EXISTENCE OF SOLUTIONS FOR NONLOCAL ELLIPTIC SYSTEMS WITH EXPONENTIAL NONLINEARITY

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Abstract. In this paper, we establish the existence of solutions for a Kirchhoff-type system with Dirichlet boundary condition and nonlinearities having exponential critical growth. Our approach is based on the Trudinger-Moser inequality and on a minimax theorem.

1. Introduction

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^2 \). In this article, we study the existence of positive solutions to the following nonlinear Kirchhoff type system

\[
\begin{align*}
-m(\int_{\Omega} |\nabla u|^2 \, dx) \Delta u &= f(u, v) \quad \text{in} \ \Omega, \\
-m(\int_{\Omega} |\nabla v|^2 \, dx) \Delta v &= g(u, v) \quad \text{in} \ \Omega, \\
u = v = 0 & \quad \text{on} \ \partial \Omega,
\end{align*}
\]

where the nonlinear terms \( f \) and \( g \) are allowed to have exponential critical growth, and \( m : \mathbb{R}^+ \to \mathbb{R}^+ \), is a continuous function that satisfy some conditions which will be stated later on. By means of the Trudinger-Moser inequality, we shall consider the variational situation in which \( \nabla F(u, v) = (f(u, v), g(u, v)) \), for some function \( F : \mathbb{R}^2 \to \mathbb{R} \) of class \( C^2 \), where \( \nabla F \) stands for the gradient of \( F \) in the variables \( w = (u, v) \in \mathbb{R}^2 \).

We make the following assumptions on the function \( m \)

(M_1) There exist real numbers \( m_0, m_1, m_2 > 0 \) and \( \kappa \geq 1 \) such that

\[
m_0 \leq m(t) \leq m_1 t^{\kappa - 1} + m_2, \quad \text{for all} \ t \geq 0.
\]

(M_2) \( M(s) + M(t) \leq M(s + t) \ \forall s, t \geq 0 \) where \( M(t) = \int_0^t m(x) \, dx \)

(M_3) \( m(t)/t \) is nonincreasing for \( t > 0 \).


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A typical example of a function satisfying the conditions \((M_1)-(M_3)\) is given by \(m(t) = m_0 + bt\) with \(b > 0\) and for all \(t \geq 0\). As a consequence of \((M_3)\), a straightforward computation shows that

\[
M(t) - \frac{1}{2}m(t)t \text{ is nondecreasing for } t \geq 0. \tag{1.2}
\]

System \((1.1)\) is related to the stationary version of a model established by Kirchhoff [12]. More precisely, Kirchhoff proposed the following model

\[
ρ \frac{∂^2 u}{∂t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{∂u}{∂x} \right|^2 dx \right) \frac{∂^2 u}{∂x^2} = 0 \tag{1.3}
\]

which extends D’Alembert’s wave equation with free vibrations of elastic strings, where \(ρ\) denotes the mass density, \(P_0\) denotes the initial tension, \(h\) denotes the area of the cross section, \(E\) denotes the Young modulus of the material, and \(L\) denotes the length of the string.

In the recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [3], [4], [10], [19], [8] in which the authors have used the variational method and the topological method to get the existence of solutions. In [8], by a direct variational approach, the authors establish the existence of a positive ground state solution for a nonlocal Kirchhoff of the type

\[
\begin{aligned}
-m(\int_Ω |\nabla u|^2 dx)Δu &= f(x,u) \quad \text{in } Ω, \\
u &\bigg|_{Ω} = 0 \quad \text{on } ∂Ω.
\end{aligned}
\]

Motivated by above and the ideas introduced in [18], in this work, we will study the existence of nontrivial solutions for problem \((1.1)\).

Let us introduce the precise assumptions under which our problem is studied.

\((H_0)\) \(f, g\) are \(C^1\) functions such that \(f(s,t) > 0, g(s,t) > 0\) for all \(s,t > 0\), and \(f(s,t) = g(s,t) = 0\) if \(s \leq 0\) or \(t \leq 0\).

\((H_1)\) \(f(s,t) = o(\sqrt{s^2 + t^2})^\mu\) and \(g(s,t) = o(\sqrt{s^2 + t^2})^\mu\) as \(|(s,t)| \to 0\), for some \(\mu \in [0,4]\).

\((H_2)\) \(f\) and \(g\) have \(α_0\)-exponential critical growth, i.e., there exists \(α_0 > 0\) such that

\[
\lim_{s^2 + t^2 \to +∞} \frac{f(s,t)}{e^{α_0(s^2 + t^2)/2}} = \lim_{s^2 + t^2 \to +∞} \frac{g(s,t)}{e^{α_0(s^2 + t^2)/2}} = \begin{cases} 0, & ∀α > α_0 \\
+∞, & ∀α < α_0 \end{cases}.
\]

\((H_3)\) There exists \(θ > 4\) such that

\[
0 < θF(s,t) \leq f(s,t)s + g(s,t)t, \quad ∀(s,t) ∈ ℝ^2 \setminus \{(0,0)\}.
\]

\((H_4)\) For every \(ν > 1\), there exists a constant \(K_0\) such that

\[
F(s,t) ≥ K_0(s^2 + t^2)^ν \quad \text{for all } s,t > 0
\]
\((H_5)\) \quad \frac{f(s,t)}{s^3} \leq \frac{f(\overline{s},\overline{t})}{\overline{s}^3} \quad \text{and} \quad \frac{g(s,t)}{t^3} \leq \frac{g(\overline{s},\overline{t})}{\overline{t}^3} \quad \text{for} \quad \overline{s} \geq s > 0, \overline{t} \geq t > 0.

\((H_6)\) For all \(s,t > 0,\)

\[3f(s,t) < \frac{\partial f(s,t)}{\partial s} s + \frac{\partial g(s,t)}{\partial s} t \quad \text{and} \quad 3g(s,t) < \frac{\partial f(s,t)}{\partial t} s + \frac{\partial g(s,t)}{\partial t} t.\]

We observe that condition \((H_6)\) implies

\[f(s,t) + g(s,t) - 4F(s,t) < f(\overline{s},\overline{t}) + g(\overline{s},\overline{t}) - 4F(\overline{s},\overline{t}) \quad \text{for} \quad \overline{s} > s > 0 \quad \text{and} \quad \overline{t} > t > 0.\]

An example of a function satisfying the above assumptions with \(\alpha_0 = 1\) is

\[F(s,t) = \begin{cases} (s^4 + t^4)e^{(s^2 + t^2)/2} \quad \text{if} \quad s > 0, \ t > 0 \\ 0 \quad \text{otherwise}. \end{cases}\]

Now, we are ready to state our main result

**Theorem 1.** Under assumption \((M_1)-(M_4)\) and \((H_0)-(H_5)\), Problem \((1.1)\) admits at least one nontrivial solution \((u_0,v_0) \in H^1_0(\Omega) \times H^1_0(\Omega)\).

This work is organised as follows: In Section 2, we present the variational setting in which our problem will be treated, and some preliminary results. Section 3 is devoted to show that the energy functional has the mountain pass geometry and in Section 4 we obtain an estimate for the minimax level associated to our functional. Finally, we prove our main result in Section 5.

### 2. Preliminaries

As mentioned in the introduction, the nonlinearities \(f\) and \(g\) are allowed to have exponential critical growth which allows to treat the problem by variational methods. This growth is given by the so-called Trudinger-Moser inequality (see [17], [22]), which says that if \(u\) is a \(H^1_0(\Omega)\) function then there exists a constant \(C > 0\) such that

\[
\sup_{\|u\|_{H^1_0(\Omega)} = 1} \int_{\Omega} e^{\alpha u^2} \, dx \leq C |\Omega| \quad \text{if} \quad \alpha \leq 4\pi. \quad (2.1)
\]

Let \(H := H^1_0(\Omega) \times H^1_0(\Omega)\) be the Sobolev space endowed with the norm

\[
\| (u,v) \|_H := \left( \| u \|_{1,2}^2 + \| v \|_{1,2}^2 \right)^{1/2} \quad \text{where} \quad \| u \|_{1,2} = \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}
\]

and \(\| u \|_r\) denotes the norm in \(L^r(\Omega)\), i.e. \(\| u \|_r = \left( \int_{\Omega} |u|^r \, dx \right)^{1/r}\). In this work, we shall use the following adapted version of Moser-Trudinger inequality for the pair \((u,v)\) [18]:
Proof of Lemma 1. Let \((u, v) \in \mathcal{H}\), then \(\int_\Omega e^{\gamma (u^2 + v^2)} dx < +\infty\) for any \(\gamma > 0\). Moreover, there exists a constant \(C = C(\Omega)\) such that

\[
\sup_{\| (u, v) \| \leq 1} \int_\Omega e^{\gamma (u^2 + v^2)} dx \leq C, \quad \text{provided that} \quad \gamma \leq 4\pi.
\]

We shall look for solutions of (1.1) by finding critical points of the energy functional \(E : \mathcal{H} \to \mathbb{R}\) given by

\[
E(u, v) = \frac{1}{2} M\left( \int_\Omega |\nabla u|^2 dx \right) + \frac{1}{2} M\left( \int_\Omega |\nabla v|^2 dx \right) - \int_\Omega F(u, v) dx,
\]

where \(M(t) = \int_0^t m(s) ds\). Under our assumptions we have that \(E\) is well defined and it is \(C^1\) on \(\mathcal{H}\). Indeed, by \((H_1), (H_2)\) and for \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
f(u, v) \leq \varepsilon \sqrt{u^2 + v^2} \quad \text{and} \quad g(u, v) = \varepsilon \sqrt{u^2 + v^2} \quad \text{always that} \quad |(u, v)| < \delta.
\]

On the other hand, for \(\alpha > \alpha_0\), there exist constants \(C > 0\) such that

\[
f(u, v) \leq Ce^{\alpha (u^2 + v^2)/2} \quad \text{and} \quad g(u, v) \leq Ce^{\alpha (u^2 + v^2)/2} \quad \text{for all} \quad |(u, v)| \geq \delta.
\]

Thus, for all \((u, v) \in \mathcal{H}\) we have

\[
f(u, v) \leq \varepsilon \sqrt{u^2 + v^2} + Ce^{\alpha (u^2 + v^2)/2}, \tag{2.2}
\]

and

\[
g(u, v) \leq \varepsilon \sqrt{u^2 + v^2} + Ce^{\alpha (u^2 + v^2)/2}. \tag{2.3}
\]

Hence, using \((H_3), (2.2)\) and \((2.3)\), we obtain

\[
|F(u, v)| \leq \varepsilon (u^2 + v^2) + C \sqrt{u^2 + v^2} e^{\alpha (u^2 + v^2)/2}. \tag{2.4}
\]

This inequality together with Lemma 1 yields \(F(u, v) \in L^1(\Omega)\) for all \((u, v) \in \mathcal{H}\), which implies that \(E\) is well defined, for \(\alpha > \alpha_0\). Using standard arguments, we can see that \(E \in C^1(\mathcal{H}, \mathbb{R})\) with

\[
E'(u, v)(\phi, \psi) = m(\int_\Omega |\nabla u|^2 dx) \int_\Omega \nabla u. \nabla \phi \ dx + m(\int_\Omega |\nabla v|^2 dx) \int_\Omega \nabla v. \nabla \psi \ dx
\]

\[
- \int_\Omega f(u, v) \phi \ dx - \int_\Omega g(u, v) \psi \ dx,
\]

for all \((\phi, \psi) \in \mathcal{H}\).

Also, to prove our main result, we use the following version of Lion’s higher integrability lemma [18]:

Lemma 2. Let \((u_n, v_n)\) be a sequence in \(\mathcal{H}\) such that \(\|(u_n, v_n)\|_{\mathcal{H}} = 1\), for all \(n \in \mathbb{N}^*\) and \(u_n \rightharpoonup u, \quad v_n \rightharpoonup v\) in \(H^1_0(\Omega)\) for some \((u, v) \neq (0, 0)\).

Then, for \(4\pi < p < 4\pi \left(1 - \|(u, v)\|_{\mathcal{H}}^2\right)^{-1}\),

\[
\sup_{n \geq 1} \int_\Omega e^{p(u_n^2 + v_n^2)} dx < \infty.
\]
3. The Mountain Pass Geometry

In this section, we prove that the functional $E$ has the Mountain Pass Geometry. This fact is proved in the next lemmas:

**Lemma 3.** Assume $(M_1)$ and $(H_0)-(H_3)$, then there exist positive constants $\tau$ and $\rho$ such that

$$E(u, v) \geq \tau, \forall (u, v) \in \mathcal{H} : \| (u, v) \|_\mathcal{H} = \rho.$$  

**Proof.** Just as we have obtained (2.4), we deduce that

$$|F(u, v)| \leq \epsilon (u^2 + v^2) + C(u^q + v^q) e^{\alpha(u^2+v^2)/2},$$

for all $(u, v) \in \mathcal{H}$ and $q > 2$. Using Hölder’s inequality and the Sobolev embedding, we have

$$\int_\Omega |F(u, v)| \, dx \leq \epsilon \left( \|u\|_{H^1}^2 + \|v\|_{H^1}^2 \right) + C \left( \left( \int_\Omega (|u|^q + |v|^q)^{\frac{2}{q}} \right)^\frac{1}{2} \left( \int_\Omega e^{\alpha(u^2+v^2)} \, dx \right)^{1/2} \right) \leq \epsilon C \| (u, v) \|_\mathcal{H}^2 + C \left( \|u\|_{1,2}^q + \|v\|_{1,2}^q \right) \times \left( \int_\Omega e^{\alpha \| (u, v) \|_\mathcal{H}^2} \left( \frac{u}{\| (u, v) \|_\mathcal{H}} \right)^2 + \left( \frac{v}{\| (u, v) \|_\mathcal{H}} \right)^2 \, dx \right)^{1/2}.$$  

Now, for $\| (u, v) \|_\mathcal{H} = \rho$ such that $\rho^2 \leq \pi/\alpha$ and by the Moser-Trudinger inequality, we obtain

$$\int_\Omega |F(u, v)| \, dx \leq \epsilon C \| (u, v) \|_\mathcal{H}^2 + C \left( \|u\|_{1,2}^q + \|v\|_{1,2}^q \right) \leq \epsilon C \| (u, v) \|_\mathcal{H}^2 + 2C \| (u, v) \|_\mathcal{H}^q.$$  

Therefore, using $(M_1)$, we get

$$E(u, v) \geq \left( \frac{m_0}{2} - \epsilon C \right) \| (u, v) \|^2_\mathcal{H} - 2C \| (u, v) \|^q_\mathcal{H}.$$  

Consequently

$$E(u, v) \geq \left( \frac{m_0}{2} - \epsilon C \right) \rho^2 - 2C \rho^q.$$  

Now, we may fix $\epsilon > 0$ such that $\frac{m_0}{2} - \epsilon C > 0$. Thus, for $\rho > 0$ sufficiently small there exists $\tau := \left( \frac{m_0}{2} - \epsilon C \right) \rho^2 - 2C \rho^q > 0$ such that

$$E(u, v) \geq \tau, \forall (u, v) \in \mathcal{H} \text{ with } \| (u, v) \|_\mathcal{H} = \rho.$$  

The proof of Lemma is complete. □

**Lemma 4.** Assume $(M_1)$ and $(H_4)$. Then, there exists $(e_1, e_2) \in \mathcal{H}$ such that

$$E(e_1, e_2) < 0 \text{ and } \| (e_1, e_2) \| \geq \rho.$$

Proof. Using $(M_1)$ and $(H_4)$, we obtain

\[
E(u,v) \leq \frac{m_1}{2\kappa} \left( \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla v|^2 \, dx \right) - K_0 \int_\Omega (u^2 + v^2) \, dx
\]

Let $(u_0,v_0) \in \mathcal{H}$ with $u_0,v_0 > 0$ in $\Omega$ and $\|(u_0,v_0)\|_{\mathcal{H}} = 1$. Thus, we have

\[
E(tu_0,tv_0) \leq \frac{m_1}{\kappa} t^{2\kappa} + m_2 \|(u,v)\|_{\mathcal{H}}^2 - K_0 \int_\Omega (u^2 + v^2) \, dx,
\]

for all $t > 0$, which yields $E(tu,tv) \to -\infty$ as $t \to +\infty$, provided $v > \kappa$. Setting $(e_1,e_2) = (tu_0, tv_0)$ with $T > 0$ large enough, the proof is complete. □

4. On the mini-max level

In view of Lemmas 3 and 4, we may apply a version of the Mountain Pass theorem without Palais-Smale condition to obtain a sequence $(u_n,v_n) \in \mathcal{H}$ such that

\[ E(u_n,v_n) \to \rho_0 \text{ and } E'(u_n,v_n) \to 0, \]

where

\[ \rho_0 = \inf_{\varphi \in \Gamma} \max_{t \in [0,1]} E(\varphi(t)), \tag{4.1} \]

with

\[ \Gamma = \{ \varphi \in C([0,1],\mathcal{H}) : \varphi(0) = (0,0), \ \varphi(1) = (e_1,e_2) \}. \]

Let $d$ be the inner radius of $\Omega$, that is, it is the radius of the largest open ball contained in $\Omega$. So $B_d(x_0) \subset \Omega$. We may assume that $x_0 = 0$. In order to get more information about the minimax level, it was crucial in our argument to consider the following concentrating functions $\psi_n(x) = \tilde{\psi}_n(x/d)$, $n \in \mathbb{N}$ where

\[
\tilde{\psi}_n(x) = \frac{1}{2\sqrt{\pi}} \begin{cases} 
(\log n)^{1/2} & \text{for } 0 \leq |x| \leq 1/n \\
\frac{\log(1/|x|)}{(\log n)^{1/2}} & \text{for } 1/n \leq |x| \leq 1 \\
0 & \text{for } |x| \geq 1.
\end{cases}
\]

Then, $\psi_n$ has support in $B_d(0)$ and $(\psi_n,\psi_n)$ is such that $\|(\psi_n,\psi_n)\|_{\mathcal{H}} = 1$ $\forall n \in \mathbb{N}$. We can now prove the following upper bounded for $\rho_0$. 


LEMMA 5. With \( \rho_0 \) defined as in (4.1), we have \( \rho_0 < \frac{1}{2}M(4\pi/\alpha_0) \).

Proof. Suppose \( \rho_0 \geq \frac{1}{2}M(4\pi/\alpha_0) \) and we derive a contradiction. As \( E \) possesses the mountain pass geometry, for each \( n \) there exist \( t_n, s_n > 0 \) such that
\[
E(t_n \psi_n, s_n \psi_n) = \sup_{t, s > 0} E(t \psi_n, s \psi_n) \geq \frac{1}{2}M(4\pi/\alpha_0), \forall n \in \mathbb{N}.
\]
From this inequality and using that \( F(u, v) \geq 0 \) for all \( (u, v) \in \mathbb{R}_+^2 \), we obtain
\[
\frac{1}{2}(M(t_n ||\psi_n||^2_{1,2}) + M(s_n ||\psi_n||^2_{1,2})) \geq \frac{1}{2}M(4\pi/\alpha_0).
\]
Using (\( M_2 \)) and since \( M \) is an increasing bijection, we have
\[
t_n + s_n \geq 4\pi/\alpha_0.
\]
On the other hand, \( (t_n, s_n) \) is a critical point of \( E(t \psi_n, s \psi_n) \), so
\[
E'(t \psi_n, s \psi_n)_{(t, s) = (t_n, s_n)} = 0
\]
and therefore
\[
m(\frac{t_n^2}{2})t_n^2 + m(\frac{s_n^2}{2})s_n^2 = \int_{\Omega} \left( f(t_n \psi_n, s_n \psi_n)t_n \psi_n + g(t_n \psi_n, s_n \psi_n)s_n \psi_n \right) dx.
\]
Now, using that \( t_n \psi_n \to \infty, s_n \psi_n \to \infty \) on \( \{ |x| \leq \delta/n \} \) and (\( H2 \)), we obtain
\[
m(\frac{t_n^2}{2})t_n^2 + m(\frac{s_n^2}{2})s_n^2 \geq \int_{\Omega \setminus \{ |x| \leq d/n \}} e^{\alpha_0(\frac{\alpha^2 + \beta^2}{2})} \psi_n(t_n + s_n) \psi_n dx
\]
\[
= \frac{\sqrt{\pi d^2}}{2n^2} e^{\frac{\alpha_0(\alpha^2 + \beta^2)}{2}(\log n)^{1/2}} (t_n + s_n)(\log n)^{1/2}
\]
\[
= \frac{\sqrt{\pi d^2}}{2} e^{\alpha_0(\frac{\alpha^2 + \beta^2}{8\pi}) - 2} \log n (t_n + s_n)(\log n)^{1/2},
\]
and from (\( M_1 \)), we can conclude that
\[
\frac{m_1}{2^{K-1}}(t_n^2 + s_n^2)^{\kappa} \geq \frac{\sqrt{\pi d^2}}{2} e^{\alpha_0(\frac{\alpha^2 + \beta^2}{8\pi}) - 2} \log n (t_n + s_n)(\log n)^{1/2}.
\]
Note that, we can see
\[
\frac{m_1}{2^{K-1}}(t_n^2 + s_n^2)^{\kappa} \to 0 \text{ if } t_n^2 + s_n^2 \to +\infty.
\]
It follows from this and (4.3), we infer that
\[
t_n^2 + s_n^2 \to 16\pi/\alpha_0.
\]
Moreover, using (4.3) again, we obtain
\[
\frac{m_1}{2^{K-1}}(t_n^2 + s_n^2)^{\kappa} \geq \frac{\sqrt{\pi d^2}}{2} (t_n + s_n)(\log n)^{1/2}
\]
This in turn implies that \( t_n^2 + s_n^2 \to \infty \) as \( n \to \infty \), which contradicts (4.4). \( \square \)
5. Proof of main result

First, we consider the Nehari manifold associated to the problem (1.1) as
\[ \mathcal{N} = \{ (u,v) \in \mathcal{H} \setminus \{ (0,0) \} : \langle E'(u,v), (u,v) \rangle = 0 \} \]
and the number \( A := \inf_{(u,v) \in \mathcal{N}} E(u,v) \).

**Lemma 6.** Assume that the conditions \((H_0), (H_5)\) and \((M_3)\) hold. Then \( \rho_0 \leq A \).

**Proof.** Given \((u,v) \in \mathcal{N}, \) let us define
\[ h(t) := E(tu, tv) = \frac{1}{2} M \left( t^2 \| u \|_{1,2}^2 \right) + \frac{1}{2} M \left( t^2 \| v \|_{1,2}^2 \right) - \int_{\Omega} F(tu, tv) dx, \quad \forall t > 0. \]
The function \( h \) is differentiable and
\[
\begin{align*}
\frac{d}{dt} h(t) &= \langle E'(tu, tv), (u, v) \rangle \\
&= m \left( t^2 \| u \|_{1,2}^2 \right) t \| u \|_{1,2}^2 + m \left( t^2 \| v \|_{1,2}^2 \right) t \| v \|_{1,2}^2 - \int_{\Omega} f(tu, tv) u dx - \int_{\Omega} g(tu, tv) v dx, \\
&\quad \forall t > 0.
\end{align*}
\]
Since \( \langle E'(u, v), (u, v) \rangle = 0 \), for all \((u, v) \in \mathcal{N}, \) we get
\[
\begin{align*}
\frac{d}{dt} h(t) &= t^3 \| u \|_{1,2}^3 \left( \frac{m \left( t^2 \| u \|_{1,2}^2 \right)}{t^2 \| u \|_{1,2}^2} - \frac{m \left( \| u \|_{1,2}^2 \right)}{\| u \|_{1,2}^2} \right) \\
&\quad + t^3 \| v \|_{1,2}^3 \left( \frac{m \left( t^2 \| v \|_{1,2}^2 \right)}{t^2 \| v \|_{1,2}^2} - \frac{m \left( \| v \|_{1,2}^2 \right)}{\| v \|_{1,2}^2} \right) \\
&\quad + t^3 \int_{\Omega} \left( \frac{f(u, v)}{u^3} - \frac{f(tu, tv)}{t^3 v^3} \right) u^4 dx + t^3 \int_{\Omega} \left( \frac{g(u, v)}{v^3} - \frac{g(tu, tv)}{t^3 v^3} \right) v^4 dx.
\end{align*}
\]
Then \( h'(1) = 0 \) and from \((M_3)\) and \((H_5)\), we conclude that \( h'(t) \geq 0 \) for \( 0 < t < 1 \) and \( h'(t) \leq 0 \) for \( 0 < t < 1 \). Hence
\[ E(u, v) = \max_{t \geq 0} E(tu, tv). \]

Now, defining \( \gamma : [0, 1] \rightarrow \mathcal{H}, \gamma(t) = (te_1, te_2), \) we have \( \gamma \in \Gamma \) and therefore
\[ \rho_0 \leq \max_{t \in [0, 1]} E(\gamma(t)) \leq \max_{t \geq 0} E(tu, tv) = E(u, v), \]
which implies \( \rho_0 \leq A. \) \( \square \)

Next, we prove that \( E \) satisfies Palais-Smale condition.
PROPOSITION 1. Assume \((M_1)-(M_4)\) and \((H_0)-(H_5)\). Then the functional \(E\) satisfies Palais-Smale condition for all \(\rho_0 < \frac{1}{2} M \left( \frac{4\pi}{\alpha_0} \right)\).

In order to prove this proposition, we shall use the following result of convergence, whose proof can be found in [18].

**Lemma 7.** Let \(\{(u_n,v_n)\} \subset \mathcal{H}\) be a Palais-Smale sequence. Then, there exists \((u_0,v_0)\) such that, up to a subsequence, \(u_n \rightharpoonup u_0\) and \(v_n \rightharpoonup v_0\) in \(H_0^1(\Omega)\), and

\[
\lim_{n \to \infty} \int_{\Omega} f(u_n,v_n) dx = \int_{\Omega} f(u_0,v_0) dx,
\]
\[
\lim_{n \to \infty} \int_{\Omega} g(u_n,v_n) dx = \int_{\Omega} g(u_0,v_0) dx.
\]

**Proof of proposition 1.** Let \((u_n,v_n)\) be a sequence in \(\mathcal{H}\) verifying

\[
E(u_n,v_n) \to \rho_0 \quad \text{and} \quad E'(u_n,v_n) \to 0,
\]

which implies

\[
\frac{1}{2} M \left( \int_{\Omega} |\nabla u_n|^2 dx \right) + \frac{1}{2} M \left( \int_{\Omega} |\nabla v_n|^2 dx \right) - \int_{\Omega} F(u_n,v_n) dx \to \rho_0 \quad (5.1)
\]

\[
m \int_{\Omega} |\nabla u_n|^2 dx \int_{\Omega} |\nabla v_n|^2 dx + m \int_{\Omega} |\nabla u_n|^2 dx \int_{\Omega} |\nabla v_n|^2 dx - \int_{\Omega} f(u_n,v_n) u_n dx - \int_{\Omega} g(u_n,v_n) v_n dx \leq \epsilon_n \|(u_n,v_n)\|_{\mathcal{H}},
\]

\[
(5.2)
\]

where \(\epsilon_n \to 0\). It follows from (5.1) and (5.2) using \((H_3)\), \((M_1)\) and \((M_3)\) we obtain

\[
C + \|(u_n,v_n)\|_{\mathcal{H}} \geq E(u_n,v_n) - \frac{1}{\theta} |E'(u_n,v_n)(u_n,v_n)| \geq \left( \frac{\theta - 4}{4\theta} \right) \|(u_n,v_n)\|_{\mathcal{H}}^2.
\]

Hence \((u_n,v_n)\) is bounded in \(\mathcal{H}\). Now we take a subsequence denoted again by \((u_n,v_n)\) such that, for some \((u_0,v_0) \in \mathcal{H}\), we have

\[
(u_n,v_n) \rightharpoonup (u_0,v_0) \quad \text{in} \quad \mathcal{H}
\]
\[
u_n \to u_0 \quad \text{and} \quad v_n \to v_0 \quad \text{in} \quad L^q(\Omega), \quad \forall q \geq 1.
\]
\[
u_n(x) \to u_0(x) \quad \text{and} \quad v_n(x) \to v_0(x) \quad \text{a.e. in} \quad \Omega.
\]

Now, we can apply Lemma 7 to conclude that

\[
\int_{\Omega} f(u_n,v_n) dx \to \int_{\Omega} f(u_0,v_0) dx \quad \text{and} \quad \int_{\Omega} g(u_n,v_n) dx \to \int_{\Omega} g(u_0,v_0) dx,
\]

\[
(5.4)
\]
and therefore using \((H_3), (5.2)\) and generalized Lebesgue dominated convergence theorem, we obtain
\[
\int_\Omega F(u_n, v_n) dx \longrightarrow \int_\Omega F(u_0, v_0) dx. \tag{5.5}
\]

At this point, we affirm that \((u_0, v_0) \neq (0, 0)\). We suppose that \((u_0, v_0) = (0, 0)\) and we derive a contradiction. Since \((u_0, v_0) = (0, 0)\), we have \(\int_\Omega F(u_n, v_n) dx \longrightarrow 0\) and so
\[
\frac{1}{2} M \left( \int_\Omega |\nabla u_n|^2 dx \right) + \frac{1}{2} M \left( \int_\Omega |\nabla v_n|^2 dx \right) \rightarrow \rho < M(4\pi/\alpha_0),
\]
and therefore \(\frac{\|u_n,v_n\|^2}{2} < 4\pi/\alpha_0\). Thus, there exist \(N \in \mathbb{N}\) and \(\gamma > 0\) such that
\[
\frac{\alpha_0}{2} \|u_n,v_n\|^2 < \gamma < 4\pi \quad \text{for all} \quad n \geq N.
\]

Now, choose \(p > 1\) close to \(1\) and \(\alpha > \alpha_0\) close to \(\alpha_0\) so that we still have
\[
p \frac{\alpha}{2} \|u_n,v_n\|^2 < \gamma < 4\pi.
\]

From this and by using \((2.2), (2.3), \text{Hölder inequality, lemma 1 and (5.3)}\) we get
\[
\left|\int_\Omega f(u_n, v_n) u_n \, dx + \int_\Omega g(u_n, v_n) v_n \, dx\right|
\leq C_1 \left( \int_\Omega |u_n|^2 \, dx + \int_\Omega |v_n|^2 \, dx \right)
+ C_2 \left( \int_\Omega \left( |u_n| + |v_n| \right)e^{\frac{q_2}{2}(u_n^2 + v_n^2)} \, dx \right)
\leq C_1 \left( \|u_n\|^2 + \|v_n\|^2 \right)
+ C_3 \left( \int_\Omega \left( |u_n| + |v_n| \right)^{p/(p-1)} \right)^{(p-1)/p}
\times \left( \int_\Omega e^{p\frac{q_2}{2}(u_n^2 + v_n^2)} \, dx \right)^{1/p}
\leq C_1 \left( \|u_n\|^2 + \|v_n\|^2 \right)
+ C_4 \left( \|u_n\|_{p/(p-1)} + \|v_n\|_{p/(p-1)} \right)
\times \left( e^{p\frac{q_2}{2}(u_n^2 + v_n^2)} \left( \frac{u_n}{\|u_n,v_n\|_{\mathcal{Q}}} \right)^2 + \left( \frac{v_n}{\|u_n,v_n\|_{\mathcal{Q}}} \right)^2 \right)^{1/p}
\rightarrow 0.
\]

Since
\[
E'(u_n,v_n)(u_n,v_n) = m\left( \|u_n\|_{1,2} \right) \|u_n\|_{1,2} + m\left( \|v_n\|_{1,2} \right) \|v_n\|_{1,2}
- \int_\Omega f(u_n,v_n) u_n \, dx - \int_\Omega g(u_n,v_n) v_n \, dx
\]
and \(E'(u_n,v_n)(u_n,v_n) \rightarrow 0\) it follows that
\[
m\left( \|u_n\|_{1,2} \right) \|u_n\|_{1,2} + m\left( \|v_n\|_{1,2} \right) \|v_n\|_{1,2} \rightarrow 0.
\]

Hence, by \((M_1)\) we have \(\|u_n,v_n\|^2_{\mathcal{Q}} \rightarrow 0\) and therefore \(E(u_n,v_n) \rightarrow 0\), what is absurd and thus we must \((u_0,v_0) \neq (0,0)\). Next, we will make some assertions.
Assertion 1. $E(u_0, v_0) \geq 0$

First, we claim that
\[
m(\|u_0\|_{1,2}^2)\|u_0\|_{1,2}^2 + m(\|v_0\|_{1,2}^2)\|v_0\|_{1,2}^2 \geq \int_{\Omega} f(u_0, v_0)u_0dx + \int_{\Omega} g(u_0, v_0)v_0dx. \quad (5.6)
\]

Suppose by contradiction that
\[
m(\|u_0\|_{1,2}^2)\|u_0\|_{1,2}^2 + m(\|v_0\|_{1,2}^2)\|v_0\|_{1,2}^2 < \int_{\Omega} f(u_0, v_0)u_0dx + \int_{\Omega} g(u_0, v_0)v_0dx,
\]
that is
\[
E'(u_0, v_0)(u_0, v_0) < 0.
\]

Now, using $(M_1)$, $(H_1)$ and Sobolev imbedding, we can see that
\[
E'(tu_0, tv_0)(u_0, v_0) > 0 \text{ for } t \text{ sufficiently small.}
\]

Thus, there exists $\tau \in (0, 1)$ such that $E'(\tau u_0, \tau v_0)(u_0, v_0) = 0$, which implies that $\tau(u_0, v_0) \in \mathcal{N}$. Then, according to (1.2), $(H_6)$, semicontinuity of norm and Fatou lemma we obtain
\[
\rho_0 \leq A \leq E(\tau u_0, \tau v_0) = E(\tau u_0, \tau v_0) - \frac{1}{4} E'(\tau u_0, \tau v_0)(\tau u_0, \tau v_0)
\]
\[
= \frac{1}{2} M(\|\tau u_0\|_{1,2}^2) - \frac{1}{4} m(\|\tau u_0\|_{1,2}^2)\|\tau u_0\|_{1,2}^2
\]
\[
+ \frac{1}{2} M(\|\tau v_0\|_{1,2}^2) - \frac{1}{4} m(\|\tau v_0\|_{1,2}^2)\|\tau v_0\|_{1,2}^2
\]
\[
+ \frac{1}{4} \int_{\Omega} (f(\tau u_0, \tau v_0)\tau u_0 + g(\tau u_0, \tau v_0)\tau v_0 - 4F(\tau u_0, \tau v_0)) dx
\]
\[
< \frac{1}{2} M(\|u_0\|_{1,2}^2) - \frac{1}{4} m(\|u_0\|_{1,2}^2)\|u_0\|_{1,2}^2 + \frac{1}{2} M(\|v_0\|_{1,2}^2)
\]
\[
- \frac{1}{4} m(\|v_0\|_{1,2}^2)\|v_0\|_{1,2}^2
\]
\[
+ \frac{1}{4} \int_{\Omega} (f(u_0, v_0)u_0 + g(u_0, v_0)v_0 - 4F(u_0, v_0)) dx
\]
\[
\leq \liminf_{n \to \infty} \left[ \frac{1}{2} M(\|u_n\|_{1,2}^2) - \frac{1}{4} m(\|u_n\|_{1,2}^2)\|u_n\|_{1,2}^2 \right]
\]
\[
+ \liminf_{n \to \infty} \left[ \frac{1}{2} M(\|v_n\|_{1,2}^2) - \frac{1}{4} m(\|v_n\|_{1,2}^2)\|v_n\|_{1,2}^2 \right]
\]
\[
+ \liminf_{n \to \infty} \left[ \frac{1}{4} \int_{\Omega} (f(u_n, v_n)u_n + g(u_n, v_n)v_n - 4F(u_n, v_n)) dx \right]
\]
\[
\leq \lim_{n \to \infty} \left[ E(u_n, v_n) - \frac{1}{4} E'(u_n, v_n)(u_n, v_n) \right] = \rho_0,
\]
which is absurd.
Next, we claim that \( E(u_0, v_0) \geq 0 \). By (5.6), (1.2) and (1.4) one has
\[
E(u_0, v_0) \geq \frac{1}{2} M \left( \| u_0 \|_{1,2}^2 \right) - \frac{1}{4} m \left( \| u_0 \|_{1,2}^2 \right) \| u_0 \|_{1,2}^2 + \frac{1}{2} M \left( \| v_0 \|_{1,2}^2 \right) \\
- \frac{1}{4} m \left( \| v_0 \|_{1,2}^2 \right) \| v_0 \|_{1,2}^2 \\
+ \frac{1}{4} \int_{\Omega} \left( f(u_0, v_0) u_0 + g(u_0, v_0) v_0 - 4F(u_0, v_0) \right) dx
\]
\[
\geq 0.
\]
This completes the proof of assertion 1.

**Assertion 2.** \((u_n, v_n) \to (u_0, v_0)\) in \( \mathcal{H} \).

As \((u_n, v_n)\) is bounded, up to a subsequence, \( \|(u_n, v_n)\|_{\mathcal{H}} \to r > 0 \), with
\[
r^2 = r_1^2 + r_2^2, \quad \| u_n \|_{1,2} \to r_1 \quad \text{and} \quad \| v_n \|_{1,2} \to r_2.
\]
By using (5.5) and semicontinuity of norm, we have
\[
E(u_0, v_0) \leq \rho_0. \quad (5.7)
\]
In this case we claim that \( E(u_0, v_0) = \rho_0 \). So it remains to prove (5.7), assume by contradiction that \( E(u_0, v_0) < \rho_0 \). Then, \( \| u_0 \|_{1,2} < r/2 \) and \( \| v_0 \|_{1,2} < r/2 \). Let
\[
U_n = \frac{2u_n}{\| (u_n, v_n) \|_{\mathcal{H}}}, \quad V_n = \frac{2v_n}{\| (u_n, v_n) \|_{\mathcal{H}}}, \quad U_0 = \frac{2u_0}{r} \quad \text{and} \quad V_0 = \frac{2v_0}{r}.
\]
We have
\[
U_n \to U_0 \quad \text{in} \quad H_0^1(\Omega) \quad \text{and} \quad V_n \to V_0 \quad \text{in} \quad H_0^1(\Omega)
\]
\[
\| U_0 \|_{1,2} < 1 \quad \text{and} \quad \| V_0 \|_{1,2} < 1.
\]
Thus, by lemma 2
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} e^p \left( U_n^2 + V_n^2 \right) dx < \infty, \quad \forall p < \frac{4\pi}{1 - \| U_0 \|_{1,2}^2 - \| V_0 \|_{1,2}^2}. \quad (5.8)
\]
On the other hand,
\[
2\rho_0 - 2E(u_0, v_0) = M(r_1^2 + r_2^2) - M(\| u_0 \|_{1,2}^2 + \| v_0 \|_{1,2}^2).
\]
Using this equality, lemma 5 and the fact that \( E(u_0, v_0) \geq 0 \), we get
\[
M(r_1^2) + M(r_2^2) < M\left( \frac{4\pi}{\alpha_0} \right) + M(\| u_0 \|_{1,2}^2) + M(\| v_0 \|_{1,2}^2).
\]
From \((M_1)\) and \((M_2)\) it follows that
\[
M(r_1^2) < M\left( \frac{4\pi}{\alpha_0} + \| u_0 \|_{1,2}^2 + \| v_0 \|_{1,2}^2 \right),
\]
and

\[ M\left(\frac{r^2}{2}\right) < M\left(\frac{4\pi}{\alpha_0} + \|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2\right), \]

which implies that

\[ r_1^2 < \frac{4\pi}{\alpha_0} + \|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2 \quad \text{and} \quad r_2^2 < \frac{4\pi}{\alpha_0} + \|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2, \]  

(5.10)

and therefore

\[ r_1^2 + r_2^2 = r^2 < \frac{8\pi}{\alpha_0} + 2(\|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2). \]  

(5.11)

Now, we observe that

\[ r^2 = \frac{r^2 - 2(\|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2)}{1 - \|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2}, \]

and from (5.11), it follows that

\[ r^2 < \frac{4\pi}{\alpha_0} (1 - \|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2)^{-1}. \]

Then, we can choose \( p > 4\pi \) such that \( \alpha_0 \frac{\|(u_n, v_n)\|_{2,\Omega}}{2} < p < 4\pi (1 - \|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2)^{-1} \) for \( n \) sufficiently large. Now, taking \( q > 1 \) close to \( 1 \) and \( \alpha > \alpha_0 \) close to \( \alpha_0 \) such that

\[ q\alpha \frac{\|(u_n, v_n)\|_{2,\Omega}^2}{2} < p < 4\pi (1 - \|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2)^{-1}, \quad \text{for } n \text{ large enough} \]

and invoking (5.8), for some \( C > 0 \), we concluded that

\[ \int_{\Omega} e^{q\alpha \frac{\|(u_n, v_n)\|_{2,\Omega}^2}{2}} \, dx = \int_{\Omega} e^{q\alpha \frac{\|(u_n, v_n)\|_{2,\Omega}^2}{4}} (u_n^2 + v_n^2) \, dx \leq \int_{\Omega} e^p (u_n^2 + v_n^2) \, dx \leq C. \]  

(5.12)

Hence, using Hölder inequality, (5.12) and (5.3) we reach

\[ \left| \int_{\Omega} f(u_n, v_n) (u_n - u_0) \, dx \right| \to 0 \quad \text{as } n \to \infty, \]

and

\[ \left| \int_{\Omega} g(u_n, v_n) (v_n - v_0) \, dx \right| \to 0 \quad \text{as } n \to \infty. \]

Since \( E'(u_n, v_n)(u_n - u_0, 0) = o(1) \), we get

\[ m(\|u_n\|_{1,2})^2 \|u_n\|_{1,2}^2 - m(\|u_n\|_{1,2}^2) \|\nabla u_n \nabla u_0 \, dx \to m(r_1^2) r_1^2 - m(r_1^2) \|u_0\|_{1,2}^2 = 0. \]
It follows that
\[ \|u_0\|_{1,2} = r_1. \]
Similarly, we obtain
\[ \|v_0\|_{1,2} = r_2, \]
which implies that \( \|(u_n, v_n)\|_{\mathcal{H}} \to \|(u_0, v_0)\|_{\mathcal{H}} \) and so \( (u_n, v_n) \to (u_0, v_0) \) in \( \mathcal{H} \). In view of the continuity of \( E \), we must have \( E(u_0, v_0) = \rho_0 \) what is an absurde. Thus, the proof of Proposition 1 is complete. \( \square \)

**Finalizing the proof of Theorem 1.** It follows from the hypotheses in Theorem 1 that \( E \) satisfies Palais-Smale condition for all \( \rho_0 < \frac{1}{2}M(4\pi/\alpha_0) \), see proposition 1. To finish the proof of theorem 1, we use lemmas 2 and 3 and apply the Mountain pass Theorem. \( \square \)

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