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EXISTENCE OF SOLUTIONS FOR NONLOCAL ELLIPTIC SYSTEMS WITH EXPONENTIAL NONLINEARITY

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Abstract. In this paper, we establish the existence of solutions for a Kirchhoff-type system with Dirichlet boundary condition and nonlinearities having exponential critical growth. Our approach is based on the Trudinger-Moser inequality and on a minimax theorem.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$. In this article, we study the existence of positive solutions to the following nonlinear Kirchhoff type system

$$\begin{cases} -m\Big(\int\limits_{\Omega}|\nabla u|^2dx\Big)\Delta u = f(u,v) & \text{in } \Omega,\\ -m\Big(\int\limits_{\Omega}|\nabla v|^2dx\Big)\Delta v = g(u,v) & \text{in } \Omega,\\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the nonlinear terms f and g are allowed to have exponential critical growth, and $m:\mathbb{R}^+\to\mathbb{R}^+$, is a continuous function that satisfy some conditions which will be stated later on. By means of the Trudinger-Moser inequality, we shall consider the variational situation in which $\nabla F(u,v) = \big(f(u,v),g(u,v)\big)$, for some function $F:\mathbb{R}^2\to\mathbb{R}$ of class C^2, where ∇F stands for the gradient of F in the variables $w=(u,v)\in\mathbb{R}^2$.

We make the following assumptions on the function m

(M_1) There exist real numbers m_0, $m_1, m_2 > 0$ and $\kappa \geqslant 1$ such that

$$m_0 \leqslant m(t) \leqslant m_1 t^{\kappa-1} + m_2, \quad \text{for all } t \geqslant 0.$$

(M_2) $M(s)+M(t) \leqslant M(s+t)$ $\forall s,t \geqslant 0$ where $M(t) = \int\limits_0^t m(x)dx$

(M_3) $m(t)/t$ is noninreasing for $t>0$.

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A typical example of a function satisfying the conditions (M_1)–(M_3) is given by $m(t) = m_0 + bt$ with $b > 0$ and for all $t \geqslant 0$. As a consequence of (M_3), a straightforward computation shows that

$$M(t) - \frac{1}{2}m(t)t \text{ is nondecreasing for } t \geqslant 0. \tag{1.2}$$

System (1.1) is related to the stationary version of a model established by Kirchhoff [12]. More precisely, Kirchhoff proposed the following model

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

which extends D'Alembert's wave equation with free vibrations of elastic strings, where ρ denotes the mass density, P_0 denotes the initial tension, h denotes the area of the cross section, E denotes the Young modulus of the material, and L denotes the length of the string.

In the recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [3], [4], [10], [19], [8] in which the authors have used the variational method and the topological method to get the existence of solutions. In [8], by a direct variational approach, the autors establish the exietence of a positive ground state solution for a nonlocal Kirchhoff of the type

$$\begin{cases} -m\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Motivated by above and the ideas introduced in [18], in this work, we will study the existence of nontrivial solutions for problem (1.1).

Let us introduce the precise assumptions under which our problem is studied.

(H_0) f, g are C^1 functions such that $f(s,t) > 0$, $g(s,t) > 0$ for all $s,t > 0$, and $f(s,t) = g(s,t) = 0$ if $s \leqslant 0$ or $t \leqslant 0$.

(H_1) $f(s,t) = o\big(\sqrt{s^2+t^2}\big)^\mu$ and $g(s,t) = o\big(\sqrt{s^2+t^2}\big)^\mu$ as $|(s,t)| \to 0$, for some $\mu \in [0,4)$.

(H_2) f and g have α_0-exponential critical growth, i.e., there exists $\alpha_0 > 0$ such

$$\lim_{s^2+t^2\to+\infty} \frac{f(s,t)}{e^{\alpha(s^2+t^2)/2}} = \lim_{s^2+t^2\to+\infty} \frac{g(s,t)}{e^{\alpha(s^2+t^2)/2}} = \begin{cases} 0, & \forall \alpha > \alpha_0 \\ +\infty, & \forall \alpha < \alpha_0 \end{cases}.$$

(H_3) There exists $\theta > 4$ such that

$$0 < \theta F(s,t) \leqslant f(s,t)s + g(s,t)t, \quad \forall (s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

(H_4) For every $\nu > 1$, there exists a constant K_0 such that

$$F(s,t) \geqslant K_0\big(s^2+t^2\big)^\nu \text{ for all } s,t > 0$$

(H_5)

$$\frac{f(\underline{s},\underline{t})}{\underline{s}^3} \leqslant \frac{f(\overline{s},\overline{t})}{\overline{s}^3} \text{ and } \frac{g(\underline{s},\underline{t})}{\underline{t}^3} \leqslant \frac{g(\overline{s},\overline{t})}{\overline{t}^3} \text{ for } \overline{s} \geqslant \underline{s} > 0,\ \overline{t} \geqslant \underline{t} > 0.$$

(H_6) For all $s,t > 0$,

$$3f(s,t) < \frac{\partial f(s,t)}{\partial s}s + \frac{\partial g(s,t)}{\partial s}t \text{ and } 3g(s,t) < \frac{\partial f(s,t)}{\partial t}s + \frac{\partial g(s,t)}{\partial t}t.$$

We observe that condition (H_6) implies

$$f(\underline{s},\underline{t}) + g(\underline{s},\underline{t}) - 4F(\underline{s},\underline{t}) < f(\overline{s},\overline{t}) + g(\overline{s},\overline{t}) - 4F(\overline{s},\overline{t}) \text{ for } \overline{s} > \underline{s} > 0 \text{ and } \overline{t} > \underline{t} > 0. \tag{1.4}$$

An example of a function satisfying the above assumptions with $\alpha_0 = 1$ is

$$F(s,t) = \begin{cases} (s^4 + t^4)e^{(s^2+t^2)/2} & \text{if } s > 0,\ t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now, we are ready to state our main result

THEOREM 1. *Under assumption* (M_1)–(M_4) *and* (H_0)–(H_5), *Problem* (1.1) *admits at least one nontrivial solution* $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$.

This work is organised as follows: In Section 2, we present the variational setting in which our problem will be treated, and some preliminary results. Section 3 is devoted to show that the energy functional has the mountain pass geometry and in Section 4 we obtain an estimate for the minimax level associated to our functional. Finally, we prove our main result in Section 5.

2. Preliminaries

As mentioned in the introduction, the nonlinearities f and g are allowed to have exponential critical growth which allows to treat the problem by variational methods. This growth is given by the so-called Trudinger-Moser inequality (see [17], [22]), which says that if u is a $H_0^1(\Omega)$ function then there exists a constant $C > 0$ such that

$$\sup_{\|u\|_{H_0^1(\Omega)} \leqslant 1} \int_\Omega e^{\alpha u^2} dx \leqslant C|\Omega| \quad \text{if } \alpha \leqslant 4\pi. \tag{2.1}$$

Let $\mathscr{H} := H_0^1(\Omega) \times H_0^1(\Omega)$ be the Sobolev space endowed with the norm

$$\|(u,v)\|_{\mathscr{H}} := \left(\|u\|_{1,2}^2 + \|v\|_{1,2}^2\right)^{1/2} \quad \text{where } \|u\|_{1,2} = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}$$

and $\|u\|_r$ denotes the norm in $L^r(\Omega)$, i.e. $\|u\|_r = \big(\int_\Omega |u|^r dx\big)^{1/r}$. In this work, we shall use the following adapted version of Moser-Trudinger inequality for the pair (u,v) [18]:

LEMMA 1. *Let $(u,v)\in\mathscr{H}$, then $\int_\Omega e^{\gamma\left(u^2+v^2\right)}dx<+\infty$ for any $\gamma>0$. Moreover, there exists a constant $C=C(\Omega)$ such that*

$$\sup_{\|(u,v)\|_{\mathscr{H}}=1}\int_\Omega e^{\gamma\left(u^2+v^2\right)}dx\leqslant C,\ \textit{provided that } \gamma\leqslant 4\pi.$$

We shall look for solutions of (1.1) by finding critical points of the energy functional $E:\mathscr{H}\to\mathbb{R}$ given by

$$E(u,v)=\frac{1}{2}M\Big(\int_\Omega|\nabla u|^2dx\Big)+\frac{1}{2}M\Big(\int_\Omega|\nabla v|^2dx\Big)-\int_\Omega F(u,v)dx,$$

where $M(t)\int\limits_0^t m(s)ds$. Under our assumptions we have that E is well defined and it is C^1 on $\mathscr{H}$. Indeed, by (H_1), (H_2) and for $\varepsilon>0$ there exists $\delta>0$ such that

$$f(u,v)\leqslant\varepsilon\sqrt{u^2+v^2}\text{ and }g(u,v)=\varepsilon\sqrt{u^2+v^2}\text{ always that }\left|(u,v)\right|<\delta.$$

On the other hand, for $\alpha>\alpha_{0,}$ there exist constants $C>0$ such that

$$f(u,v)\leqslant Ce^{\alpha(u^2+v^2)/2}\ \text{ and } g(u,v)\leqslant Ce^{\alpha(u^2+v^2)/2}\ \text{ for all }\left|(u,v)\right|\geqslant\delta.$$

Thus, for all $(u,v)\in\mathscr{H}$ we have

$$f(u,v)\leqslant\varepsilon\sqrt{u^2+v^2}+Ce^{\alpha(u^2+v^2)/2},\tag{2.2}$$

and

$$g(u,v)\leqslant\varepsilon\sqrt{u^2+v^2}+Ce^{\alpha(u^2+v^2)/2}.\tag{2.3}$$

Hence, using (H_3), (2.2) and (2.3), we obtain

$$\left|F(u,v)\right|\leqslant\varepsilon\left(u^2+v^2\right)+C\sqrt{u^2+v^2}e^{\alpha(u^2+v^2)/2}.\tag{2.4}$$

This inequality together with Lemma 1 yields $F(u,v)\in L^1\left(\Omega\right)$ for all $(u,v)\in\mathscr{H}$, which implies that E is well defined, for $\alpha>\alpha_0$. Using standard arguments, we can see that $E\in C^1(\mathscr{H},\mathbb{R})$ with

$$E'(u,v)(\phi,\psi)=m\Big(\int\limits_\Omega\left|\nabla u\right|^2dx\Big)\int_\Omega\nabla u.\nabla\phi\ dx+m\Big(\int\limits_\Omega\left|\nabla v\right|^2dx\Big)\int_\Omega\nabla v.\nabla\psi\ dx$$
$$-\int_\Omega f(u,v)\phi\ dx-\int_\Omega g(u,v)\psi\ dx,$$

for all $(\phi,\psi)\in\mathscr{H}$.

Also, to prove our main result, we use the following version of Lion's higher integrability lemma [18]:

LEMMA 2. *Let (u_n,v_n) be a sequence in $\mathscr{H}$ such that $\|(u_n,v_n)\|_{\mathscr{H}}=1$, for all $n\in\mathbb{N}^*$ and $u_n\rightharpoonup u$, $v_n\rightharpoonup v$ in $H_0^1(\Omega)$ for some $(u,v)\neq(0,0)$.*

Then, for $4\pi<p<4\pi\left(1-\|(u,v)\|^2_{\mathscr{H}}\right)^{-1}$, $\sup\limits_{n\geqslant1}\int_\Omega e^{p\left(u_n^2+v_n^2\right)}dx<\infty$.

3. The Mountain Pass Geometry

In this section, we prove that the functional E has the Mountain Pass Geometry. This fact is proved in the next lemmas:

LEMMA 3. *Assume* (M_1) *and* (H_0)–(H_3), *then there exist positive constants* τ *and* ρ *such that*

$$E(u,v) \geqslant \tau,\ \forall (u,v) \in \mathscr{H} : \|(u,v)\|_{\mathscr{H}} = \rho.$$

Proof. Just as we have obtained (2.4), we deduce that

$$\big|F(u,v)\big| \leqslant \varepsilon\big(u^2+v^2\big) + C\big(u^q+v^q\big)e^{\alpha(u^2+v^2)/2},$$

for all $(u,v) \in \mathscr{H}$ *and* $q > 2$. Using Hölder's inequality and the Sobolev embedding, we have

$$\begin{aligned}\int_\Omega \big|F(u,v)\big|dx &\leqslant \varepsilon\Big(\|u\|_2^2 + \|v\|_2^2\Big) + C\Big(\int_\Omega \Big(|u|^q+|v|^q\Big)^2\Big)^{1/2}\Big(\int_\Omega e^{\alpha(u^2+v^2)}dx\Big)^{1/2}\\ &\leqslant \varepsilon C\|(u,v)\|_{\mathscr{H}}^2 + C\Big(\|u\|_{1,2}^q + \|v\|_{1,2}^q\Big)\\ &\quad\times\Big(\int_\Omega e^{\alpha\|(u,v)\|_{\mathscr{H}}^2\left(\left(\frac{u}{\|(u,v)\|_{\mathscr{H}}}\right)^2+\left(\frac{v}{\|(u,v)\|_{\mathscr{H}}}\right)^2\right)}dx\Big)^{1/2}.\end{aligned}$$

Now, for $\|(u,v)\|_{\mathscr{H}} = \rho$ such that $\rho^2 \leqslant \pi/\alpha$ and by the Moser-Trudinger inequality, we obtain

$$\begin{aligned}\int_\Omega \big|F(u,v)\big|dx &\leqslant \varepsilon C\|(u,v)\|_{\mathscr{H}}^2 + C\Big(\|u\|_{1,2}^q + \|v\|_{1,2}^q\Big)\\ &\leqslant \varepsilon C\|(u,v)\|_{\mathscr{H}}^2 + 2C\|(u,v)\|_{\mathscr{H}}^q.\end{aligned}$$

Therfore, using (M_1), we get

$$E(u,v) \geqslant \Big(\frac{m_0}{2} - \varepsilon C\Big)\|(u,v)\|_{\mathscr{H}}^2 - 2C\|(u,v)\|_{\mathscr{H}}^q.$$

Consequently

$$E(u,v) \geqslant \Big(\frac{m_0}{2} - \varepsilon C\Big)\rho^2 - 2C\rho^q.$$

Now, we may fix $\varepsilon > 0$ such that $\frac{m_0}{2} - \varepsilon C > 0$. Thus, for $\rho > 0$ sufficiently small there exists $\tau := \Big(\frac{m_0}{2} - \varepsilon C\Big)\rho^2 - 2C\rho^q > 0$ such that

$$E(u,v) \geqslant \tau,\ \forall (u,v) \in \mathscr{H} \text{ with } \|(u,v)\|_{\mathscr{H}} = \rho.$$

The proof of Lemma is complete. □

LEMMA 4. *Assume* (M_1) *and* (H_4). *Then, there exists* $(e_1,e_2) \in \mathscr{H}$ *such that*

$$E(e_1,e_2) < 0 \text{ and } \big\|(e_1,e_2)\big\| > \rho.$$

Proof. Using (M_1) and (H_4), we obtain

$$\begin{aligned} E(u,v) &\leqslant \frac{m_1}{2\kappa}\Big(\Big(\int_\Omega |\nabla u|^2 dx\Big)^\kappa + \Big(\int_\Omega |\nabla v|^2 dx\Big)^\kappa\Big) + m_2\Big(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\Big) \\ &\quad - K_0 \int_\Omega \big(u^2+v^2\big)^\nu dx. \\ &\leqslant \frac{m_1}{\kappa}\|(u,v)\|_{\mathscr{H}}^{2\kappa} + m_2\|(u,v)\|_{\mathscr{H}}^2 - K_0\int_\Omega \big(u^2+v^2\big)^\nu dx. \end{aligned}$$

Let $(u_0,v_0)\in\mathscr{H}$ with $u_0,v_0>0$ in Ω and $\|(u_0,v_0)\|_{\mathscr{H}}=1$. Thus, we have

$$E(tu_0,tv_0) \leqslant \frac{m_1}{\kappa}t^{2\kappa} + m_2t^2 - t^{2\nu}K_0\int_\Omega \big(u_0^2+v_0^2\big)^\nu dx,$$

for all $t>0$, which yields $E(tu,tv)\to-\infty$ as $t\to+\infty$, provided $\nu>\kappa$. Setting $(e_1,e_2)=(\bar{t}u_0,\bar{t}v_0)$ with $\bar{t}>0$ large enough, the proof is complete. □

4. On the mini-max level

In view of Lemmas 3 and 4, we may apply a version of the Mountain Pass theorem without Palais-Smale condition to obtain a sequence $(u_n,v_n)\in\mathscr{H}$ such that

$$E(u_n,v_n)\to\rho_0 \quad\text{and}\quad E'(u_n,v_n)\to 0,$$

where

$$\rho_0 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} E\big(\gamma(t)\big), \tag{4.1}$$

with

$$\Gamma = \big\{\gamma\in C\big([0,1],\mathscr{H}\big) : \gamma(0)=(0,0),\ \ \gamma(1)=(e_1,e_2)\ \big\}.$$

Let d be the inner radius of Ω, that is, it is the radius of the largest open ball contained in Ω. So $B_d(x_0)\subset\Omega$. We may assume that $x_0=0$. In order to get more information about the minimax level, it was crucial in our argument to consider the following concentrating functions $\psi_n(x)=\tilde{\psi}_n(x/d)$, $n\in\mathbb{N}$ where

$$\tilde{\psi}_n(x) = \frac{1}{2\sqrt{\pi}}\begin{cases} (\log n)^{1/2} & \text{for } 0\leqslant|x|\leqslant 1/n \\ \frac{\log(1/|x|)}{(\log n)^{1/2}} & \text{for } 1/n\leqslant|x|\leqslant 1 \\ 0 & \text{for } |x|\geqslant 1. \end{cases}$$

Then, ψ_n has support in $B_d(0)$ and (ψ_n,ψ_n) is such that $\|(\psi_n,\psi_n)\|_{\mathscr{H}}=1\ \forall n\in\mathbb{N}$. We can now prove the following upper bounded for ρ_0.

LEMMA 5. *With ρ_0 defined as in (4.1), we have $\rho_0 < \frac{1}{2}M\big(4\pi/\alpha_0\big)$.*

Proof. Suppose $\rho_0 \geqslant \frac{1}{2}M\big(4\pi/\alpha_0\big)$ and we derive a contradiction. As E possesses the mountain pass geometry, for each n there exist $t_n,\ s_n > 0$ such that

$$E(t_n\psi_n, s_n\psi_n) = \sup_{t,\,s>0} E(t\psi_n, s\psi_n) \geqslant \frac{1}{2}M\big(4\pi/\alpha_0\big),\ \ \forall n \in \mathbb{N}.$$

From this inequality and using that $F(u,v) \geqslant 0$ for all $(u,v) \in \mathbb{R}^2_+$, we obtain

$$\frac{1}{2}\big(M\big(t_n\|\psi_n\|^2_{1,2}\big) + M\big(s_n\|\psi_n\|^2_{1,2}\big)\big) \geqslant \frac{1}{2}M\big(4\pi/\alpha_0\big).$$

Using $\big(M_2\big)$ and since M is an increasing bijection, we have

$$t_n + s_n \geqslant 4\pi/\alpha_0. \tag{4.2}$$

On the other hand, (t_n, s_n) is a critical point of $E(t\psi_n, s\psi_n)$, so

$$E'(t\psi_n, s\psi_n)_{|_{(t,s)=(t_n,s_n)}} = 0$$

and therefore

$$m\Big(\frac{t_n^2}{2}\Big)t_n^2 + m\Big(\frac{s_n^2}{2}\Big)s_n^2 = \int_\Omega \Bigg(f(t_n\psi_n, s_n\psi_n)t_n\psi_n + g(t_n\psi_n, s_n\psi_n)s_n\psi_n\Bigg)dx.$$

Now, using that $t_n\psi_n \to \infty,\ s_n\psi_n \to \infty$ on $\big\{|x| \leqslant \delta/n\big\}$ and $(H2)$, we obtain

$$\begin{aligned} m\big(\frac{t_n^2}{2}\big)t_n^2 + m\big(\frac{s_n^2}{2}\big)s_n^2 &\geqslant \int_{\Omega\cap\big\{|x|\leqslant d/n\big\}} e^{\alpha_0\big(\frac{t_n^2+s_n^2}{2}\big)\psi_n^2}(t_n+s_n)\psi_n dx \\ &= \frac{\sqrt{\pi}d^2}{2n^2}e^{\alpha_0\big(\frac{t_n^2+s_n^2}{2}\big)\frac{\log n}{4\pi}}(t_n+s_n)(\log n)^{1/2} \\ &= \frac{\sqrt{\pi}d^2}{2}e^{\big(\frac{\alpha_0(t_n^2+s_n^2)}{8\pi}-2\big)\log n}(t_n+s_n)(\log n)^{1/2}, \end{aligned}$$

and from $\big(M_1\big)$, we can conclude that

$$\frac{m_1}{2^{\kappa-1}}\big(t_n^2+s_n^2\big)^\kappa \geqslant \frac{\sqrt{\pi}d^2}{2}e^{\big(\frac{\alpha_0(t_n^2+s_n^2)}{8\pi}-2\big)\log n}(t_n+s_n)(\log n)^{1/2}. \tag{4.3}$$

Note that, we can see

$$\frac{\frac{m_1}{2^{\kappa-1}}\big(t_n^2+s_n^2\big)^\kappa}{e^{\big(\frac{\alpha_0(t_n^2+s_n^2)}{8\pi}-2\big)\log n}(t_n+s_n)(\log n)^{1/2}} \to 0 \ \text{ if } \ t_n^2+s_n^2 \to +\infty.$$

It follows from this and (4.3), we infer that

$$t_n^2 + s_n^2 \to 16\pi/\alpha_0. \tag{4.4}$$

Moreover, using (4.3) again, we obtain

$$\frac{m_1}{2^{\kappa-1}}\big(t_n^2+s_n^2\big)^\kappa \geqslant \frac{\sqrt{\pi}d^2}{2}(t_n+s_n)(\log n)^{1/2}$$

This in turn implies that $t_n^2+s_n^2 \to \infty$ as $n \to \infty$, which contradicts (4.4). $\square$

5. Proof of main result

First, we consider the Nehari manifold asssociated to the problem (1.1) as

$$\mathscr{N} = \left\{(u,v) \in \mathscr{H} \diagdown \{(0,0)\} : \left\langle E'(u,v),(u,v)\right\rangle = 0\right\}$$

and the number $A := \inf\limits_{(u,v)\in\mathscr{N}} E(u,v)$.

LEMMA 6. *Assume that the conditions (H_0), (H_5) and (M_3) hold. Then $\rho_0 \leqslant A$.*

Proof. Given $(u,v) \in N$, let us define

$$h(t) := E(tu,tv) = \frac{1}{2}M\left(t^2\|u\|_{1,2}^2\right) + \frac{1}{2}M\left(t^2\|v\|_{1,2}^2\right) - \int_\Omega F(tu,tv)dx, \quad \forall t > 0.$$

The function h is differentiable and

$$\begin{aligned} h'(t) &= \left\langle E'(tu,tv),(u,v)\right\rangle \\ &= m\left(t^2\|u\|_{1,2}^2\right)t\|u\|_{1,2}^2 + m\left(t^2\|v\|_{1,2}^2\right)t\|v\|_{1,2}^2 - \int_\Omega f(tu,tv)udx - \int_\Omega g(tu,tv)vdx, \\ &\qquad \forall t > 0. \end{aligned}$$

Since $\left\langle E'(u,v),(u,v)\right\rangle = 0$, for all $(u,v) \in \mathscr{N}$, we get

$$\begin{aligned} h'(t) = t^3\|u\|_{1,2}^3 &\left(\frac{m\left(t^2\|u\|_{1,2}^2\right)}{t^2\|u\|_{1,2}^2} - \frac{m\left(\|u\|_{1,2}^2\right)}{\|u\|_{1,2}^2}\right) \\ &+ t^3\|v\|_{1,2}^3\left(\frac{m\left(t^2\|v\|_{1,2}^2\right)}{t^2\|v\|_{1,2}^2} - \frac{m\left(\|v\|_{1,2}^2\right)}{\|v\|_{1,2}^2}\right) \\ &+ t^3\int_\Omega\left(\frac{f(u,v)}{u^3} - \frac{f(tu,tv)}{t^3u^3}\right)u^4dx + t^3\int_\Omega\left(\frac{g(u,v)}{v^3} - \frac{g(tu,tv)}{t^3v^3}\right)v^4dx. \end{aligned}$$

Then $h'(1) = 0$ and from (M_3) and (H_5), we conclude that $h'(t) \geqslant 0$ for $0 < t < 1$ and $h'(t) \leqslant 0$ for $0 < t < 1$. Hence

$$E(u,v) = \max_{t\geqslant 0} E(tu,tv).$$

Now, defining $\gamma : [0,1] \to \mathscr{H}$, $\gamma(t) = (te_1, te_2)$, we have $\gamma \in \Gamma$ and therfore

$$\rho_0 \leqslant \max_{t\in[0,1]} E(\gamma(t)) \leqslant \max_{t\geqslant 0} E(tu,tv) = E(u,v),$$

which implies $\rho_0 \leqslant A$. $\square$

Next, we prove that E satisfies Palais-Smale condition.

PROPOSITION 1. *Assume* $(M_1)-(M_4)$ *and* $(H_0)-(H_5)$. *Then the functional* E *satisfies Palais-Smale condition for all* $\rho_0 < \frac{1}{2}M\big(4\pi/\alpha_0\big)$.

In order to prove this proposition, we shall use the following result of convergence, whose proof can be found in [18].

LEMMA 7. *Let* $\{(u_n,v_n)\} \subset \mathscr{H}$ *be a Palais-Smale sequence. Then,* $\exists\,(u_0,v_0)$ *such that, up to a subsequence,* $u_n \rightharpoonup u_0$ *and* $v_n \rightharpoonup v_0$ *in* $H_0^1(\Omega)$, *and*

$$\lim_{n\to\infty}\int_\Omega f(u_n,v_n) = \int_\Omega f(u_0,v_0)dx,$$

$$\lim_{n\to\infty}\int_\Omega g(u_n,v_n)dx = \int_\Omega g(u_0,v_0)dx.$$

Proof of proposition 1. Let (u_n,v_n) be a sequence in $\mathscr{H}$ verifying

$$E(u_n,v_n)\to\rho_0 \quad \text{and} \quad E'(u_n,v_n)\to 0,$$

which implies

$$\frac{1}{2}M\Big(\int_\Omega|\nabla u_n|^2dx\Big)+\frac{1}{2}M\Big(\int_\Omega|\nabla v_n|^2dx\Big)-\int_\Omega F(u_n,v_n)dx\to\rho_0 \tag{5.1}$$

$$\begin{aligned} &m\Big(\int_\Omega\big|\nabla u_n\big|^2dx\Big)\int_\Omega|\nabla u_n|^2\,dx+m\Big(\int_\Omega\big|\nabla v_n\big|^2dx\Big)\int_\Omega|\nabla v_n|^2\,dx\\ &-\int_\Omega f(u_n,v_n)u_n\,dx-\int_\Omega g(u_n,v_n)v_n\,dx\leqslant\varepsilon_n\|(u_n,v_n)\|_{\mathscr{H}}, \end{aligned}\tag{5.2}$$

where $\varepsilon_n\to 0$. It follows from (5.1) and (5.2) using (H_3), $\big(M_1\big)$ and $\big(M_3\big)$ we obtain

$$\begin{aligned} C+\|(u_n,v_n)\|_{\mathscr{H}} &\geqslant E(u_n,v_n)-\frac{1}{\theta}\big|E'(u_n,v_n)(u_n,v_n)\big|\\ &\geqslant\Big(\frac{\theta-4}{4\theta}\Big)\|(u_n,v_n)\|^2_{\mathscr{H}}. \end{aligned}$$

Hence (u_n,v_n) is bounded in $\mathscr{H}$. Now we take a subsequence denoted again by (u_n,v_n) such that, for some $(u_0,v_0)\in\mathscr{H}$, we have

$$\begin{gathered} (u_n,v_n)\rightharpoonup(u_0,v_0)\ \text{ in }\mathscr{H}\\ u_n\to u_0\text{ and }v_n\to v_0\text{ in }L^q\big(\Omega\big),\ \forall q\geqslant 1.\\ u_n\big(x\big)\to u_0\big(x\big)\text{ and }v_n\big(x\big)\to v_0\big(x\big)\text{ a.e. in }\Omega. \end{gathered}\tag{5.3}$$

Now, we can apply Lemma 7 to conclude that

$$\int_\Omega f(u_n,v_n)dx\longrightarrow\int_\Omega f(u_0,v_0)dx\ \text{ and }\int_\Omega g(u_n,v_n)dx\longrightarrow\int_\Omega g(u_0,v_0)dx, \tag{5.4}$$

and therefore using (H_3), (5.2) and generalized Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega F(u_n,v_n)dx \longrightarrow \int_\Omega F(u_0,v_0)dx. \tag{5.5}$$

At this point, we affirm that $(u_0,v_0)\neq(0,0)$. We suppose that $(u_0,v_0)=(0,0)$ and we derive a contradiction. Since $(u_0,v_0)=(0,0)$, we have $\int_\Omega F(u_n,v_n)dx \longrightarrow 0$ and so

$$\frac{1}{2}M\Big(\int_\Omega |\nabla u_n|^2 dx\Big)+\frac{1}{2}M\Big(\int_\Omega |\nabla v_n|^2 dx\Big)\to\rho<M\big(4\pi/\alpha_0\big),$$

and therfore $\frac{\|(u_n,v_n)\|^2_{\mathscr{H}}}{2}<4\pi/\alpha_0$. Thus,there exist $N\in\mathbb{N}$ and $\gamma>0$ such that

$$\frac{\alpha_0}{2}\|(u_n,v_n)\|^2_{\mathscr{H}}<\gamma<4\pi \text{ for all } n\geqslant N.$$

Now, choose $p>1$ close to 1 and $\alpha>\alpha_0$ close to α_0 so that we still have

$$p\frac{\alpha}{2}\|(u_n,v_n)\|^2_{\mathscr{H}}<\gamma<4\pi.$$

From this and by using (2.2), (2.3), Hölder inequality, lemma 1 and (5.3) we get

$$\begin{aligned}
&|\int_\Omega f(u_n,v_n)u_n\,dx+\int_\Omega g(u_n,v_n)v_n\,dx|\\
&\leqslant C_1\Big(\int_\Omega |u_n|^2dx+\int_\Omega |v_n|^2dx\Big)+C_2\Big(\int_\Omega \big(|u_n|+|v_n|\big)e^{\frac{\alpha}{2}(u_n^2+v_n^2)}dx\Big)\\
&\leqslant C_1\Big(\|u_n\|_2^2+\|v_n\|_2^2\Big)+C_3\Big(\int_\Omega \Big(|u_n|+|v_n|\Big)^{p/(p-1)}\Big)^{(p-1)/p}\Big(\int_\Omega e^{p\frac{\alpha}{2}(u_n^2+v_n^2)}dx\Big)^{1/p}\\
&\leqslant C_1\Big(\|u_n\|_2^2+\|v_n\|_2^2\Big)+C_4\Big(\|u_n\|_{p/(p-1)}+\|v_n\|_{p/(p-1)}\Big)\\
&\quad\times\Big(\int_\Omega e^{p\frac{\alpha}{2}\|(u_n,v_n)\|^2_{\mathscr{H}}\big(\big(\frac{u_n}{\|(u_n,v_n)\|_{\mathscr{H}}}\big)^2+\big(\frac{v_n}{\|(u_n,v_n)\|_{\mathscr{H}}}\big)^2\big)}dx\Big)^{1/p}\\
&\leqslant C_1\Big(\|u_n\|_2^2+\|v_n\|_2^2\Big)+C_5\Big(\|u_n\|_{p/(p-1)}+\|v_n\|_{p/(p-1)}\Big)\underset{n\to\infty}{\to}0.
\end{aligned}$$

Since

$$\begin{aligned}
E'(u_n,v_n)(u_n,v_n)&=m\Big(\|u_n\|_{1,2}\Big)\|u_n\|_{1,2}+m\Big(\|v_n\|_{1,2}\Big)\|v_n\|_{1,2}\\
&\quad-\int_\Omega f(u_n,v_n)u_n\,dx-\int_\Omega g(u_n,v_n)v_n\,dx
\end{aligned}$$

and $E'(u_n,v_n)(u_n,v_n)\to 0$ it follows that

$$m\Big(\|u_n\|_{1,2}\Big)\|u_n\|_{1,2}+m\Big(\|v_n\|_{1,2}\Big)\|v_n\|_{1,2}\to 0.$$

Hence, by (M_1) we have $\|(u_n,v_n)\|^2_{\mathscr{H}}\to 0$ and therfore $E(u_n,v_n)\to 0$, what is absurd and thus we must $(u_0,v_0)\neq(0,0)$. Next, we will make some assertions.

Assertion 1. $E(u_0,v_0) \geqslant 0$

First, we claim that

$$m\left(\|u_0\|_{1,2}^2\right)\|u_0\|_{1,2}^2+m\left(\|v_0\|_{1,2}^2\right)\|v_0\|_{1,2}^2 \geqslant \int_\Omega f(u_0,v_0)u_0dx+\int_\Omega g(u_0,v_0)v_0dx. \quad (5.6)$$

Suppose by contradiction that

$$m\left(\|u_0\|_{1,2}^2\right)\|u_0\|_{1,2}^2+m\left(\|v_0\|_{1,2}^2\right)\|v_0\|_{1,2}^2 < \int_\Omega f(u_0,v_0)u_0dx+\int_\Omega g(u_0,v_0)v_0dx,$$

that is

$$E'(u_0,v_0)(u_0,v_0)<0.$$

Now, using $\left(M_1\right)$, (H_1) and Sobolev imbedding, we can see that

$$E'(tu_0,tv_0)(u_0,v_0)>0 \text{ for } t \text{ sufficiently small.}$$

Thus,there exists $\tau \in \left(0,1\right)$ such that $E'(\tau u_0,\tau v_0)(u_0,v_0)=0$, which implies that $(\tau u_0,\tau v_0) \in \mathcal{N}$. Then, according to (1.2), $\left(H_6\right)$, semicontinuity of norm and Fatou lemma we obtain

$$\begin{aligned}
\rho_0 \leqslant A \leqslant E(\tau u_0,\tau v_0) &= E(\tau u_0,\tau v_0)-\frac{1}{4}E'(\tau u_0,\tau v_0)(\tau u_0,\tau v_0)\\
&= \frac{1}{2}M\left(\|\tau u_0\|_{1,2}^2\right)-\frac{1}{4}m\left(\|\tau u_0\|_{1,2}^2\right)\|\tau u_0\|_{1,2}^2\\
&\quad +\frac{1}{2}M\left(\|\tau v_0\|_{1,2}^2\right)-\frac{1}{4}m\left(\|\tau v_0\|_{1,2}^2\right)\|\tau v_0\|_{1,2}^2\\
&\quad +\frac{1}{4}\int_\Omega\Big(f(\tau u_0,\tau v_0)\tau u_0+g(\tau u_0,\tau v_0)\tau v_0-4F(\tau u_0,\tau v_0)\Big)dx\\
&< \frac{1}{2}M\left(\|u_0\|_{1,2}^2\right)-\frac{1}{4}m\left(\|u_0\|_{1,2}^2\right)\|u_0\|_{1,2}^2+\frac{1}{2}M\left(\|v_0\|_{1,2}^2\right)\\
&\quad -\frac{1}{4}m\left(\|v_0\|_{1,2}^2\right)\|v_0\|_{1,2}^2\\
&\quad +\frac{1}{4}\int_\Omega\Big(f(u_0,v_0)u_0+g(u_0,v_0)v_0-4F(u_0,v_0)\Big)dx\\
&\leqslant \liminf_{n\to\infty}\Big[\frac{1}{2}M\left(\|u_n\|_{1,2}^2\right)-\frac{1}{4}m\left(\|u_n\|_{1,2}^2\right)\|u_n\|_{1,2}^2\Big]\\
&\quad +\liminf_{n\to\infty}\Big[\frac{1}{2}M\left(\|v_n\|_{1,2}^2\right)-\frac{1}{4}m\left(\|v_n\|_{1,2}^2\right)\|v_n\|_{1,2}^2\Big]\\
&\quad +\liminf_{n\to\infty}\Big[\frac{1}{4}\int_\Omega\Big(f(u_n,v_n)u_n+g(u_n,v_n)v_n-4F(u_n,v_n)\Big)dx\Big]\\
&\leqslant \lim_{n\to\infty}\Big[E(u_n,v_n)-\frac{1}{4}E'(u_n,v_n)(u_n,v_n)\Big]=\rho_0,
\end{aligned}$$

which is absurd.

Next, we claim that $E(u_0,v_0) \geqslant 0$. By (5.6), (1.2) and (1.4) one has

$$\begin{aligned} E(u_0,v_0) \geqslant\ & \frac{1}{2}M\Big(\|u_0\|_{1,2}^2\Big) - \frac{1}{4}m\Big(\|u_0\|_{1,2}^2\Big)\|u_0\|_{1,2}^2 + \frac{1}{2}M\Big(\|v_0\|_{1,2}^2\Big) \\ & -\frac{1}{4}m\Big(\|v_0\|_{1,2}^2\Big)\|v_0\|_{1,2}^2 \\ & +\frac{1}{4}\int_\Omega \Big(f(u_0,v_0)u_0 + g(u_0,v_0)v_0 - 4F(u_0,v_0)\Big)dx \\ \geqslant\ & 0. \end{aligned}$$

This completes the proof of assertion 1.

Assertion 2. $(u_n,v_n) \to (u_0,v_0)$ in $\mathscr{H}$.

As (u_n,v_n) is bounded, up to a subsequence, $\|(u_n,v_n)\|_{\mathscr{H}} \to r > 0$, with

$$r^2 = r_1^2 + r_2^2, \quad \|u_n\|_{1,2} \to r_1 \text{ and } \|v_n\|_{1,2} \to r_2.$$

By using (5.5) and semicontinuity of norm, we have

$$E(u_0,v_0) \leqslant \rho_0. \tag{5.7}$$

In this case we claim that $E(u_0,v_0) = \rho_0$. So it remains to prove (5.7), assume by contradiction that $E(u_0,v_0) < \rho_0$. Then, $\|u_0\|_{1,2} < r/2$ and $\|v_0\|_{1,2} < r/2$. Let

$$U_n = \frac{2u_n}{\|(u_n,v_n)\|_{\mathscr{H}}}, \quad V_n = \frac{2v_n}{\|(u_n,v_n)\|_{\mathscr{H}}}, \quad U_0 = \frac{2u_0}{r} \text{ and } V_0 = \frac{2v_0}{r}.$$

We have

$$\begin{gathered} U_n \rightharpoonup U_0 \text{ in } H_0^1(\Omega) \text{ and } V_n \rightharpoonup V_0 \text{ in } H_0^1(\Omega) \\ \|U_0\|_{1,2} < 1 \text{ and } \|V_0\|_{1,2} < 1. \end{gathered}$$

Thus, by lemma 2

$$\sup_{n\in\mathbb{N}} \int_\Omega e^{p\left(U_n^2+V_n^2\right)}dx < \infty, \quad \forall p < \frac{4\pi}{1-\|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2}. \tag{5.8}$$

On the other hand,

$$2\rho_0 - 2E(u_0,v_0) = M(r_1^2) + M(r_2^2) - \big(M(\|u_0\|_{1,2}^2) + M(\|v_0\|_{1,2}^2)\big). \tag{5.9}$$

Using this equality, lemma 5 and the fact that $E(u_0,v_0) \geqslant 0$, we get

$$M(r_1^2) + M(r_2^2) < M\Big(\frac{4\pi}{\alpha_0}\Big) + M(\|u_0\|_{1,2}^2) + M(\|v_0\|_{1,2}^2).$$

From (M_1) and (M_2) it follows that

$$M(r_1^2) < M\Big(\frac{4\pi}{\alpha_0} + \|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2\Big),$$

and

$$M\left(r_2^2\right) < M\left(\frac{4\pi}{\alpha_0} + \|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2\right),$$

which implies that

$$r_1^2 < \frac{4\pi}{\alpha_0} + \|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2 \text{ and } r_2^2 < \frac{4\pi}{\alpha_0} + \|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2, \tag{5.10}$$

and therfore

$$r_1^2 + r_2^2 = r^2 < \frac{8\pi}{\alpha_0} + 2\left(\|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2\right). \tag{5.11}$$

Now, we observe that

$$r^2 = \frac{r^2 - 2\left(\|u_0\|_{1,2}^2 + \|v_0\|_{1,2}^2\right)}{1 - \|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2},$$

and from (5.11), it follows that

$$\frac{r^2}{2} < \frac{4\pi}{\alpha_0}\left(1 - \|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2\right)^{-1}.$$

Then, we can choose $p > 4\pi$ such that $\alpha_0 \frac{\|(u_n,v_n)\|_{\mathscr{H}}^2}{2} < p < 4\pi\left(1 - \|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2\right)^{-1}$ for n sufficiently large. Now, taking $q > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0 such that

$$q\alpha \frac{\|(u_n,v_n)\|_{\mathscr{H}}^2}{2} \leqslant p < 4\pi\left(1 - \|U_0\|_{1,2}^2 - \|V_0\|_{1,2}^2\right)^{-1}, \text{ for } n \text{ large enough}$$

and invoking (5.8), for some $C > 0,$ we concluded that

$$\begin{aligned} \int_\Omega e^{q\alpha\left(\frac{u_n^2+v_n^2}{2}\right)} dx &= \int_\Omega e^{q\alpha \frac{\|(u_n,v_n)\|_{\mathscr{H}}^2}{4}\left(U_n^2+V_n^2\right)} dx \\ &\leqslant \int_\Omega e^{p\left(U_n^2+V_n^2\right)} dx \leqslant C. \end{aligned} \tag{5.12}$$

Hence, using Hölder inequality, (5.12) and (5.3) we reach

$$\left|\int_\Omega f(u_n,v_n)\left(u_n - u_0\right) dx\right| \to 0 \text{ as } n \to \infty,$$

and

$$\left|\int_\Omega g(u_n,v_n)\left(v_n - v_0\right) dx\right| \to 0 \text{ as } n \to \infty.$$

Since $E'(u_n,v_n)(\left(u_n - u_0\right),0) = o\left(1\right),$ we get

$$m\left(\|u_n\|_{1,2}^2\right)\|u_n\|_{1,2}^2 - m\left(\|u_n\|_{1,2}^2\right)\int_\Omega \nabla u_n \nabla u_0 \, dx \to m\left(r_1^2\right)r_1^2 - m\left(r_1^2\right)\|u_0\|_{1,2}^2 = 0.$$

It follows that

$$\|u_0\|_{1,2} = r_1.$$

Similarly, we obtain

$$\|v_0\|_{1,2} = r_2,$$

which implies that $\|(u_n,v_n)\|_{\mathscr{H}} \to \|(u_0,v_0)\|_{\mathscr{H}}$ and so $(u_n,v_n) \to (u_0,v_0)$ in $\mathscr{H}$. In view of the continuity of E, we must have $E(u_0,v_0) = \rho_0$ what is an absurde. Thus, the proof of Proposition 1 is complete. $\square$

Finalizing the proof of Theorem 1. It follows from the hypotheses in Theorem 1 that E satisfies Palais-Smale condition for all $\rho_0 < \frac{1}{2}M\big(4\pi/\alpha_0\big)$, see proposition 1. To finish the proof of theorem 1, we use lemmas 2 and 3 and apply the Mountain pass Theorem. $\square$

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