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BOUNDARY VALUE PROBLEM FOR HYBRID GENERALIZED HILFER FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This manuscript is concerned with the existence of solutions for a class of boundary value problems for nonlinear fractional hybrid differential equations involving generalized Hilfer fractional derivative. The main result is based on a fixed point theorem due to Dhage, which is illustrated with examples.

1. Introduction

Fractional calculus is a branch of classical mathematics, which generalizes the integer order differentiation and integration of a function to non-integer order. For its growing interest in theory and its applications, for instance, see [2, 3, 4, 14]. For some fundamental results in the theory of fractional calculus and fractional differential equations, one can see [1, 5, 6, 7, 8, 11] and the references therein. Some recent works on hybrid fractional differential equations can be found in [9, 12, 15, 20, 21, 22, 23].

Benchohra *et al.* [13] studied the problem:

$$\begin{cases} \left({}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r}x\right)(t) = f\left(t, x(t), \left({}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r}x\right)(t)\right), & t \in I := [a,T],\ a > 0, \\ x(T) = c \in R, \end{cases}$$

where ${}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r}$ is the generalized Hilfer fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$.

In [17], Hilal and Kajouni discussed the following hybrid problem:

$$\begin{cases} {}^{c}\mathscr{D}_{0^+}^{\vartheta}\left(\frac{u(t)}{f(t,u(t))}\right) = g(t,u(t)), \quad t \in I := [0,T], \\ c_1\left(\frac{u(0)}{f(0,u(0))}\right) + c_2\left(\frac{u(T)}{f(T,u(T))}\right) = c_3. \end{cases}$$

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The aim of the present paper is to investigate a hybrid problem given by

$$ {}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r}\left(\frac{x(t)}{f(t,x(t))}\right) = \varphi(t,x(t)),\ \ t\in(a,b], \tag{1} $$

$$ c_1\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(a^+) + c_2\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(b) = c_3, \tag{2} $$

where ${}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r}$, ${}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}$ respectively denote the generalized Hilfer derivative operator of order $\vartheta\in(0,1)$ and type $r\in[0,1]$, and generalized fractional integral of order $1-\xi$, $(\xi=\vartheta+r-\vartheta r)$, $c_1,c_2,c_3\in R$, $c_1+c_2\neq 0$, $f\in C([a,b]\times R, R\setminus\{0\})$ and $\varphi\in C([a,b]\times R,R)$.

The present paper is organized as follows. In Section 2, some notations are introduced and some preliminaries about generalized Hilfer fractional derivative and related results are recalled. In Section 3, we first present an auxiliary result and then prove the existence of solutions for the problem (1)–(2) by applying a fixed point theorem due to Dhage [16], together with mixed Lipschitz and Carathéodory conditions. Finally, in the last section, we give two examples.

2. Preliminaries

Let $0<a<b$, $J=[a,b]$. Introduce the space

$$ C_{\xi,\alpha}(J) = \left\{x:(a,b]\to R : t\to \Psi_{\xi}(t,a)x(t)\in C(J,R)\right\},\ \ 0\leqslant\xi<1, $$

where $\Psi_{\xi}(t,a)=\left(\frac{t^{\alpha}-a^{\alpha}}{\alpha}\right)^{1-\xi}$, and

$$ C^n_{\xi,\alpha}(J) = \left\{x\in C^{n-1}(J) : x^{(n)}\in C_{\xi,\alpha}(J)\right\},\ \ n\in N, $$
$$ C^0_{\xi,\alpha}(J) = C_{\xi,\alpha}(J), $$

with

$$ \|x\|_{C_{\xi,\alpha}} = \sup_{t\in J}\left|\Psi_{\xi}(t,a)x(t)\right|. $$

Denote by $X^p_c(a,b)$, $c\in R$, $1\leqslant p\leqslant\infty$, the space of the complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X^p_c}<\infty$, where the norm is defined by

$$ \|f\|_{X^p_c} = \left(\int_a^b |t^c f(t)|^p\frac{dt}{t}\right)^{\frac{1}{p}},\ \ (1\leqslant p<\infty,\ c\in R). $$

DEFINITION 1. [18] Let $\vartheta\in R_+$, $c\in R$ and $h\in X^p_c(a,b)$. The generalized fractional integral of order ϑ is defined by

$$ \left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}h\right)(t) = \int_a^t \tau^{\alpha-1}\bar{\Psi}_{\vartheta}(t,\tau)h(\tau)d\tau,\ \ t>a,\ \ \alpha>0, $$

where $\bar{\Psi}_{\vartheta}(t,\tau) = \frac{1}{\Gamma(\vartheta)}\left(\frac{t^{\alpha}-\tau^{\alpha}}{\alpha}\right)^{\vartheta-1}$.

DEFINITION 2. [18] Let $\vartheta \in R_+ \setminus \mathbb{N}$ and $\alpha > 0$. The generalized fractional derivative of order ϑ is defined by

$$\begin{aligned}\left({}^{\alpha}\mathbb{D}^{\vartheta}_{a^+}h\right)(t) &= \delta^n_{\alpha}({}^{\alpha}\mathbb{J}^{n-\vartheta}_{a^+}h)(t)\\ &= \left(t^{1-\alpha}\frac{d}{dt}\right)^n \int_a^t \tau^{\alpha-1}\bar{\Psi}_{n-\vartheta}(t,\tau)h(\tau)d\tau, \ \ t > a, \ \ \alpha > 0,\end{aligned}$$

where $n = [\vartheta]+1$ and $\delta^n_{\alpha} = \left(t^{1-\alpha}\frac{d}{dt}\right)^n$.

We define the spaces

$$C^{\vartheta,r}_{\xi,\alpha}(J) = \left\{x \in C_{\xi,\alpha}(J), \ {}^{\alpha}\mathbb{D}^{\vartheta,r}_{a^+}x \in C_{\xi,\alpha}(J)\right\},$$

and

$$C^{\xi}_{\xi,\alpha}(J) = \left\{x \in C_{\xi,\alpha}(J), \ {}^{\alpha}\mathbb{D}^{\xi}_{a^+}x \in C_{\xi,\alpha}(J)\right\},$$

where $\xi = \vartheta + r - \vartheta r$, $0 < \vartheta$, r, $\xi < 1$. Since ${}^{\alpha}\mathbb{D}^{\vartheta,r}_{a^+}x = {}^{\alpha}\mathbb{J}^{r(1-\vartheta)}_{a^+}\,{}^{\alpha}\mathbb{D}^{\xi}_{a^+}x$, it follows that

$$C^{\xi}_{\xi,\alpha}(J) \subset C^{\vartheta,r}_{\xi,\alpha}(J) \subset C_{\xi,\alpha}(J).$$

LEMMA 1. ([16]) *Let B be a closed, convex, bounded and nonempty subset of a Banach algebra $(X, \|\cdot\|)$, and let $\mathscr{P} : X \to X$ and $\mathscr{Q} : B \to X$ be two operators such that*

1) *$\mathscr{P}$ is Lipschitzian with Lipschitz constant η;*

2) *$\mathscr{Q}$ is completely continuous;*

3) *$x = \mathscr{P}x\mathscr{Q}y \Rightarrow x \in B$ for all $y \in B$;*

4) *$\eta\beta < 1$, where $\beta = \|\mathscr{Q}(B)\| = \sup\{\|\mathscr{Q}(y)\| : y \in B\}$.*

Then the operator equation $\mathscr{P}x\mathscr{Q}x = x$ has a solution in B.

3. Existence of solutions

We consider the following fractional differential equation

$$ {}^{\alpha}\mathbb{D}^{\vartheta,r}_{a^+}\left(\frac{x(t)}{f(t,x(t))}\right) = v(t), \quad t \in (a,b], \tag{3}$$

where $0 < \vartheta < 1$, $0 \leqslant r \leqslant 1$, $\alpha > 0$, with the condition

$$c_1\left({}^{\alpha}\mathbb{J}^{1-\xi}_{a^+}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(a^+) + c_2\left({}^{\alpha}\mathbb{J}^{1-\xi}_{a^+}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(b) = c_3, \tag{4}$$

The following theorem deals with an integral representation of the problem (3)–(4).

THEOREM 1. *If* $v(\cdot)\in C_{\xi,\alpha}(J)$, *and* $f\in C(J\times R, R\setminus\{0\})$, *then* x *satisfies* (3)–(4) *if and only if it satisfies*

$$x(t)=f(t,x(t))\left[\bar{\Psi}_{\xi}(t,a)\left(\frac{1}{c_1+c_2}\left[c_3-c_2\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi+\vartheta}v(\tau)\right)(b)\right]\right)+\left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}v(\tau)\right)(t)\right]. \tag{5}$$

Proof. Assume that x satisfies the equations (3) and (4) and introduce a function $g:t\mapsto\left(\frac{x(t)}{f(t,x(t))}\right)\in C_{\xi,\alpha}^{\xi}(J)$, such that

$$t\mapsto\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}g(\tau)\right)(t)\in C_{\xi,\alpha}(J),$$

and

$${}^{\alpha}\mathbb{D}_{a^+}^{\xi}g(t)=\left(\delta_{\alpha}\ {}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}g(\tau)\right)(t),$$

where

$$t\mapsto\left(\delta_{\alpha}\ {}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}g(\tau)\right)(t)\in C_{\xi,\alpha}(J).$$

Thus

$$t\mapsto\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}g(\tau)\right)(t)\in C_{\xi,\alpha}^{1}(J).$$

Hence

$$\left({}^{\alpha}\mathbb{J}_{a^+}^{\xi}\ {}^{\alpha}\mathbb{D}_{a^+}^{\xi}g(\tau)\right)(t)=g(t)-\bar{\Psi}_{\xi}(t,a)\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}g(\tau)\right)(a).$$

Then

$$\begin{aligned}\left({}^{\alpha}\mathbb{J}_{a^+}^{\xi}\ {}^{\alpha}\mathbb{D}_{a^+}^{\xi}g(\tau)\right)(t)&=\left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}\ {}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r}g(\tau)\right)(t)\\&=\left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}v(\tau)\right)(t).\end{aligned}$$

In consequence, we get

$$\frac{x(t)}{f(t,x(t))}=\bar{\Psi}_{\xi}(t,a)\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}g(\tau)\right)(a)+\left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}v(\tau)\right)(t),$$

which implies that

$$x(t)=f(t,x(t))\left[\bar{\Psi}_{\xi}(t,a)\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(a)+\left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}v(\tau)\right)(t)\right]. \tag{6}$$

Thus

$$\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(b)=\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(a)+\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi+\vartheta}v(\tau)\right)(b).$$

Then, by using condition (4), we have

$$\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(a)=\frac{c_3}{c_1+c_2}-\frac{c_2}{c_1+c_2}\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi+\vartheta}v(\tau)\right)(b). \tag{7}$$

Substituting (7) into (6), we obtain (5).

Conversely, let x satisfy (5) and $g : t \longrightarrow \left(\frac{x(t)}{f(t,x(t))}\right) \in C^{\xi}_{\xi,\alpha}(J)$. Then we obtain

$$({}^{\alpha}\mathbb{D}^{\xi}_{a^+} g(\tau))(t) = \left({}^{\alpha}\mathbb{D}^{r(1-\vartheta)}_{a^+} v(\tau)\right)(t). \tag{8}$$

Since $g \in C^{\xi}_{\xi,\alpha}(J)$, it follows by definition of $C^{\xi}_{\xi,\alpha}(J)$ that ${}^{\alpha}\mathbb{D}^{\xi}_{a^+} g \in C_{\xi,\alpha}(J)$, and hence (8) implies that

$$({}^{\alpha}\mathbb{D}^{\xi}_{a^+} g(\tau))(t) = \left(\delta_\alpha \, {}^{\alpha}\mathbb{J}^{1-r(1-\vartheta)}_{a^+} v(\tau)\right)(t) = \left({}^{\alpha}\mathbb{D}^{r(1-\vartheta)}_{a^+} v(\tau)\right)(t) \in C_{\xi,\alpha}(J). \tag{9}$$

As $v(\cdot) \in C_{\xi,\alpha}(J)$, we have

$$\left({}^{\alpha}\mathbb{J}^{1-r(1-\vartheta)}_{a^+} v\right) \in C_{\xi,\alpha}(J). \tag{10}$$

Moreover, it follows from (9) and (10) that

$$\left({}^{\alpha}\mathbb{J}^{1-r(1-\vartheta)}_{a^+} v\right) \in C^{1}_{\xi,\alpha}(J).$$

Consequently, we get

$$\begin{aligned}\left({}^{\alpha}\mathbb{D}^{\vartheta,r}_{a^+} g(\tau)\right)(t) &= {}^{\alpha}\mathbb{J}^{r(1-\vartheta)}_{a^+} \left({}^{\alpha}\mathbb{D}^{\xi}_{a^+} g(\tau)\right)(t) \\ &= v(t) - \bar{\Psi}_{r(1-\vartheta)}(t,a)\left({}^{\alpha}\mathbb{J}^{1-r(1-\vartheta)}_{a^+} v(\tau)\right)(a) \\ &= v(t),\end{aligned}$$

that is, (3) holds. Now, applying ${}^{\alpha}\mathbb{J}^{1-\xi}_{a^+}$ on both sides of (5), we obtain

$$\left({}^{\alpha}\mathbb{J}^{1-\xi}_{a^+} g(\tau)\right)(t) = \frac{1}{c_1+c_2}\left[c_3 - c_2\left({}^{\alpha}\mathbb{J}^{1-\xi+\vartheta}_{a^+} v(\tau)\right)(b)\right] + \left({}^{\alpha}\mathbb{J}^{1-\xi+\vartheta}_{a^+} v(\tau)\right)(t). \tag{11}$$

Inserting $t = a^+$ and $t = b$ in (11), we obtain

$$\left({}^{\alpha}\mathbb{J}^{1-\xi}_{a^+}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(a^+) = \frac{c_3}{c_1+c_2} - \frac{c_2}{c_1+c_2}\left({}^{\alpha}\mathbb{J}^{1-\xi+\vartheta}_{a^+} v(\tau)\right)(b), \tag{12}$$

$$\begin{aligned}\left({}^{\alpha}\mathbb{J}^{1-\xi}_{a^+}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(b) = \frac{c_3}{c_1+c_2} &- \frac{c_2}{c_1+c_2}\left({}^{\alpha}\mathbb{J}^{1-\xi+\vartheta}_{a^+} v(\tau)\right)(b) \\ &+ \left({}^{\alpha}\mathbb{J}^{1-\xi+\vartheta}_{a^+} v(\tau)\right)(b).\end{aligned} \tag{13}$$

From (12) and (13), we find that

$$c_1\left({}^{\alpha}\mathbb{J}^{1-\xi}_{a^+}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(a^+) + c_2\left({}^{\alpha}\mathbb{J}^{1-\xi}_{a^+}\left(\frac{x(\tau)}{f(\tau,x(\tau))}\right)\right)(b) = c_3,$$

that is, (4) is satisfied. This completes the proof. □

The following hypotheses will be used in the sequel.

$(A1)$ $\varphi : J \times R \to R$ is continuous on J and

$$\varphi(\cdot, x(\cdot)) \in C^{r(1-\vartheta)}_{\xi,\alpha}(J), \text{ for any } x \in C_{\xi,\alpha}(J).$$

$(A2)$ $f : J \times R \to R \setminus \{0\}$ is continuous and there exists function $p \in C(J, [0,\infty))$ that

$$|f(t,x) - f(t,\bar{x})| \leqslant p(t)\Psi_\xi(t,a)|x - \bar{x}|$$

for any $x, \bar{x} \in R$ and $t \in (a,b]$.

$(A3)$ There exists a function $\lambda \in C(J, [0,\infty))$ such that

$$|\varphi(t,x)| \leqslant \lambda(t)|x| \text{ for } t \in (a,b], \text{ and } x \in R.$$

$(A4)$ There exists a number $R > 0$ such that

$$R \geqslant \frac{f^*}{1-\ell}\left[\frac{|\phi_1|}{\Gamma(\xi)} + \lambda^* R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)} + \frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right],$$

$$\ell = p^*\left[\frac{|\phi_1|}{\Gamma(\xi)} + \lambda^* R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)} + \frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right] < 1,$$

where

$$p^* = \sup_{t \in J} p(t), \quad \lambda^* = \sup_{t \in J} \lambda(t), \quad f^* = \sup_{t \in J} |f(t,0)|$$

and

$$\phi_1 = \frac{c_3}{c_1 + c_2}, \quad \phi_2 = \frac{c_2}{c_1 + c_2}.$$

THEOREM 2. *Assume that* $(A1)$–$(A4)$ *hold. If*

$$\Psi_\xi(b,a)\ell < 1, \tag{14}$$

then (1)–(2) *admits at least one solution in* $C_{\xi,\alpha}(J)$.

Proof. We define a subset Ω of $C_{\xi,\alpha}(J)$ by

$$\Omega = \{x \in C_{\xi,\alpha}(J) : \|x\|_{\xi,\alpha} \leqslant R\}.$$

Consider the operators $\mathscr{S} : C_{\xi,\alpha}(J) \to C_{\xi,\alpha}(J)$ and $\mathscr{T} : \Omega \to C_{\xi,\alpha}(J)$ defined by

$$(\mathscr{S}x)(t) = f(t, x(t)), \quad t \in (a,b], \tag{15}$$

$$(\mathscr{T}x)(t) = \bar{\Psi}_\xi(t,a)\left(\phi_1 - \phi_2\left({}^\alpha\mathbb{J}^{1-\xi+\vartheta}_{a^+}\varphi(\tau, x(\tau))\right)(b)\right) + \left({}^\alpha\mathbb{J}^{\vartheta}_{a^+}\varphi(\tau, x(\tau))\right)(t), \tag{16}$$

$t \in (a,b]$ and set

$$\mathfrak{I}x = \mathscr{S}x\mathscr{T}x.$$

Step 1: The operator $\mathscr{S}$ is a Lipschitz on $C_{\xi,\alpha}(J)$.

Let $x,y \in C_{\xi,\alpha}(J)$ and $t \in (a,b]$. Then, by $(A2)$, we have

$$\begin{aligned}\left|((\mathscr{S}x)(t)-(\mathscr{S}y)(t))\Psi_\xi(t,a)\right| &\leqslant \Psi_\xi(t,a)|f(t,x(t))-f(t,y(t))|,\\ &\leqslant p(t)\Psi_\xi(t,a)\|x(t)-y(t)\|_{\xi,\alpha},\\ &\leqslant p^*\Psi_\xi(b,a)\|x(t)-y(t)\|_{\xi,\alpha},\end{aligned}$$

which, for each $t \in (a,b]$, implies that

$$\|\mathscr{S}x-\mathscr{S}y\|_{\xi,\alpha} \leqslant p^*\Psi_\xi(b,a)\|x(t)-y(t)\|_{\xi,\alpha}.$$

Step 2: The operator $\mathscr{T}$ is completely continuous on Ω.

We firstly show that the operator $\mathscr{T}$ is continuous on Ω. Let $\{x_n\}$ be sequence in Ω such that $x_n \to x$ in Ω. Let $x,y \in C_{\xi,\alpha}(J)$. Then, for each $t \in (a,b]$, we have

$$\begin{aligned}\left|(\mathscr{T}x_n)(t)-(\mathscr{T}x)(t))\Psi_\xi(t,a)\right| \leqslant{}& \frac{|\phi_2|}{\Gamma(\xi)}\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi+\vartheta}|\varphi(\tau,x_n(\tau))-\varphi(\tau,x(\tau))|\right)(b)\\ &+\Psi_\xi(t,a)\left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}|\varphi(\tau,x_n(\tau))-\varphi(\tau,x(\tau))|\right)(t).\end{aligned}$$

Since $x_n \to x$ and φ is a continuous function on J, therefore, $\varphi(\tau,x_n(\tau)) \to \varphi(\tau,x(\tau))$ as $n \to \infty$ for each $t \in (a,b]$. So, by Lebesgue's dominated convergence theorem, we have

$$\|\mathscr{T}x_n-\mathscr{T}x\|_{C_{\xi,\alpha}} \to 0 \text{ as } n \to \infty.$$

Hence $\mathscr{T}$ is continuous. Let $x \in \Omega$. Then

$$\begin{aligned}&\left|\Psi_\xi(t,a)(\mathscr{T}x)(t)\right|\\ &\leqslant \frac{|\phi_1|}{\Gamma(\xi)}+\frac{|\phi_2|}{\Gamma(\xi)}\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi+\vartheta}|\varphi(\tau,x(\tau))|\right)(b)+\Psi_\xi(t,a)\left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}|\varphi(\tau,x(\tau))|\right)(t)\\ &\leqslant \frac{|\phi_1|}{\Gamma(\xi)}+\lambda^*\|x\|_{C_{\xi,\alpha}}\left[|\phi_2|\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi+\vartheta}\bar{\Psi}_\xi(\tau,a)\right)(b)+\Psi_\xi(t,a)\Gamma(\xi)\left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta}\bar{\Psi}_\xi(\tau,a)\right)(t)\right]\\ &\leqslant \frac{|\phi_1|}{\Gamma(\xi)}+\lambda^*R\left[|\phi_2|\bar{\Psi}_{1+\vartheta}(b,a)+\Psi_\xi(t,a)\Gamma(\xi)\bar{\Psi}_{\vartheta+\xi}(t,a)\right]\\ &\leqslant \frac{|\phi_1|}{\Gamma(\xi)}+\lambda^*R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)}+\frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^{\vartheta}.\end{aligned}$$

Thus

$$\|\mathscr{T}x\|_{C_{\xi,\alpha}} \leqslant \frac{|\phi_1|}{\Gamma(\xi)}+\lambda^*R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)}+\frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^{\vartheta}.$$

Next we prove that the operator $\mathscr{T}\Omega$ equicontinuous. We take $x \in \Omega$ and $a < \varepsilon_1 < \varepsilon_2 \leqslant b$. Then

$$\begin{aligned}
&\left|\Psi_\xi(\varepsilon_1,a)(\mathscr{T}x)(\varepsilon_1) - \Psi_\xi(\varepsilon_2,a)(\mathscr{T}x)(\varepsilon_2)\right| \\
&\leqslant \left|\Psi_\xi(\varepsilon_1,a)\left({}^\alpha\mathbb{J}^\vartheta_{a^+}\varphi(\tau,x(\tau))\right)(\varepsilon_1) - \Psi_\xi(\varepsilon_2,a)\left({}^\alpha\mathbb{J}^\vartheta_{a^+}\varphi(\tau,x(\tau))\right)(\varepsilon_2)\right| \\
&\leqslant \int_a^{\varepsilon_1} |\Psi_\xi(\varepsilon_1,a)\bar{\Psi}_\vartheta(\varepsilon_1,\tau) - \Psi_\xi(\varepsilon_2,a)\bar{\Psi}_\vartheta(\varepsilon_2,\tau)|\left|\tau^{\alpha-1}\varphi(\tau,x(\tau))\right| d\tau \\
&\quad + \Psi_\xi(\varepsilon_2,a)\left({}^\alpha\mathbb{J}^\vartheta_{\varepsilon_1^+}|\varphi(\tau,x(\tau))|\right)(\varepsilon_2) \\
&\leqslant R\lambda^*\Gamma(\xi)\int_a^{\varepsilon_1} \tau^{\alpha-1}|\Psi_\xi(\varepsilon_1,a)\bar{\Psi}_\vartheta(\varepsilon_1,\tau) - \Psi_\xi(\varepsilon_2,a)\bar{\Psi}_\vartheta(\varepsilon_2,\tau)|\bar{\Psi}_\xi(\tau,a)d\tau, \\
&\quad + R\lambda^*\Gamma(\xi)\Psi_\xi(\varepsilon_2,a)\bar{\Psi}_{\vartheta+\xi}(\varepsilon_2,\varepsilon_1).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\left|\Psi_\xi(\varepsilon_1,a)(\mathscr{T}x)(\varepsilon_1) - \Psi_\xi(\varepsilon_2,a)(\mathscr{T}x)(\varepsilon_2)\right| \\
&\leqslant R\lambda^*\Gamma(\xi)\int_a^{\varepsilon_1} \tau^{\alpha-1}|\Psi_\xi(\varepsilon_1,a)\bar{\Psi}_\vartheta(\varepsilon_1,\tau) - \Psi_\xi(\varepsilon_2,a)\bar{\Psi}_\vartheta(\varepsilon_2,\tau)|\bar{\Psi}_\xi(\tau,a)d\tau, \\
&\quad + R\lambda^*\Gamma(\xi)\Psi_\xi(\varepsilon_2,a)\bar{\Psi}_{\vartheta+\xi}(\varepsilon_2,\varepsilon_1).
\end{aligned}$$

Since

$$\left|\Psi_\xi(\varepsilon_1,a)(\mathscr{T}x)(\varepsilon_1) - \Psi_\xi(\varepsilon_2,a)(\mathscr{T}x)(\varepsilon_2)\right| \to 0 \quad \text{as} \quad \varepsilon_1 \to \varepsilon_2,$$

therefore, $\mathscr{T}\Omega$ is equicontinuous on J. Hence, by the Arzela-Ascoli Theorem, $\mathscr{T}$ is completely continuous on Ω.

Step 3: Now we verify the third hypothesis of Lemma 1. Let $x \in C_{\xi,\alpha}(J)$ and $y \in \Omega$ be arbitrary such that $x = \mathscr{S}x\mathscr{T}y$. Then, for $t \in (a,b]$, we have

$$\begin{aligned}
&\left|\Psi_\xi(t,a)x(t)\right| \\
&= \left|\Psi_\xi(t,a)(\mathscr{S}x\mathscr{T}y)(t)\right| \\
&= \Psi_\xi(t,a)\,|(\mathscr{S}x)(t)|\,|(\mathscr{T}y)(t)| \\
&= \left[\frac{|\phi_1|}{\Gamma(\xi)} + \frac{|\phi_2|}{\Gamma(\xi)}\left({}^\alpha\mathbb{J}^{1-\xi+\vartheta}_{a^+}|\varphi(\tau,y(\tau))|\right)(b) + \Psi_\xi(t,a)\left({}^\alpha\mathbb{J}^\vartheta_{a^+}|\varphi(\tau,y(\tau))|\right)(t)\right] \\
&\quad \times |f(t,x(t))| \\
&\leqslant (|f(t,x(t)) - f(t,0)| + |f(t,0)|) \\
&\quad \times \left[\frac{|\phi_1|}{\Gamma(\xi)} + \lambda^* R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)} + \frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right] \\
&\leqslant \left(p^*\|x\|_{C_{\xi,\alpha}} + f^*\right)\left[\frac{|\phi_1|}{\Gamma(\xi)} + \lambda^* R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)} + \frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right].
\end{aligned}$$

Thus

$$\|x\|_{C_{\xi,\alpha}} = \frac{f^*\left[\frac{|\phi_1|}{\Gamma(\xi)}+\lambda^*R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)}+\frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^\vartheta\right]}{1-p^*\left[\frac{|\phi_1|}{\Gamma(\xi)}+\lambda^*R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)}+\frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^\vartheta\right]} \leqslant R.$$

This shows that the third hypothesis of Lemma 1 is satisfied.

Step 4: Now we show that $p^*\Psi_\xi(b,a)L<1$, where

$$L=\|\mathscr{T}(\Omega)\|_{C_{\xi,\alpha}}=\sup\{\|\mathscr{T}y\|_{C_{\xi,\alpha}}:y\in\Omega\}.$$

Since

$$L\leqslant\frac{|\phi_1|}{\Gamma(\xi)}+\lambda^*R\left[\frac{|\phi_2|}{\Gamma(1+\vartheta)}+\frac{\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\right]\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^\vartheta,$$

we have that $p^*\Psi_\xi(b,a)L\leqslant\ell\Psi_\xi(b,a)<1$, that is, the last hypothesis of Lemma 1 is satisfied. Thus the operator equation $\mathfrak{I}x=\mathscr{S}x\mathscr{T}x=x$ has at least one solution $x^*\in C_{\xi,\alpha}$, which is a fixed point of the operator $\mathfrak{I}$.

Step 5: For a fixed point $x^*\in C_{\xi,\alpha}(J)$, we show that

$$g:t\to\frac{x^*(t)}{f(t,x^*(t))}\in C^{\xi}_{\xi,\alpha}(J).$$

Since x^* is a fixed point of operator $\mathfrak{I}$ in $C_{\xi,\alpha}(J)$, we have

$$\mathfrak{I}x^*(t)=f(t,x^*(t))\Big[\bar{\Psi}_\xi(t,a)\Big(\phi_1-\phi_2\Big({}^\alpha\mathbb{J}^{1-\xi+\vartheta}_{a^+}\varphi(\tau,x^*(\tau))\Big)(b)\Big)+\Big({}^\alpha\mathbb{J}^{\vartheta}_{a^+}\varphi(\tau,x^*(\tau))\Big)(t)\Big].$$

Applying ${}^\alpha\mathbb{D}^\xi_{a^+}$ to both sides, we get

$${}^\alpha\mathbb{D}^\xi_{a^+}\left(\frac{x^*(t)}{f(t,x^*(t))}\right)=\Big({}^\alpha\mathbb{D}^\xi_{a^+}\,{}^\alpha\mathbb{J}^\vartheta_{a^+}\varphi(\tau,x^*(\tau))\Big)(t)=\Big({}^\alpha\mathbb{D}^{r(1-\vartheta)}_{a^+}\varphi(\tau,x^*(\tau))\Big)(t).$$

Thus ${}^\alpha\mathbb{D}^\xi_{a^+}g\in C_{\xi,\alpha}(J)$. Clearly $g\in C_{\xi,\alpha}(J)$. As $f\in C(J\times R\to R\setminus\{0\})$, we have $g\in C^\xi_{\xi,\alpha}(J)$. In view of the foregoing arguments, Lemma 1 implies that (1)–(2) has at least one solution in $C_{\xi,\alpha}(J)$. This completes the proof. □

4. Examples

EXAMPLE 1. Consider the terminal problem

$$
{}^{\frac{1}{2}}\mathbb{D}_{e^+}^{\frac{1}{2},0}\left(\frac{x(t)}{f(t,x(t))}\right)=\frac{(\sqrt{t}-\sqrt{e})^{\frac{1}{2}}x(t)}{105e^{-t+\pi}(1+|\cos(t)|(\sqrt{t}-\sqrt{e})^{\frac{1}{2}}|x(t)|)}, \quad \text{for each } t\in(e,\pi], \tag{17}
$$

$$
\left({}^{\frac{1}{2}}\mathbb{J}_{e^+}^{\frac{1}{2}}\left(\frac{x(\tau)}{f(t,x(\tau))}\right)\right)(\pi)=\frac{1}{2}, \tag{18}
$$

where $J=[e,\pi]$, $a=e$, $b=\pi$ and

$$
f(t,x(t))=\frac{\sqrt{t}-\sqrt{e}}{52e^{\pi-t}}\left(|\sin(t)|x(t)+\tan^{-1}(t)+\pi\right),\ \ t\in J,\ \ x\in C_{\frac{1}{2},\frac{1}{2}}(J).
$$

Set

$$
\varphi(t,x)=\frac{(\sqrt{t}-\sqrt{e})^{\frac{1}{2}}x}{105e^{-t+\pi}(1+|\cos(t)|(\sqrt{t}-\sqrt{e})^{\frac{1}{2}}|x|)},\ \ t\in J,\ \ x\in R.
$$

We have

$$
C_{\xi,\alpha}^{r(1-\vartheta)}(J)=C_{\frac{1}{2},\frac{1}{2}}^{0}(J)=\left\{u:(e,\pi]\to R:\sqrt{2}(\sqrt{t}-\sqrt{e})^{\frac{1}{2}}u\in C(J,R)\right\},
$$

with $\xi=\vartheta=\frac{1}{2}$, $\alpha=\frac{1}{2}$, $r=0$. Clearly, the continuous function $\varphi\in C_{\frac{1}{2},\frac{1}{2}}^{0}(J)$. Hence the condition $(A1)$ is satisfied.

Further, we have

$$
|f(t,x)-f(t,\bar{x})|\leqslant\frac{(\sqrt{t}-\sqrt{e})|\sin(t)|}{52e^{\pi-t}}|x-\bar{x}|,
$$

which shows that condition $(A2)$ is satisfied with

$$
p(t)=\frac{(\sqrt{t}-\sqrt{e})^{\frac{1}{2}}|\sin(t)|}{52\sqrt{2}e^{\pi-t}} \quad \text{and} \quad p^*\leqslant\frac{1}{52\sqrt{2}}.
$$

Let $x\in R$. Then we have

$$
|\varphi(t,x)|\leqslant\frac{(\sqrt{t}-\sqrt{e})^{\frac{1}{2}}|x|}{105e^{-t+\pi}},\ t\in J,
$$

and so the condition $(A3)$ is satisfied with

$$
\lambda(t)=\frac{(\sqrt{t}-\sqrt{e})^{\frac{1}{2}}}{105e^{-t+\pi}}, \quad \text{and} \quad \lambda^*\leqslant\frac{1}{105}.
$$

Also, the condition $(A4)$ and the condition (14) of Theorem 2 are satisfied as

$$
2655\approx\frac{283920\sqrt{2\pi}-5460}{104\sqrt{2}(\sqrt{\pi}-\sqrt{e})^{\frac{1}{2}}(2+\pi)}\leqslant R<\left(1-\frac{1}{104\sqrt{2\pi}}\right)\frac{5460\sqrt{\pi}(2+\pi)^{-1}}{(\sqrt{\pi}-\sqrt{e})^{\frac{1}{2}}}\approx 5330.
$$

Thus the problem (17) – (18) has at least one solution in $C_{\frac{1}{2},\frac{1}{2}}(J)$.

EXAMPLE 2. Consider the initial value problem:

$$ {}^{1}\mathbb{D}_{1^+}^{\frac{1}{2},0}\left(\frac{x(t)}{f(t,x(t))}\right)=\frac{e^{t-2}\sqrt{t-1}\ln(t)x(t)}{333(1+\|x\|_{C_{\frac{1}{2},1}})},\quad \text{for each } t\in(1,2], \tag{19} $$

$$ \left({}^{1}\mathbb{J}_{1^+}^{\frac{1}{2}}\left(\frac{x(\tau)}{f(t,x(\tau))}\right)\right)(1^+)=1, \tag{20} $$

where $J=[1,2]$, $a=1$, $b=2$ and

$$ f(t,x(t))=\frac{\sqrt{t-1}|\tan^{-1}(t)|x(t)}{111e^{-t+3}}+\frac{\ln(|\cos(t)|+\sqrt{t})}{e^3\sqrt{t}},\ \ t\in J,\ \ x\in C_{\frac{1}{2},1}(J). $$

Set

$$ \varphi(t,x(t))=\frac{e^{t-2}\sqrt{t-1}\ln(t)x(t)}{333(1+\|x\|_{C_{\frac{1}{2},1}})},\ \ t\in J,\ \ x\in C_{\frac{1}{2},1}(J). $$

We have

$$ C_{\xi,\alpha}^{r(1-\vartheta)}(J)=C_{\frac{1}{2},1}^{0}(J)=\left\{u:(1,2]\to R:(\sqrt{t-1})u\in C(J,R)\right\}, $$

with $\xi=\vartheta=\frac{1}{2}$, $\alpha=1$, $r=0$. Clearly, $\varphi\in C_{\frac{1}{2},1}^{0}(J)$. Hence the condition $(A1)$ is satisfied. Observe that

$$ |f(t,x)-f(t,\bar{x})|\leqslant\frac{\sqrt{t-1}|\tan^{-1}(t)|}{111e^{-t+3}}|x-\bar{x}|, $$

which implies that condition $(A2)$ is satisfied with

$$ p(t)=\frac{|\tan^{-1}(t)|}{111e^{-t+3}},\quad \text{and}\quad p^*=\frac{1}{222e}. $$

Let $x\in R$. Then we have

$$ |\varphi(t,x)|\leqslant\frac{e^{t-2}\sqrt{t-1}\ln(t)|x|}{333},\ t\in J, $$

and so the condition $(A3)$ is satisfied with

$$ \lambda(t)=\frac{e^{t-2}\sqrt{t-1}\ln(t)}{333},\quad \text{and}\quad \lambda^*=\frac{\ln(2)}{333}. $$

As in the last example, the condition $(A4)$ and the condition (14) of Theorem 2 hold for a proper value of R. Thus the problem (19) – (20) has at least one solution in $C_{\frac{1}{2},1}(J)$.

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