde{Z}^-,\tilde{Z}^+] & = & 2F \\
(\tilde{O}_6) & \{\tilde{Z}^-,\tilde{Z}^+\} & = & \frac{1}{2}F^2+\frac{3}{2} & & & \\
(\tilde{O}_7) & & & & \{Z^+,Z^+\} & = & 2\tilde{Z}^+ \\
 & & & & \{Z^-,Z^-\} & = & 2\tilde{Z}^- \\
(\tilde{O}_8) & & & & [\tilde{Z}^-,Z^+] & = & 2Z^- \\
 & & & & [\tilde{Z}^+,Z^+] & = & 0 \\
(\tilde{O}_9) & & & & [\tilde{Z}^+,Z^-] & = & -2Z^+ \\
 & & & & [\tilde{Z}^-,Z^-] & = & 0.
\end{array} \tag{8}$$

Proof. We only show the right side of (O_2) and $(\tilde{O}_5),(\tilde{O}_6)$. Here we have

$$[F,Z^+] = GHX^+ - X^+GH \;=\; G[H,X^+] \;=\; 2G^2X^+ \;=\; 2Z^+,$$

then

$$\begin{aligned}
[\tilde{Z}^-,\tilde{Z}^+] &= [(X^-)^2,(X^+)^2] \;=\; X^-[X^-,(X^+)^2] + [X^-,(X^+)^2]X^- \\
&= X^-X^+[X^-,X^+] + X^-[X^-,X^+]X^+ + X^+[X^-,X^+]X^- + X^+[X^-,X^+]X^+X^- \\
&= 2(X^-X^+ + X^+X^-)G \;=\; 2HG \;=\; 2F
\end{aligned}$$

and finally

$$\begin{aligned}
\tfrac{1}{2}F^2 + \tfrac{3}{2} &= \tfrac{1}{2}H^2 + \tfrac{3}{2}G^2 \\
&= \tfrac{1}{2}(X^-X^+ - X^+X^-)^2 + \tfrac{3}{2}(X^-X^+ + X^+X^-)^2 \\
&= 2(X^-X^+)^2 + 2(X^+X^-)^2 - X^-(X^+)^2X^- - -X^+(X^-)^2X^+ \\
&= X^-(2X^+X^-X^- - (X^+)^2X^-) + X^+(X^-X^+X^- - (X^-)^2X^+) \\
&\overset{(V_2),(V_3)}{=} (X^-)^2(X^+)^2 + (X^+)^2(X^-)^2 \;=\; \{\tilde{Z}^-,\tilde{Z}^+\}. \quad \square
\end{aligned}$$

REMARK 5. There is a representation of $\mathscr{V}_0$ for $p=1$ and $G^2=1$ on a space of single variable smooth functions, given by

$$\begin{aligned}
Z^+f(x) &= x\cdot f(x) & \qquad Ff(x) &= f(x) + 2xf'(x) \\
Z^-f(x) &= f'(x) & \qquad Gf(x) &= f(x).
\end{aligned}$$

We see that the subalgebra generated by the two elements Z^+,Z^- and relation (O_5) is again the Weyl algebra of Remark 2.

3.2. The Lie superalgebra $\mathfrak{osp}(1|2)$

$\mathscr{V}$ has the structure of a $\mathbb{Z}_2$-graded algebra, i.e. there exists a decomposition of the underlying vector space that is compatible with the algebra structure as follows

$$\mathscr{V} = \mathscr{V}^{[0]} \oplus \mathscr{V}^{[1]} \qquad \text{and} \qquad \mathscr{V}^{[a]} \cdot \mathscr{V}^{[b]} \;\subseteq\; \mathscr{V}^{[(a+b) \bmod 2]}.$$

The elements in $\mathscr{V}^{[0]} \cup \mathscr{V}^{[1]}$ are called homogeneous. The elements in $\mathscr{V}^{[0]}$ are called even, they have degree 0, whereas the elements in $\mathscr{V}^{[1]}$ are odd and have degree 1.

The $\mathbb{Z}_2$-grading in $\mathscr{V}$ is defined by the affiliations

$$P,C,G,H \;\in\; \mathscr{V}^{[0]}, \qquad\qquad X^+,X^- \;\in\; \mathscr{V}^{[1]}.$$

The relations (V_0)–(V_4), see (1), preserve this $\mathbb{Z}_2$-grading.

We recall from [6] that a Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $V = V^{[0]} \oplus V^{[1]}$, equipped with a bilinear product $[\cdot,\cdot]_s : V \times V \to V$, such that

$$\begin{aligned}
[A,B]_s + (-1)^{ab}[B,A]_s &= 0 \\
(-1)^{ac}[A,[B,C]_s]_s + (-1)^{cb}[C,[A,B]_s]_s + (-1)^{ba}[B,[C,A]_s]_s &= 0,
\end{aligned}$$

whenever $A,B,C \in \mathscr{V}^{[0]} \cup \mathscr{V}^{[1]}$ with $\deg A = a$, $\deg B = b$ and $\deg C = c$.

As with any $\mathbb{Z}_2$-graded algebra, the algebra $\mathscr{V}$ bears in a natural way the structure of a Lie superalgebra. One has to define the bilinear product by

$$[A,B]_s \ := \begin{cases} [A,B], & \text{if } A \text{ or } B \text{ is even,} \\ \{A,B\}, & \text{if } A \text{ and } B \text{ are odd.} \end{cases}$$

To be more concrete, the 12 commutator and anticommutator relations in the right column of (8) constitute the structure of the classical Lie superalgebra $\mathfrak{osp}(1|2)$. Its basis consists of two odd elements Z^+, Z^- and three even elements F, $\tilde{Z}^+$, $\tilde{Z}^-$.

The even elements $\tilde{Z}^-, \tilde{Z}^+, F$ in the p-deformed algebra $\mathscr{V}$ together with the relations on the right side of $(\tilde{O}_2), (\tilde{O}_3), (\tilde{O}_5)$ generate the Lie algebra $\mathfrak{sl}(2)$. The three elements Z^+, Z^-, F with the relations on the right side of $(O_2), (O_3), (O_6)$ constitute the so called parabosonic algebra, see [2, Section 6.2].

There is a representation of $\mathfrak{osp}(1|2)$ by 3×3 matrices, see [6, p. 15]. By some easy computations one can check that the 12 relations in the right column of (8) are fulfilled, if the representation is defined by the assignments

$$Z^- \mapsto \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}, \qquad Z^+ \mapsto \begin{pmatrix} 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{Z}^- \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}, \qquad \tilde{Z}^+ \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Altogether we see that the algebra $\mathscr{V}_0$ with additional relations $p = 1$ and $G^2 = 1$ is isomorphic to the Lie superalgebra $\mathfrak{osp}(1|2)$, extended by a central involution G. The other way round we can state that the p-deformed algebra $\mathscr{V}_0$ is a reasonable p-deformation, henceforward q-deformation, of the Lie superalgebra $\mathfrak{osp}(1|2)$.

3.3. The algebra $\mathscr{V}_0$, where $p = -1$

The relations (T_1)–(T_6) from (4), now have the following form

$$\begin{array}{lrcllrcl}
(T_1) & [G,H] & = & 0 & & & & \\
(T_2) & \{H,X^+\} & = & 0 & & \{G,X^+\} & = & -2X^+H \\
(T_3) & \{H,X^-\} & = & 0 & & \{G,X^-\} & = & 2X^-H \\
(T_4) & P & = & H^2 \ = \ 1 & & & & \\
(T_5) & G & = & [X^-,X^+] & & & & \\
(T_6) & H & = & \{X^-,X^+\}. & & & &
\end{array}$$

So H is in the supercenter, i.e. commutes with all even elements and anticommutes with all odd elements.

THEOREM 6. *We add to* $\mathscr{V}_0$ *with* $p=-1$ *the relation* $H^2=1$ *and perform an "H-transformation".*

$$\begin{array}{rclrclcl} F & := & HG, & Z^+ & := & X^+, & & Z^- := HX^-, \\ & & & \tilde{Z}^+ & := & (Z^+)^2 = (X^+)^2, & & \tilde{Z}^- := (Z^-)^2 = -(X^-)^2. \end{array}$$

Then one can thoroughly derive from the above set of relations (T_1)–(T_6) *the equivalent set* (O_1)–(O_6) *and additionally the "tilde" set* $(\tilde{O}_2),(\tilde{O}_3),(\tilde{O}_5)$–$(\tilde{O}_9)$

$$\begin{array}{lrclrcl}
(O_1) & [H,F] & = & 0 & & & \\
(O_2) & \{H,Z^+\} & = & 0 & [F,Z^+] & = & 2Z^+ \\
(O_3) & \{H,Z^-\} & = & 0 & [F,Z^-] & = & -2Z^- \\
(O_4) & P=H^2 & = & 1 & & & \\
(O_5) & [Z^-,Z^+] & = & 1 & & & \\
(O_6) & & & & \{Z^-,Z^+\} & = & F \\
(\tilde{O}_2) & [H,\tilde{Z}^+] & = & 0 & [F,\tilde{Z}^+] & = & 4\tilde{Z}^+ \\
(\tilde{O}_3) & [H,\tilde{Z}^-] & = & 0 & [F,\tilde{Z}^-] & = & -4\tilde{Z}^- \\
(\tilde{O}_5) & & & & [\tilde{Z}^-,\tilde{Z}^+] & = & 2F \\
(\tilde{O}_6) & \{\tilde{Z}^-,\tilde{Z}^+\} & = & \frac{1}{2}F^2+\frac{3}{2} & & & \\
(\tilde{O}_7) & & & & \{Z^+,Z^+\} & = & 2\tilde{Z}^+ \\
 & & & & \{Z^-,Z^-\} & = & 2\tilde{Z}^- \\
(\tilde{O}_8) & & & & [\tilde{Z}^-,Z^+] & = & 2Z^- \\
 & & & & [\tilde{Z}^+,Z^+] & = & 0 \\
(\tilde{O}_9) & & & & [\tilde{Z}^+,Z^-] & = & -2Z^+ \\
 & & & & [\tilde{Z}^-,Z^-] & = & 0.
\end{array} \tag{9}$$

Proof. The proof is analogous to the one of Theorem 5. One has to appropriately swap signs. □

REMARK 6. As in the case $p=1$, the algebra $\mathscr{V}_0$, with additional relations $p=-1$ and $H^2=1$ contains the Lie superalgebra $\mathfrak{osp}(1|2)$ with the relations presented on the right had side of (9). Altogether this algebra with relations (V_1)–(V_6) is isomorphic to the Lie superalgebra $\mathfrak{osp}(1|2)$, extended by a supercentral involution H.

This algebra is sometimes denoted by $\mathfrak{osp}_{-1}(1|2)$ or $\mathfrak{sl}_{-1}(2)$. Within this algebra it is possible to define further interesting elements or algebras, such as Dunkl operators, the Bannai-Ito algebra or Racah algebra, that evoke a lot of research activities, see [1] or [8].

REMARK 7. There is a representation of $\mathscr{V}_0$ for $p=-1$ and $H^2=1$ on a space of single variable smooth functions

$$\begin{array}{rclrcl} Z^+f(x) & = & x\cdot f(x) & Ff(x) & = & f(x)+2xf'(x) \\ Z^-f(x) & = & f'(x) & Gf(x) & = & f(-x). \end{array}$$

4. An Embedding Theorem

THEOREM 7. (Embedding $\tilde{\eta} : \tilde{\mathscr{V}} \to \mathscr{V}_0$) *Let $\tilde{\mathscr{V}}$ be a copy of $\mathscr{V}$ that fulfills the relations $(\tilde{V}_0)$–$(\tilde{V}_3)$. There is an embedding, i.e. an injective algebra morphism*

$$\begin{array}{rccccc} \tilde{\eta} : & \tilde{\mathscr{V}} & \hookrightarrow & \mathscr{V}_0 & & \\ & \tilde{p} & \mapsto & 8p^4 - 8p^2 + 1 & \tilde{C} \mapsto & 8p^3 P \\ & \tilde{X}^+ & \mapsto & (X^+)^2 & \tilde{G} \mapsto & 2pGH \\ & \tilde{X}^- & \mapsto & (X^-)^2 & \tilde{H} \mapsto & (p+\frac{1}{2})G^2 + (p-\frac{1}{2})H^2 \end{array} \tag{10}$$

This algebra morphism is — up to a multiplicative factor — the one presented in Prop. 3 (ii) in [7, p. 137], raised to the p-deformation setting.

Proof. We have to check that the relations $(\tilde{V}_0)$–$(\tilde{V}_3)$ are preserved under the embedding. As usual, we adopt the convention to identify the elements of $\tilde{\mathscr{V}}$ with their respective images in $\mathscr{V}_0$.

The relation $(\tilde{V}_0)$ is trivial, since P is central in $\mathscr{V}_0$. For the remaining proof note that we again and again use the identity in $\mathscr{V}_0$

$$(V_1) \qquad X^-X^+X^+X^- = X^+X^-X^-X^+.$$

Then, also with $(V_2),(V_3)$, we derive

$$\begin{aligned} &[\tilde{X}^-\tilde{X}^+, \tilde{X}^+\tilde{X}^-] \\ =\ & X^-X^-X^+X^+X^+X^+X^-X^- - X^+X^+X^-X^-X^-X^-X^+X^+ \\ =\ & (2pX^-X^+X^- - X^+X^-X^-)\,X^+X^+\,(2pX^-X^+X^- - X^-X^-X^+) - \\ & (2pX^+X^-X^+ - X^-X^+X^+)\,X^-X^-\,(2pX^+X^-X^+ - X^+X^+X^-) \\ =\ & 4p^2\,(X^-X^+X^-X^+X^+X^-X^+X^- - X^+X^-X^+X^-X^-X^+X^-X^+) + \\ & 2pX^-X^+(X^+X^-X^-X^+ - X^-X^+X^+X^-)X^-X^+ + \\ & 2p\,X^+X^-(X^-X^+X^+X^- - X^+X^-X^-X^+)X^+X^- + \\ & \quad (X^-X^+X^+X^-X^-X^+X^+X^- - X^+X^-X^-X^+X^+X^-X^-X^+) \\ =\ & 4p^2\,(X^-X^+X^+X^-X^-X^+X^+X^- - X^+X^-X^-X^+X^+X^-X^-X^+) \ = \ 0. \end{aligned}$$

Further note that

$$\begin{aligned} &(X^-)^2(X^+)^2 - (X^+)^2(X^-)^2 \\ &= [2pX^-X^+X^- - X^+(X^-)^2]X^+ - X^+[2pX^-X^+X^- - X^+(X^-)^2] \\ &= 2p[(X^-X^+)^2 - (X^+X^-)^2] \end{aligned} \tag{11}$$

$$\begin{aligned} &(X^-)^2(X^+)^2 + (X^+)^2(X^-)^2 + 2X^+(X^-)^2X^+ \\ &= [2pX^-X^+X^- - X^+(X^-)^2]X^+ \\ &\qquad + X^+[2pX^-X^+X^- - X^+(X^-)^2] + 2X^+(X^-)^2X^+ \\ &= 2p[(X^-X^+)^2 + (X^+X^-)^2]. \end{aligned} \tag{12}$$

Then with a rather tricky decomposition of the expression

$$8p^3(X^+)^2P \;=\; (8p^3-4p)X^+PX^+ + 2p(X^+)^2P + 2pP(X^+)^2$$

we prove $(\tilde{V}_2)$

$$\begin{aligned}
& (\tilde{X}^+)^2\tilde{X}^- + \tilde{X}^-(\tilde{X}^+)^2 - 2\tilde{p}\tilde{X}^+\tilde{X}^-\tilde{X}^+ - \tilde{X}^+\tilde{C} \\
= \; & (X^+)^4(X^-)^2 + (X^-)^2(X^+)^4 - 2(8p^4-8p^2+1)(X^+)^2(X^-)^2(X^+)^2 - 8p^3(X^+)^2P \\
= \; & (X^+)^2\Big[(X^-)^2(X^+)^2 - 2p[(X^-X^+)^2-(X^+X^-)^2]\Big] \\
& \quad + \Big[(X^+)^2(X^-)^2 + 2p[(X^-X^+)^2-(X^+X^-)^2]\Big](X^+)^2 \\
& \quad - (16p^4-16p^2+2)(X^+)^2(X^-)^2(X^+)^2 \\
& \quad - (8p^3-4p)X^+\,[(X^-X^+)^2+(X^+X^-)^2-2pX^+(X^-)^2X^+]\,X^+ \\
& \quad - 2p\,(X^+)^2\,[(X^-X^+)^2+(X^+X^-)^2-2pX^+(X^-)^2X^+] \\
& \quad - 2p\,[(X^-X^+)^2+(X^+X^-)^2-2pX^+(X^-)^2X^+]\,(X^+)^2 \\
= \; & -4p(X^+)^2(X^-X^+)^2 - 4pX^+(X^-X^+)^2 \\
& \quad - (16p^4-16p^2)(X^+)^2(X^-)^2(X^+)^2 \\
& \quad - (4p^2-2)X^+\,[(X^-)^2(X^+)^2+(X^+)^2(X^-)^2+2X^+(X^-)^2X^+]\,X^+ \\
& \quad + (16p^4-8p^2)(X^+)^2(X^-)^2(X^+)^2 \\
& \quad + 4p^2(X^+)^3(X^-)^2X^+ + 4p^2X^+(X^-)^2(X^+)^3 \\
= \; & 2(X^+)^2[X^+(X^-)^2+(X^-)^2X^+-2pX^-X^+X^-]X^+ \\
& \quad + 2X^+[X^+(X^-)^2+(X^-)^2X^+-2pX^-X^+X^-](X^+)^2 \\
= \; & 0.
\end{aligned}$$

Finally the expressions for $\tilde{G}$ and $\tilde{H}$ can be directly checked by using the relations (11) and (12). $\square$

REMARK 8. When we put $p=1$ and $G^2=1$ in the above theorem and incorporate the transformation (8), then the embedding (10) gets the form

$$\begin{array}{cccccc}
\tilde{\eta}: & \tilde{\mathscr{V}} & \hookrightarrow & \mathscr{V}_0 & & \\
\\
\tilde{p} & \mapsto & 1 & \tilde{C} & \mapsto & 8 \\
\tilde{X}^+ & \mapsto & \tilde{Z}^+ & \tilde{G} & \mapsto & 2F \\
\tilde{X}^- & \mapsto & \tilde{Z}^- & \tilde{H} & \mapsto & \frac{1}{2}F^2+\frac{3}{2}.
\end{array}$$

The relations $(\tilde{V}_2)$, $(\tilde{V}_3)$ and $(\tilde{V}_5)$, $(\tilde{V}_6)$ imply the corresponding relations in $\mathscr{V}_0$, where $p=1, G^2=1$, that appear in (8). Hence the embedding of the theorem generalizes the embedding $U(\mathfrak{osp}(1|2)) \hookrightarrow U(\mathfrak{sl}(2))$ to the p-deformed or q-deformed setting.

REMARK 9. When we put $p=-1$ and $H^2=1$ in Theorem 7 and incorporate the transformation (9), then the embedding (10) gets the form

$$\begin{array}{cccccc}
\tilde{\eta}: & \tilde{\mathscr{V}} & \hookrightarrow & \mathscr{V}_0 & & \\
\\
\tilde{p} & \mapsto & 1 & \tilde{C} & \mapsto & -8 \\
\tilde{X}^+ & \mapsto & \tilde{Z}^+ & \tilde{G} & \mapsto & -2F \\
\tilde{X}^- & \mapsto & -\tilde{Z}^- & \tilde{H} & \mapsto & -\frac{1}{2}F^2-\frac{3}{2}.
\end{array}$$

Again the relations $(\tilde{V}_2)$, $(\tilde{V}_3)$, $(\tilde{V}_5)$, $(\tilde{V}_6)$ in $\mathscr{V}_0/(p=-1,H^2=1)$ imply the corresponding relations appearing in (9).

REMARK 10. We only mention without any proof that there is another embedding, called the Jordan-Schwinger homomorphism, given by

$$\begin{array}{llllll} \hat{\eta}: & \hat{\mathscr{V}} & \hookrightarrow & \mathscr{V}_0^{\otimes 2} & & \\ & \hat{p} & \mapsto & 2p^2-1 & & \\ & \hat{X}^+ & \mapsto & X^+Y^- & \hat{G} \mapsto & \frac{1}{2}H_xG_y - \frac{1}{2}G_xH_y \\ & \hat{X}^- & \mapsto & X^-Y^+ & \hat{H} \mapsto & \frac{1}{2}G_xG_y - \frac{1}{2}H_xH_y \\ & \hat{C} & \mapsto & \multicolumn{3}{l}{-(p+1)G_xG_y-(p-1)H_xH_y-C_xH_y-C_yH_x.} \end{array}$$

Here the algebra $\mathscr{V}^{\otimes 2}$ is the tensor product of two algebras $\mathscr{V}$, it has a presentation with generators

$$X^+,X^-,C_x, \quad Y^+,Y^-,C_y$$

and the relations (V_0)–(V_3), separately for the x- and y-elements. Any x-element commutes with any y-element.

When we invoke the definitions (5) and (7), then we have the additional assignment

$$\hat{\eta}: \quad \hat{K} \mapsto S_xS'_y, \qquad \hat{K}' \mapsto S'_xS_y.$$

So we have an embedding of the universal enveloping algebra $\hat{\mathscr{U}} \hookrightarrow \mathscr{W}_0^{\otimes 2}$, cf. the presentations of $\mathscr{V}$ in (8) or (6), respectively. This was the topic in the article [3].

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