Differential
Equations
& Applications
Volume 14, Number 3 (2022), 447–467
doi:10.7153/dea-2022-14-31

ON THE PERIODIC SOLUTIONS FOR NONLINEAR VOLTERRA–FREDHOLM INTEGRO–DIFFERENTIAL EQUATIONS WITH ψ–HILFER FRACTIONAL DERIVATIVE

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(*Communicated by J. R. Wang*)

Abstract. In this research paper, we present some results about the existence and uniqueness of periodic solutions for a great nonlinear class of Volterra-Fredholm integro-differential equations equipped with fractional integral conditions, involving ψ-Hilfer fractional operator. This investigation is carried out by means of the coincidence degree theory of Mawhin. A typical example is also presented.

1. Introduction

Fractional Calculus is one of the most showing areas and has attracted the heed of many scholars in a deep rang of fields [1, 2, 3, 14, 15, 16, 20, 25]. Many research have published in the domain related to the study of fractional differential equations by using different methods and approaches [6, 7, 8, 9, 10, 11].

Several researchers have investigated different extension of some classical fractional operators. In 2018, Vanterler *et al.* discussed the so-called ψ-Hilfer fractional derivative [23]. For some new research related to the study of some class of fractional differential equations involving the generalized Hilfer fractional derivative, see [4, 22] and the references therein.

In [21], Tidke studied the existence and uniqueness using the fixed point theory of mixed Volterra-Fredholm integro-differential problem

$$\begin{cases} \mathfrak{u}'(t) = f\left(\mathfrak{u}(t), \int_0^t \kappa(t,s,\mathfrak{u}(s))ds, \int_0^{\mathfrak{b}} h(t,s,\mathfrak{u}(s))ds\right), & t \in [0,\mathfrak{b}] \\ \mathfrak{u}(0) + g(\mathfrak{u}) = \mathfrak{u}_0. \end{cases}$$

By means of fixed-point theorem for the mixed integro-differential equations with Caputo fractional derivative of order $0 < \alpha \leqslant 1$, Anguraj *et al.* [5] studied the existence and uniqueness of solution for the following problem with integral boundary conditions,

$$\begin{cases} \frac{d^\alpha \mathfrak{u}(t)}{dt^\alpha} = f\left(t, \mathfrak{u}(t), \int_0^t \kappa(t,s,\mathfrak{u}(s))ds, \int_0^1 h(t,s,\mathfrak{u}(s))ds\right), & t \in [0,1] \\ \mathfrak{u}(0) = \int_0^1 g(s)\mathfrak{u}(s)ds. \end{cases}$$

Mathematics subject classification (2020): 34A08, 34A12, 34B40, 45J05.
Keywords and phrases: Generalized Hilfer fractional derivative, existence, uniqueness, periodic solutions, coincidence degree theory, Volterra-Fredholm integro-differential equation.

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Very recently, some interesting results about periodic solutions for different classes of differential equations have been provided (see [13, 19, 24] and the references therein).

Motivated by the above researches and using the technique of the coincidence degree theory of Mawhin, in this work, we consider the following nonlinear class of Volterra-Fredholm integro-differential fractional equation

$$\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau) = \mathscr{F}\left(\tau,\mathfrak{u}(\tau),\mathscr{G}\mathfrak{u}(\tau),\mathscr{H}\mathfrak{u}(\tau)\right),\ \tau \in (\mathfrak{a},\mathfrak{b}], \tag{1.1}$$

with the fractional integral conditions

$$\mathfrak{J}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a}) = \mathfrak{J}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{b}), \tag{1.2}$$

where

$$\mathscr{G}\mathfrak{u}(\tau) = \int_{\mathfrak{a}}^{\tau} \mathrm{g}(\tau,s,\mathfrak{u}(s))ds \quad \text{and} \quad \mathscr{H}\mathfrak{u}(\tau) = \int_{\mathfrak{a}}^{\mathfrak{b}} \mathrm{h}(\tau,s,\mathfrak{u}(s))ds, \tag{1.3}$$

and

$$\mathscr{F} : (\mathfrak{a},\mathfrak{b}] \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}, \qquad \mathrm{g} : \Delta \times \mathfrak{R} \to \mathfrak{R} \text{ and } \mathrm{h} : \Delta_0 \times \mathfrak{R} \to \mathfrak{R},$$

are continuous functions with $\mathfrak{J} := [\mathfrak{a},\mathfrak{b}]$, $(-\infty < \mathfrak{a} < \mathfrak{b} < +\infty)$, $\Delta_0 = \mathfrak{J} \times \mathfrak{J}$ and $\Delta = \{(\tau,s) : \mathfrak{a} \leqslant s \leqslant \tau \leqslant \mathfrak{b}\}$. $\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}$ denote the generalized ψ-Hilfer fractional derivative of order $0 < \alpha \leqslant 1$ and type $\beta \in [0,1]$. $\mathfrak{J}_{\mathfrak{a}^+}^{1-\nu;\psi}$ is the generalized fractional integral in the sense of Riemann-Liouville of order $1-\nu$, $(\nu = \alpha + \beta - \alpha\beta)$.

To the best of our Knowledge, the results obtained are news and they cannot be find via fixed point theory approaches.

In this research, we investigated some new existence and uniqueness results for a wide class of Volterra-Fredholm integro-differential equations with fractional integral conditions, involving ψ-Hilfer fractional derivative, by using the coincidence degree theory of Mawhin introduced in [12, 17]. Our results enlarge and complement the results mentioned above. Thus, in Theorem 4 we prove the existence by choosing a suitable operators and applying the coincidence degree theory of Mawhin, while in Theorem 5 we present some sufficient conditions ensuring the existence and uniqueness of periodic solutions for our problem (1.1)–(1.2). Finally, the work close with an important illustrative example.

2. Basic concepts

In this paper, we consider $C(\mathfrak{J},\mathfrak{R}), AC(\mathfrak{J},\mathfrak{R})$ and $C^{\mathfrak{m}}(\mathfrak{J},\mathfrak{R})$ the spaces of continuous, absolutely continuous and $\mathfrak{m}$ times continuously differentiable functions on $\mathfrak{J}$, respectively. We note $L^{\mathfrak{p}}(\mathfrak{J},\mathfrak{R})$, $\mathfrak{p} \geqslant 1$, the space of Lebesgue integrable functions on $\mathfrak{J}$.

The weighted spaces of continuous functions are defined by

$$C_{\nu;\psi}(\mathfrak{J},\mathfrak{R}) = \{\mathfrak{u} : (\mathfrak{a},\mathfrak{b}] \to \mathfrak{R} : (\psi(\tau) - \psi(\mathfrak{a}))^{\nu}\mathfrak{u}(\tau) \in C(\mathfrak{J},\mathfrak{R})\},$$

$$C^{\mathfrak{m}}_{\nu;\psi}(\mathfrak{J},\mathfrak{R}) = \{\mathfrak{u} \in C^{\mathfrak{m}-1}(\mathfrak{J},\mathfrak{R}) : \mathfrak{u}^{(\mathfrak{m})} \in C_{\nu;\psi}(\mathfrak{J},\mathfrak{R})\},\ \mathfrak{m} \in \mathfrak{N},$$
$$C^{0}_{\nu;\psi}(\mathfrak{J},\mathfrak{R}) = C_{\nu;\psi}(\mathfrak{J},\mathfrak{R}),$$

with the norms

$$\|\mathfrak{u}\|_{C_{\nu;\psi}} = \|(\psi(\cdot)-\psi(\mathfrak{a}))^{\nu}\mathfrak{u}(\cdot)\|_{\infty} = \sup_{\tau\in\mathfrak{J}}|(\psi(\tau)-\psi(\mathfrak{a}))^{\nu}\mathfrak{u}(\tau)|$$

and

$$\|\mathfrak{u}\|_{C^{\mathfrak{m}}_{\nu;\psi}} = \sum_{k=0}^{\mathfrak{m}-1}\|\mathfrak{u}^{(k)}\|_{\infty} + \|\mathfrak{u}^{(\mathfrak{m})}\|_{C_{\nu;\psi}},$$

where $\|\cdot\|_{\infty}$ denotes the suppremum norm on $C(\mathfrak{J},\mathfrak{R})$.

These spaces satisfy the properties below.

- $C_{0,\psi}(\mathfrak{J},\mathfrak{R}) = C(\mathfrak{J},\mathfrak{R})$.
- $C^{\mathfrak{m}}_{\nu,\psi}(\mathfrak{J},\mathfrak{R}) \subset AC^{\mathfrak{m}}(\mathfrak{J},\mathfrak{R})$.

DEFINITION 1. [16] Let $(\mathfrak{a},\mathfrak{b})$, $(-\infty \leqslant \mathfrak{a} < \mathfrak{b} \leqslant \infty)$ be a finite or infinite interval of the real line $\mathfrak{R}$ and $\alpha > 0$. Also let ψ, be an increasing and positive monotone function on $(\mathfrak{a},\mathfrak{b}]$, having a continuous derivative ψ' on $(\mathfrak{a},\mathfrak{b})$. The left sided fractional integral of a function $\mathfrak{u}$ with respect to another function ψ on $[\mathfrak{a},\mathfrak{b}]$ is defined by

$$\mathfrak{I}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau) = \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\mathfrak{u}(s)ds.$$

LEMMA 1. [16] *Let $\alpha > 0$ and $\beta > 0$. Then, we have*

$$\mathfrak{I}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{I}^{\beta;\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau) = \mathfrak{I}^{\alpha+\beta;\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau),\ \textit{for all}\ \tau \in (\mathfrak{a},\mathfrak{b})$$

LEMMA 2. [16] *Let $\alpha > 0$, $\rho > 0$ and $\tau \in (\mathfrak{a},\mathfrak{b})$. If $\mathfrak{u}(\tau) = (\psi(\tau)-\psi(a))^{\rho-1}$, then*

$$\mathfrak{I}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau) = \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha+\rho-1}.$$

DEFINITION 2. [23] Let $\mathfrak{n}-1 < \alpha < \mathfrak{n}$ with $\mathfrak{n} \in \mathbb{N}$ and $\mathfrak{u},\psi \in C^{\mathfrak{n}}(\mathfrak{J},\mathfrak{R})$ two functions such that ψ is increasing and $\psi'(\tau) \neq 0$, for any $\tau \in \mathfrak{J}$. The ψ-Hilfer fractional derivative $\mathfrak{D}^{\alpha,\beta;\psi}_{\mathfrak{a}^+}(\cdot)$ of function of order α and type $0 \leqslant \beta \leqslant 1$, is defined by

$$\mathfrak{D}^{\alpha,\beta;\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau) = \mathfrak{I}^{\beta(\mathfrak{n}-\alpha);\psi}_{\mathfrak{a}^+}\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{\mathfrak{n}}\mathfrak{I}^{(1-\beta)(\mathfrak{n}-\alpha);\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau),\ \tau \in \mathfrak{J}.$$

In particular, when $0 < \alpha < 1$, we have

$$\mathfrak{D}^{\alpha,\beta;\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau) = \mathfrak{I}^{\beta(1-\alpha);\psi}_{\mathfrak{a}^+}\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)\mathfrak{I}^{(1-\beta)(1-\alpha);\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau),\ \tau \in \mathfrak{J}.$$

THEOREM 1. [23] *If* $\mathfrak{u} \in C^{\mathfrak{n}}(\mathfrak{J},\mathfrak{R})$, $0 \leqslant \beta \leqslant 1$ *and* $\mathfrak{n}-1 < \alpha < \mathfrak{n}$, *then*

$$\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau) = \mathfrak{u}(\tau) - \sum_{k=1}^{\mathfrak{n}} \frac{(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-k}}{\Gamma(\nu-k+1)} \left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{\mathfrak{n}-k} \mathfrak{I}_{\mathfrak{a}^+}^{(1-\beta)(\mathfrak{n}-\alpha);\psi}\mathfrak{u}(\mathfrak{a}),$$

with $\nu = \alpha+\beta(\mathfrak{n}-\alpha)$. *In particular, when* $0<\alpha<1$, *we have*

$$\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau) = \mathfrak{u}(\tau) - \frac{(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\nu)}\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a}).$$

THEOREM 2. [23] *Let* $\mathfrak{u} \in C^1(\mathfrak{J},\mathfrak{R})$, $0 \leqslant \beta \leqslant 1$ *and* $\alpha > 0$, *we have*

$$\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) = \mathfrak{u}(\tau).$$

THEOREM 3. [23] *Let* $\mathfrak{u},\mathfrak{v} \in C^{\mathfrak{n}}(\mathfrak{J},\mathfrak{R})$, $0 \leqslant \beta \leqslant 1$ *and* $\alpha > 0$. *Then*

$$\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau) = \mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{v}(\tau) \Longleftrightarrow \mathfrak{u}(\tau) = \mathfrak{v}(\tau) + \sum_{k=1}^{\mathfrak{n}} c_k(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-k},$$

where

$$c_k = \frac{1}{\Gamma(\nu+1-k)}\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{\mathfrak{n}-k}\mathfrak{I}_{\mathfrak{a}^+}^{(1-\beta)(\mathfrak{n}-\alpha);\psi}\mathfrak{u}(\mathfrak{a}),$$

and $\nu = \alpha+\beta-\alpha\beta$.

REMARK 1. Let $\mathfrak{u} \in C^{\mathfrak{n}}(\mathfrak{J},\mathfrak{R})$, $0 \leqslant \beta \leqslant 1$ and $\alpha > 0$. Then

$$\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau) = 0 \Longleftrightarrow \mathfrak{u}(\tau) = \sum_{k=1}^{\mathfrak{n}} c_k(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-k}.$$

We will present definitions and the coincidence degree theory that are essential in proofs of our results, see [12, 17].

DEFINITION 3. We consider the normed spaces $\mathscr{X}$ and $\mathscr{Y}$. A Fredholm operator of index zero is a linear operator $\mathfrak{L} : Dom(\mathfrak{L}) \subset \mathscr{X} \to \mathscr{Y}$ such that

a) $\mathfrak{dim}\ker\mathfrak{L} = \mathfrak{codim}\mathfrak{Img}\mathfrak{L} < +\infty$.

b) $\mathfrak{Img}\mathfrak{L}$ is a closed subset of $\mathscr{Y}$.

By Definition 3, there exist continuous projectors $\mathscr{Q} : \mathscr{Y} \to \mathscr{Y}$ and $\mathscr{P} : \mathscr{X} \to \mathscr{X}$ satisfying

$$\mathfrak{Img}\mathfrak{L} = \ker\mathscr{Q}, \quad \ker\mathfrak{L} = \mathfrak{Img}\mathscr{P}, \quad \mathscr{Y} = \mathfrak{Img}\mathscr{Q} \oplus \mathfrak{Img}\mathfrak{L}, \quad \mathscr{X} = \ker\mathscr{P} \oplus \ker\mathfrak{L},$$

Thus, the restriction of $\mathfrak{L}$ to $Dom\mathfrak{L} \cap \ker\mathscr{P}$, denoted by $\mathfrak{L}_{\mathscr{P}}$, is an isomorphism onto its image.

DEFINITION 4. Let $\Omega \subseteq \mathscr{X}$ be a bounded subset and $\mathfrak{L}$ be a Fredholm operator of index zero with $Dom\mathfrak{L} \cap \Omega \neq \emptyset$. Then, the operator $\mathscr{N} : \overline{\Omega} \to \mathscr{Y}$ is called to be $\mathfrak{L}$-compact in $\overline{\Omega}$ if

a) the mapping $\mathscr{Q}\mathscr{N} : \overline{\Omega} \to \mathscr{Y}$ is continuous and $\mathscr{Q}\mathscr{N}\left(\overline{\Omega}\right) \subseteq \mathscr{Y}$ is bounded.

b) the mapping $(\mathfrak{L}_{\mathscr{P}})^{-1}(id - \mathscr{Q})\mathscr{N} : \overline{\Omega} \to \mathscr{X}$ is completely continuous.

LEMMA 3. [18] *Let $\mathscr{X}, \mathscr{Y}$ be a Banach spaces, $\Omega \subset \mathscr{X}$ a bounded open set and symmetric with $0 \in \Omega$. Suppose that $\mathfrak{L} : Dom\mathfrak{L} \subset \mathscr{X} \to \mathscr{Y}$ is a Fredholm operator of index zero with $Dom\mathfrak{L} \cap \overline{\Omega} \neq \emptyset$ and $\mathscr{N} : \mathscr{X} \to \mathscr{Y}$ is a $\mathfrak{L}$-compact operator on $\overline{\Omega}$. Assume, moreover, that*

$$\mathfrak{L}x - \mathscr{N}x \neq -\zeta(\mathfrak{L}x + \mathscr{N}(-x)),$$

for any $x \in Dom\mathfrak{L} \cap \partial\Omega$ and any $\zeta \in (0,1]$, where $\partial\Omega$ is the boundary of Ω with respect to $\mathscr{X}$. If these conditions are verified, then there exist at least one solution of the equation $\mathfrak{L}x = \mathscr{N}x$ on $Dom\mathfrak{L} \cap \overline{\Omega}$.

3. Main results

Let

$$\mathscr{X} = \{\mathfrak{u} \in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R}) : \mathfrak{u}(\tau) = \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\upsilon(\tau) : \upsilon \in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R}), \tau \in (\mathfrak{a},\mathfrak{b}]\},$$

and $\mathscr{Y} = C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R})$ with the norm

$$\|\mathfrak{u}\|_{\mathscr{X}} = \|\mathfrak{u}\|_{\mathscr{Y}} = \|\mathfrak{u}\|_{C_{1-\nu;\psi}}.$$

Let us introduce the following hypotheses:

$(A1)$ The function $\mathscr{F} : (\mathfrak{a},\mathfrak{b}] \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ be such that

$$\mathscr{F}(\cdot,\mathfrak{u}(\cdot),\mathscr{G}(\mathfrak{u})(\cdot),\mathscr{H}(\mathfrak{u})(\cdot)) \in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R}) \text{ for all } \mathfrak{u} \in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R}),$$

$(A2)$ There exist a positive constants γ, η_1, η_2 with

$$\begin{aligned} &|\mathscr{F}(\tau,\mathfrak{u},\mathscr{G}(\mathfrak{u}),\mathscr{H}(\mathfrak{u})) - \mathscr{F}(\tau,\bar{\mathfrak{u}},\mathscr{G}(\bar{\mathfrak{u}}),\mathscr{H}(\bar{\mathfrak{u}}))| \\ &\leqslant \gamma|\mathfrak{u} - \bar{\mathfrak{u}}| + \eta_1|\mathscr{G}\mathfrak{u} - \mathscr{G}\bar{\mathfrak{u}}| + \eta_2|\mathscr{H}\mathfrak{u} - \mathscr{H}\bar{\mathfrak{u}}|, \end{aligned}$$

for every $\tau \in (\mathfrak{a},\mathfrak{b}]$ and $\mathfrak{u}, \bar{\mathfrak{u}} \in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R})$.

$(A3)$ There exists a constant $\rho_1 > 0$ such that

$$|\mathrm{g}(\tau,s,\upsilon) - \mathrm{g}(\tau,s,\bar{\upsilon})| \leqslant \rho_1|\upsilon - \bar{\upsilon}|,$$

for every $(\tau,s) \in \Delta$ and $\upsilon, \bar{\upsilon} \in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R})$.

$(A4)$ There exists a constant $\rho_2 > 0$ such that

$$|\mathfrak{h}(\tau,s,\upsilon) - \mathfrak{h}(\tau,s,\bar{\upsilon})| \leqslant \rho_2 |\upsilon - \bar{\upsilon}|,$$

for every $(\tau,s) \in \Delta_0$ and $\upsilon, \bar{\upsilon} \in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R})$.

To prove the main findings, we need the following Lemmas. Before to state it, we give the definition of the operator $\mathfrak{L} : Dom\mathfrak{L} \subseteq \mathscr{X} \to \mathscr{Y}$

$$\mathfrak{L}\mathfrak{u} := \mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}, \tag{3.1}$$

where

$$Dom\mathfrak{L} = \{\mathfrak{u} \in \mathscr{X} : \mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u} \in \mathscr{Y} : \mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a}) = \mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{b})\}.$$

LEMMA 4. *Using the definition of $\mathfrak{L}$ given in (3.1). Then*

$$\ker\mathfrak{L} = \left\{\mathfrak{u} \in \mathscr{X} : \mathfrak{u}(\tau) = \frac{\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a})}{\Gamma(\nu)}\left(\psi(\tau) - \psi(\mathfrak{a})\right)^{\nu-1},\ \tau \in (\mathfrak{a},\mathfrak{b}]\right\},$$

and

$$\mathfrak{Img}\mathfrak{L} = \left\{\upsilon \in \mathscr{Y} : \mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi}\upsilon(\mathfrak{b}) = 0\right\}.$$

Proof. By Remark 1, we have for all $\mathfrak{u} \in \mathscr{X}$ the equation $\mathfrak{L}\mathfrak{u} = \mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u} = 0$ in $(\mathfrak{a},\mathfrak{b}]$, has a solution given by

$$\mathfrak{u}(\tau) = \frac{\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a})}{\Gamma(\nu)}\left(\psi(\tau) - \psi(\mathfrak{a})\right)^{\nu-1},\ \tau \in (\mathfrak{a},\mathfrak{b}],$$

which implies that

$$\ker\mathfrak{L} = \left\{\mathfrak{u} \in \mathscr{X} : \mathfrak{u}(\tau) = \frac{\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a})}{\Gamma(\nu)}\left(\psi(\tau) - \psi(\mathfrak{a})\right)^{\nu-1},\ \tau \in (\mathfrak{a},\mathfrak{b}]\right\}.$$

For $\upsilon \in \mathfrak{Img}\mathfrak{L}$, there exists $\mathfrak{u} \in Dom\mathfrak{L}$ such that $\upsilon = \mathfrak{L}\mathfrak{u} \in \mathscr{Y}$. Using Theorem 1, we obtain for each $\tau \in (\mathfrak{a},\mathfrak{b}]$

$$\mathfrak{u}(\tau) = \frac{\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a})}{\Gamma(\nu)}\left(\psi(\tau) - \psi(a)\right)^{\nu-1} + \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\upsilon(\tau).$$

By using Lemma 2 we obtain that

$$\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\tau) = \mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a}) + \mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi}\upsilon(\tau).$$

Since $\mathfrak{u} \in Dom\mathfrak{L}$ then we have $\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a}) = \mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{b})$. Thus

$$\mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi}\upsilon(\mathfrak{b}) = 0.$$

Furthermore, if $\upsilon \in \mathscr{Y}$, and satisfies

$$\mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi}\upsilon(\mathfrak{b}) = 0,$$

then for any $\mathfrak{u}(\tau) = \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\upsilon(\tau)$, we get $\upsilon(\tau) = \mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)$. Therefore

$$\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{b}) = \mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a}),$$

which implies that $\mathfrak{u} \in Dom\mathfrak{L}$. So that $\upsilon \in \mathfrak{Img}\mathfrak{L}$.

So

$$\mathfrak{Img}\mathfrak{L} = \left\{\upsilon \in \mathscr{Y} : \mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi}\upsilon(\mathfrak{b}) = 0\right\}.$$

Which completes the proof. □

LEMMA 5. *Let $\mathfrak{L}$ be defined by (3.1). Then $\mathfrak{L}$ is a Fredholm operator of index zero, and the linear continuous projector operators $\mathscr{Q} : \mathscr{Y} \to \mathscr{Y}$ and $\mathscr{P} : \mathscr{X} \to \mathscr{X}$ can be written as*

$$\mathscr{Q}\upsilon(\tau) = \frac{\Gamma(2+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi}\upsilon(\mathfrak{b}),$$

and

$$\mathscr{P}(\mathfrak{u})(\tau) = \frac{\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a})}{\Gamma(\nu)}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1}.$$

Furthermore, the operator $\mathfrak{L}_{\mathscr{P}}^{-1} : \mathfrak{Img}\mathfrak{L} \to \mathscr{X} \cap \ker\mathscr{P}$ can be written by

$$\mathfrak{L}_{\mathscr{P}}^{-1}(\upsilon)(\tau) = \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\upsilon(\tau).$$

Proof. Obviously, for each $\upsilon \in \mathscr{Y}$, $\mathscr{Q}^2\upsilon = \mathscr{Q}\upsilon$ and $\upsilon = (\upsilon - \mathscr{Q}(\upsilon)) + \mathscr{Q}(\upsilon)$, where $(\upsilon - \mathscr{Q}(\upsilon)) \in \ker\mathscr{Q} = \mathfrak{Img}\mathfrak{L}$.

Using the fact that $\mathfrak{Img}\mathfrak{L} = \ker\mathscr{Q}$ and $\mathscr{Q}^2 = \mathscr{Q}$ then $\mathfrak{Img}\mathscr{Q} \cap \mathfrak{Img}\mathfrak{L} = 0$. So,

$$\mathscr{Y} = \mathfrak{Img}\mathfrak{L} \oplus \mathfrak{Img}\mathscr{Q}.$$

By the same way we get that $\mathfrak{Img}\mathscr{P} = \ker\mathfrak{L}$ and $\mathscr{P}^2 = \mathscr{P}$. It follows for each $\mathfrak{u} \in \mathscr{X}$, that $\mathfrak{u} = (\mathfrak{u} - \mathscr{P}(\mathfrak{u})) + \mathscr{P}(\mathfrak{u})$ then $\mathscr{X} = \ker\mathscr{P} + \ker\mathfrak{L}$. Clearly we have $\ker\mathscr{P} \cap \ker\mathfrak{L} = 0$. Thus

$$\mathscr{X} = \ker\mathscr{P} \oplus \ker\mathfrak{L}.$$

Therefore

$$\mathfrak{dim}\ker\mathfrak{L} = \mathfrak{dim}\mathfrak{Img}\mathscr{Q} = \mathfrak{codim}\mathfrak{Img}\mathfrak{L}.$$

Consequently $\mathfrak{L}$ is a Fredholm operator of index zero.

Now, we will show that the inverse of $\mathfrak{L}|_{Dom\mathfrak{L}\cap\ker\mathscr{P}}$ is $\mathfrak{L}_{\mathscr{P}}^{-1}$. Effectively, for $\upsilon \in \mathfrak{Img}\mathfrak{L}$, by Theorem 2 we have

$$\mathfrak{L}\mathfrak{L}_{\mathscr{P}}^{-1}(\upsilon) = \mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\left(\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\upsilon\right) = \upsilon. \tag{3.2}$$

Furthermore, for $\mathfrak{u} \in Dom\mathfrak{L} \cap \ker \mathscr{P}$ we get

$$\mathfrak{L}_{\mathscr{P}}^{-1}(\mathfrak{L}(\mathfrak{u}(\tau))) = \mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\left(\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right) = \mathfrak{u}(\tau) - \frac{\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a})}{\Gamma(\nu)}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1}.$$

Using the fact that $\mathfrak{u} \in Dom\mathfrak{L} \cap \ker \mathscr{P}$, then

$$\frac{\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a})}{\Gamma(\nu)}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1} = 0.$$

Thus,

$$\mathfrak{L}_{\mathscr{P}}^{-1}\mathfrak{L}(\mathfrak{u}) = \mathfrak{u}. \tag{3.3}$$

Using (3.2) and (3.3) together, we get $\mathfrak{L}_{\mathscr{P}}^{-1} = (\mathfrak{L}|_{Dom\mathfrak{L}\cap\ker\mathscr{P}})^{-1}$. Which completes the demonstration. □

LEMMA 6. *For all* $\mathfrak{u}, \bar{\mathfrak{u}} \in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R})$ *and* $\tau \in (\mathfrak{a},\mathfrak{b}]$ *we get:*

$$|\mathscr{G}\mathfrak{u}(\tau) - \mathscr{G}\bar{\mathfrak{u}}(\tau)| \leqslant \lambda_1 \|\mathfrak{u}-\bar{\mathfrak{u}}\|_{\mathscr{X}},$$
$$|\mathscr{H}\mathfrak{u}(\tau) - \mathscr{H}\bar{\mathfrak{u}}(\tau)| \leqslant \lambda_2 \|\mathfrak{u}-\bar{\mathfrak{u}}\|_{\mathscr{X}},$$

where

$$\lambda_1 = \frac{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu}}{\nu \min\limits_{\tau\in[\mathfrak{a},\mathfrak{b}]}\psi'(\tau)}\rho_1 \quad \text{and} \quad \lambda_2 = \frac{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu}}{\nu \min\limits_{\tau\in[\mathfrak{a},\mathfrak{b}]}\psi'(\tau)}\rho_2.$$

Proof. Using $(A3)$, we have for any $\tau \in (\mathfrak{a},\mathfrak{b}]$

$$\begin{aligned}
|\mathscr{G}\mathfrak{u}(\tau) - \mathscr{G}\bar{\mathfrak{u}}(\tau)| &\leqslant \int_{\mathfrak{a}}^{\tau} |\mathrm{g}(\tau,s,\mathfrak{u}(s)) - \mathrm{g}(\tau,s,\bar{\mathfrak{u}}(s))|ds \\
&\leqslant \rho_1\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{\mathscr{X}} \int_{\mathfrak{a}}^{\tau} (\psi(s)-\psi(\mathfrak{a}))^{\nu-1}ds \\
&\leqslant \rho_1\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{\mathscr{X}} \int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(s)-\psi(\mathfrak{a}))^{\nu-1}\frac{1}{\psi'(s)}ds \\
&\leqslant \frac{\rho_1\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{\mathscr{X}}}{\min\limits_{\tau\in[\mathfrak{a},\mathfrak{b}]}\psi'(\tau)} \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(s)-\psi(\mathfrak{a}))^{\nu-1}ds \\
&\leqslant \frac{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu}}{\nu \min\limits_{\tau\in[\mathfrak{a},\mathfrak{b}]}\psi'(\tau)}\rho_1\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{\mathscr{X}} := \lambda_1\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{\mathscr{X}}.
\end{aligned}$$

By using an argument similar and $(A4)$, we get

$$|\mathscr{H}\mathfrak{u}(\tau) - \mathscr{H}\bar{\mathfrak{u}}(\tau)| \leqslant \lambda_2 \|\mathfrak{u}-\bar{\mathfrak{u}}\|_{\mathscr{X}}.$$

Now, we define $\mathscr{N} : \mathscr{X} \to \mathscr{Y}$ by

$$\mathscr{N}\mathfrak{u}(\tau) := \mathscr{F}(\tau,\mathfrak{u}(\tau),\mathscr{G}\mathfrak{u}(\tau),\mathscr{H}\mathfrak{u}(\tau)), \tau \in (\mathfrak{a},\mathfrak{b}].$$

The operator $\mathscr{N}$ is well defined, because $\mathscr{F}$, $\mathfrak{g}$ and $\mathfrak{h}$ are continuous functions.

We can remark that the problem (1.1)–(1.2) is equivalent to the problem $\mathfrak{L}\mathfrak{u} = \mathscr{N}\mathfrak{u}$. $\square$

LEMMA 7. *Suppose that (A1), (A2), (A3) and (A4) are satisfied then, for any bounded open set $\Omega \subset \mathscr{X}$, the operator $\mathscr{N}$ is $\mathfrak{L}$-compact.*

Proof. We consider for $\mathscr{M} > 0$ the bounded open set $\Omega = \{\mathfrak{u} \in \mathscr{X} : \|\mathfrak{u}\|_{\mathscr{X}} < \mathscr{M}\}$. We split the proof into three steps:

Step 1: *$\mathscr{Q}\mathscr{N}$ is continuous.*

Let $(\mathfrak{u}_\mathfrak{n})_{\mathfrak{n}\in\mathbb{N}}$ be a sequence such that $\mathfrak{u}_\mathfrak{n} \longrightarrow \mathfrak{u}$ in $\mathscr{Y}$, then for each $\tau \in \mathfrak{J}$, we have

$$\begin{aligned}
&|\mathscr{Q}\mathscr{N}(\mathfrak{u}_\mathfrak{n})(\tau) - \mathscr{Q}\mathscr{N}(\mathfrak{u})(\tau)| \\
&\leqslant \frac{(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}} \int_\mathfrak{a}^\mathfrak{b} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)} |\mathscr{N}(\mathfrak{u}_\mathfrak{n})(s) - \mathscr{N}(\mathfrak{u})(s)|\, ds.
\end{aligned}$$

By (A2), we have

$$\begin{aligned}
&|\mathscr{Q}\mathscr{N}(\mathfrak{u}_\mathfrak{n})(\tau) - \mathscr{Q}\mathscr{N}(\mathfrak{u})(\tau)| \\
\leqslant\ & \frac{\gamma(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}} \int_\mathfrak{a}^\mathfrak{b} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)} |\mathfrak{u}_\mathfrak{n}(s) - \mathfrak{u}(s)|\, ds \\
&+ \frac{\eta_1(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}} \int_\mathfrak{a}^\mathfrak{b} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)} |\mathscr{G}(\mathfrak{u}_\mathfrak{n})(s) - \mathscr{G}(\mathfrak{u})(s)|\, ds \\
&+ \frac{\eta_2(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}} \int_\mathfrak{a}^\mathfrak{b} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)} |\mathscr{H}(\mathfrak{u}_\mathfrak{n})(s) - \mathscr{H}(\mathfrak{u})(s)|\, ds \\
\leqslant\ & \frac{\gamma\Gamma(2+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}} \|\mathfrak{u}_\mathfrak{n} - \mathfrak{u}\|_{\mathscr{Y}} \mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi} (\psi(s)-\psi(\mathfrak{a}))^{\nu-1}(\mathfrak{b}) \\
&+ \frac{\eta_1(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}} \int_\mathfrak{a}^\mathfrak{b} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)} |\mathscr{G}(\mathfrak{u}_\mathfrak{n})(s) - \mathscr{G}(\mathfrak{u})(s)|\, ds \\
&+ \frac{\eta_2(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}} \int_\mathfrak{a}^\mathfrak{b} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)} |\mathscr{H}(\mathfrak{u}_\mathfrak{n})(s) - \mathscr{H}(\mathfrak{u})(s)|\, ds.
\end{aligned}$$

Using Lemma 2 and Lemma 6, we get

$$\begin{aligned}
&|\mathscr{Q}\mathscr{N}(\mathfrak{u}_\mathfrak{n})(\tau) - \mathscr{Q}\mathscr{N}(\mathfrak{u})(\tau)| \\
&\leqslant \frac{\gamma\Gamma(2+\beta(\alpha-1))\Gamma(\nu)}{\Gamma(\alpha+1)} (\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1} \|\mathfrak{u}_\mathfrak{n} - \mathfrak{u}\|_{\mathscr{Y}} + (\eta_1\lambda_1 + \eta_2\lambda_2)\|\mathfrak{u}_\mathfrak{n} - \mathfrak{u}\|_{\mathscr{Y}} \\
&\leqslant \left[\frac{\gamma\Gamma(2+\beta(\alpha-1))\Gamma(\nu)}{\Gamma(\alpha+1)} (\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1} + (\lambda_1\eta_1 + \lambda_2\eta_2)\right] \|\mathfrak{u}_\mathfrak{n} - \mathfrak{u}\|_{\mathscr{Y}}.
\end{aligned}$$

Thus, for each $\tau \in \mathfrak{J}$, we obtain

$$\begin{aligned}&\left|(\psi(\tau)-\psi(\mathfrak{a}))^{1-\nu}\left(\mathscr{Q}\mathscr{N}(\mathfrak{u}_{\mathfrak{n}})(\tau)-\mathscr{Q}\mathscr{N}(\mathfrak{u})(\tau)\right)\right|\\ &\leqslant \left[\frac{\gamma\Gamma(2+\beta(\alpha-1))\Gamma(\nu)}{\Gamma(\alpha+1)}+(\lambda_1\eta_1+\lambda_2\eta_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}\right]\|\mathfrak{u}_{\mathfrak{n}}-\mathfrak{u}\|_{\mathscr{Y}}.\end{aligned}$$

Then, for all $\tau \in \mathfrak{J}$, we get

$$\left|(\psi(\tau)-\psi(\mathfrak{a}))^{1-\nu}\left(\mathscr{Q}\mathscr{N}(\mathfrak{u}_{\mathfrak{n}})(\tau)-\mathscr{Q}\mathscr{N}(\mathfrak{u})(\tau)\right)\right| \longrightarrow 0 \text{ as } n \longrightarrow +\infty,$$

therefore,

$$\|\mathscr{Q}\mathscr{N}(\mathfrak{u}_{\mathfrak{n}})-\mathscr{Q}\mathscr{N}(\mathfrak{u})\|_{\mathscr{Y}} \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

We deduce that $\mathscr{Q}\mathscr{N}$ is continuous.

Step 2: $\mathscr{Q}\mathscr{N}(\overline{\Omega})$ *is bounded*
For $\tau \in \mathfrak{J}$ and $\mathfrak{u} \in \overline{\Omega}$, we have

$$\begin{aligned}&|\mathscr{Q}\mathscr{N}(\mathfrak{u})(\tau)|\\
&\leqslant \frac{(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{N}(\mathfrak{u})(s)|ds\\
&\leqslant \frac{(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\\
&\quad\times\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}(\mathfrak{u})(s),\mathscr{H}(\mathfrak{u})(s))-\mathscr{F}(s,0,0,0)|ds\\
&\quad+\frac{(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{F}(s,0,0,0)|ds\\
&\leqslant \frac{(1+\beta(\alpha-1))\mathscr{F}^*}{\alpha}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}\\
&\quad+\frac{\gamma(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathfrak{u}(s)|ds\\
&\quad+\frac{\eta_1(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{G}(\mathfrak{u})(s)-\mathscr{G}(0)(s)|ds\\
&\quad+\frac{\eta_1(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{G}(0)(s)|ds\\
&\quad+\frac{\eta_2(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{H}(\mathfrak{u})(s)-\mathscr{H}(0)(s)|ds\\
&\quad+\frac{\eta_2(1+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{H}(0)(s)|ds\\
&\leqslant \frac{(1+\beta(\alpha-1))}{\alpha}\left[\mathscr{F}^*+\mathscr{M}(\gamma+\lambda_1\eta_1+\lambda_2\eta_2)\right](\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}\\
&\quad+(\mathrm{g}^*\eta_1+\mathrm{h}^*\eta_2)(\mathfrak{b}-\mathfrak{a}),\end{aligned}$$

where

$$\mathscr{F}^* = \|\mathscr{F}(.,0,0,0)\|_{C_{1-\nu;\psi}},\ \mathrm{g}^* = \sup_{(\tau,s)\in\Delta} |\mathrm{g}(\tau,s,0,0)| \text{ and } \mathrm{h}^* = \sup_{(\tau,s)\in\Delta_0} |\mathrm{h}(\tau,s,0,0)|.$$

Thus

$$\begin{aligned}\|\mathscr{Q}\mathscr{N}(\mathfrak{u})\|_{\mathscr{Y}} \leqslant \frac{(1+\beta(\alpha-1))}{\alpha}&\Big[\mathscr{F}^* + \mathscr{M}(\gamma+\lambda_1\eta_1+\lambda_2\eta_2)\Big]\\ &+(\mathrm{g}^*\eta_1+\mathrm{h}^*\eta_2)(\mathfrak{b}-\mathfrak{a})(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}.\end{aligned}$$

So, $\mathscr{Q}\mathscr{N}(\overline{\Omega})$ is a bounded set in $\mathscr{Y}$.

Step 3: $\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}:\overline{\Omega}\to\mathscr{X}$ *is completely continuous.*

We will use the Arzelà-Ascoli theorem, so we have to show that $\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}(\overline{\Omega}) \subset \mathscr{X}$ is equicontinuous and bounded. Firstly, for any $\mathfrak{u}\in\overline{\Omega}$ and $\tau\in(\mathfrak{a},\mathfrak{b}]$, we get

$$\begin{aligned}&\mathfrak{L}_{\mathscr{P}}^{-1}(\mathscr{N}\mathfrak{u}(\tau)-\mathscr{Q}\mathscr{N}\mathfrak{u}(\tau))\\ &=\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\Bigg[\mathscr{F}(\tau,\mathfrak{u}(\tau),\mathscr{G}\mathfrak{u}(\tau),\mathscr{H}\mathfrak{u}(\tau))\\ &\quad-\frac{\Gamma(2+\beta(\alpha-1))}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi}\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))(\mathfrak{b})\Bigg]\\ &=\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))ds\\ &\quad-\frac{\Gamma(2+\beta(\alpha-1))(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha+1)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\mathfrak{I}_{\mathfrak{a}^+}^{1+\beta(\alpha-1);\psi}\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))(\mathfrak{b})\Bigg].\end{aligned}$$

For all $\mathfrak{u}\in\overline{\Omega}$ and $\tau\in(\mathfrak{a},\mathfrak{b}]$, we get

$$\begin{aligned}&|\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}\mathfrak{u}(\tau)|\\ &\leqslant\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))-\mathscr{F}(s,0,0,0)|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathscr{F}(s,0,0,0)|ds\\ &\quad+\frac{1}{\Gamma(\alpha+1)}(1+\beta(\alpha-1))(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}\\ &\quad\times\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))-\mathscr{F}(s,0,0,0)|ds\\ &\quad+\frac{(1+\beta(\alpha-1))(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\alpha+1)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{F}(s,0,0,0)|ds,\\ &\leqslant\frac{\mathscr{F}^*\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha+\nu-1}+\frac{\mathscr{F}^*\Gamma(\nu)\Gamma(2+\beta(\alpha-1))}{\Gamma^2(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha+\nu-1}\\ &\quad+\frac{\gamma}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{u}(s)|ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{\eta_1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathscr{G}(\mathfrak{u})(s)-\mathscr{G}(0)(s)|ds\\
&+\frac{\eta_1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathscr{G}(0)(s)|ds\\
&+\frac{\eta_2}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathscr{H}(\mathfrak{u})(s)-\mathscr{H}(0)(s)|ds\\
&+\frac{\eta_2}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathscr{H}(0)(s)|ds\\
&+\frac{\gamma(1+\beta(\alpha-1))(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\alpha+1)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathfrak{u}(s)|ds\\
&+\frac{\eta_1(1+\beta(\alpha-1))(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\alpha+1)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{G}(\mathfrak{u})(s)-\mathscr{G}(0)(s)|ds\\
&+\frac{\eta_1(1+\beta(\alpha-1))(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\alpha+1)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{G}(0)(s)|ds\\
&+\frac{\eta_2(1+\beta(\alpha-1))(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\alpha+1)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{H}(\mathfrak{u})(s)-\mathscr{H}(0)(s)|ds\\
&+\frac{\eta_2(1+\beta(\alpha-1))(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\alpha+1)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{H}(0)(s)|ds
\end{aligned}
$$

By using Lemma 6, we get

$$
\begin{aligned}
|\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}\mathfrak{u}(\tau)| \leqslant\ & \frac{\mathscr{F}^*\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha+\nu-1}\\
&+\frac{\mathscr{F}^*\Gamma(\nu)\Gamma(2+\beta(\alpha-1))}{\Gamma^2(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha+\nu-1}\\
&+\frac{\gamma\mathscr{M}\Gamma(\nu)}{\Gamma(\nu+\alpha)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha+\nu-1}\\
&+\frac{\gamma\mathscr{M}\Gamma(\nu)\Gamma(2+\beta(\alpha-1))}{\Gamma^2(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha+\nu-1}\\
&+\frac{2\mathscr{M}}{\Gamma(\alpha+1)}(\lambda_1\eta_1+\lambda_2\eta_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}\\
&+\frac{2(\mathfrak{b}-\mathfrak{a})}{\Gamma(\alpha+1)}(\mathrm{g}^*\eta_1+\mathrm{h}^*\eta_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}.
\end{aligned}
$$

So

$$
\begin{aligned}
|\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}\mathfrak{u}(\tau)| \leqslant\ & \frac{(\mathscr{F}^*+\gamma\mathscr{M})\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha+\nu-1}\\
&+\frac{(\mathscr{F}^*+\gamma\mathscr{M})\Gamma(\nu)\Gamma(2+\beta(\alpha-1))}{\Gamma^2(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha+\nu-1}\\
&+\frac{2(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha+1)}\Big[(\lambda_1\eta_1+\lambda_2\eta_2)\mathscr{M}+(\mathrm{g}^*\eta_1+\mathrm{h}^*\eta_2)(\mathfrak{b}-\mathfrak{a})\Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}\mathfrak{u}\|_{\mathscr{X}} \\
&\leqslant \Big[\frac{(\mathscr{F}^*+\gamma\mathscr{M})\Gamma(\nu)}{\Gamma(\alpha+\nu)}+\frac{(\mathscr{F}^*+\gamma\mathscr{M})\Gamma(\nu)\Gamma(2+\beta(\alpha-1))}{\Gamma^2(\alpha+1)}\Big](\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha} \\
&+\frac{2}{\Gamma(\alpha+1)}\Big[(\lambda_1\eta_1+\lambda_2\eta_2)\mathscr{M}+(\mathrm{g}^*\eta_1+\mathrm{h}^*\eta_2)(\mathfrak{b}-\mathfrak{a})\Big](\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha+1-\nu}.
\end{aligned}
$$

This means that $\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}(\overline{\Omega})$ is uniformly bounded in $\mathscr{X}$.

It remains to show that $\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}(\overline{\Omega})$ is equicontinuous.

For $\mathfrak{a}<\tau_1<\tau_2\leqslant\mathfrak{b}$, $\mathfrak{u}\in\overline{\Omega}$, we have

$$
\begin{aligned}
&|(\psi(\tau_2)-\psi(\mathfrak{a}))^{1-\nu}\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}\mathfrak{u}(\tau_2)-(\psi(\tau_1)-\psi(\mathfrak{a}))^{1-\nu}\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}\mathfrak{u}(\tau_1)| \\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau_1}\psi'(s) \\
&\times\left|(\psi(\tau_2)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(\mathfrak{a}))^{1-\nu}-(\psi(\tau_1)-\psi(s))^{\alpha-1}(\psi(\tau_1)-\psi(\mathfrak{a}))^{1-\nu}\right| \\
&\times|\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))|\,ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(\mathfrak{a}))^{1-\nu}|\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))|\,ds \\
&+\frac{(1+\beta(\alpha-1))}{\Gamma(\alpha+1)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\beta(\alpha-1)}}\Big[(\psi(\tau_2)-\psi(\mathfrak{a}))^{1+\alpha-\nu}-(\psi(\tau_1)-\psi(\mathfrak{a}))^{1+\alpha-\nu}\Big] \\
&\times\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}|\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))|\,ds \\
\leqslant\ &\frac{(\mathscr{F}^*+\gamma\mathscr{M})}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau_1}\psi'(s) \\
&\times\left|(\psi(\tau_2)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(\mathfrak{a}))^{1-\nu}-(\psi(\tau_1)-\psi(s))^{\alpha-1}(\psi(\tau_1)-\psi(\mathfrak{a}))^{1-\nu}\right| \\
&\times(\psi(s)-\psi(\mathfrak{a}))^{\nu-1}\,ds \\
&+\frac{1}{\Gamma(\alpha)}\Big[(\lambda_1\eta_1+\lambda_2\eta_2)\mathscr{M}+(\eta_1\mathrm{g}^*+\eta_2\mathrm{h}^*)(\mathfrak{b}-\mathfrak{a})\Big]\int_{\mathfrak{a}}^{\tau_1}\psi'(s) \\
&\times\left|(\psi(\tau_2)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(\mathfrak{a}))^{1-\nu}-(\psi(\tau_1)-\psi(s))^{\alpha-1}(\psi(\tau_1)-\psi(\mathfrak{a}))^{1-\nu}\right|ds \\
&+\frac{(\mathscr{F}^*+\gamma\mathscr{M})}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(\mathfrak{a}))^{1-\nu}(\psi(s)-\psi(\mathfrak{a}))^{\nu-1}\,ds \\
&+\frac{1}{\Gamma(\alpha)}\Big[(\lambda_1\eta_1+\lambda_2\eta_2)\mathscr{M}+(\eta_1\mathrm{g}^*+\eta_2\mathrm{h}^*)(\mathfrak{b}-\mathfrak{a})\Big] \\
&\times\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(\mathfrak{a}))^{1-\nu}\,ds
\end{aligned}
$$

$$+\left[\frac{(\mathscr{F}^*+\gamma\mathscr{M})\Gamma(2+\beta(\alpha-1))\Gamma(\nu)}{\Gamma(\alpha+1)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}}+(\lambda_1\eta_1+\lambda_2\eta_2)\mathscr{M}+(\eta_1\mathrm{g}^*+\eta_2\mathrm{h}^*)(\mathfrak{b}-\mathfrak{a})\right]$$
$$\times\frac{1}{\Gamma(\alpha+1)}\left[(\psi(\tau_2)-\psi(\mathfrak{a}))^{1+\alpha-\nu}-(\psi(\tau_1)-\psi(\mathfrak{a}))^{1+\alpha-\nu}\right].$$

The operator $\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}(\overline{\Omega})$ is equicontinuous in $\mathscr{X}$ because the right-hand side of the above inequality tends to zero as $\tau_1\to\tau_2$ and the limit is independent of $\mathfrak{u}$. The Arzelà-Ascoli theorem implies that $\mathfrak{L}_{\mathscr{P}}^{-1}(id-\mathscr{Q})\mathscr{N}(\overline{\Omega})$ is relatively compact in $\mathscr{X}$. As a consequence of steps 1 to 3, we get that $\mathscr{N}$ is $\mathfrak{L}$-compact in $\overline{\Omega}$. Which completes the demonstration. □

LEMMA 8. *Assume* (A1), (A2), (A3) *and* (A4). *If the condition*

$$\frac{\gamma\Gamma(\nu)}{\Gamma(\nu+\alpha)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}+\frac{(\lambda_1\eta_1+\lambda_2\eta_2)}{\Gamma(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\alpha-\nu}<\frac{1}{2}, \tag{3.4}$$

is satisfied, then there exists $\mathscr{A}>0$, which is independent of ζ, such that,

$$\mathfrak{L}(\mathfrak{u})-\mathscr{N}(\mathfrak{u})=-\zeta[\mathfrak{L}(\mathfrak{u})+\mathscr{N}(-\mathfrak{u})]\Longrightarrow\|\mathfrak{u}\|_{\mathscr{X}}\leqslant\mathscr{A},\ \zeta\in(0,1].$$

Proof. Let $\mathfrak{u}\in\mathscr{X}$ satisfies

$$\mathfrak{L}(\mathfrak{u})-\mathscr{N}(\mathfrak{u})=-\zeta\mathfrak{L}(\mathfrak{u})-\zeta\mathscr{N}(-\mathfrak{u}),$$

then

$$\mathfrak{L}(\mathfrak{u})=\frac{1}{1+\zeta}\mathscr{N}(\mathfrak{u})-\frac{\zeta}{1+\zeta}\mathscr{N}(-\mathfrak{u}).$$

So, from the expression of $\mathfrak{L}$ and $\mathscr{N}$, we get for any $\tau\in(\mathfrak{a},\mathfrak{b}]$:

$$\begin{aligned}\mathfrak{L}\mathfrak{u}(\tau)=\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)&=\frac{1}{1+\zeta}\mathscr{F}(\tau,\mathfrak{u}(\tau),\mathscr{G}\mathfrak{u}(\tau),\mathscr{H}\mathfrak{u}(\tau))\\&\quad-\frac{\zeta}{1+\zeta}\mathscr{F}(\tau,-\mathfrak{u}(\tau),\mathscr{G}(-\mathfrak{u})(\tau),\mathscr{H}(-\mathfrak{u})(\tau)).\end{aligned}$$

By Theorem 1 we get

$$\begin{aligned}\mathfrak{u}(\tau)=&\frac{c_1(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\nu)}+\frac{1}{\zeta+1}\Big[\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\Big(\mathscr{F}(s,\mathfrak{u}(s),\mathscr{G}\mathfrak{u}(s),\mathscr{H}\mathfrak{u}(s))\Big)(\tau)\\&-\zeta\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\Big(\mathscr{F}(s,-\mathfrak{u}(s),\mathscr{G}(-\mathfrak{u})(s),\mathscr{H}(-\mathfrak{u})(s))\Big)(\tau)\Big],\end{aligned}$$

where $c_1=\mathfrak{I}_{\mathfrak{a}^+}^{1-\nu;\psi}\mathfrak{u}(\mathfrak{a})$. Thus for each $\tau\in(\mathfrak{a},\mathfrak{b}]$ we have

$$\begin{aligned}|\mathfrak{u}(\tau)|\leqslant&\frac{|c_1|(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\nu)}+\frac{2\mathscr{F}^*\Gamma(\nu)}{(\zeta+1)\Gamma(\nu+\alpha)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha+\nu-1}\\&+\frac{2(\mathrm{g}^*\eta_1+\mathrm{h}^*\eta_2)(\mathfrak{b}-\mathfrak{a})}{(\zeta+1)\Gamma(\alpha+1)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha}\\&+\frac{2\|\mathfrak{u}\|_{\mathscr{X}}}{(\zeta+1)}\left[\frac{\gamma\Gamma(\nu)}{\Gamma(\nu+\alpha)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha+\nu-1}+\frac{(\lambda_1\eta_1+\lambda_2\eta_2)}{\Gamma(\alpha+1)}(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha}\right],\end{aligned}$$

thus

$$
\begin{aligned}
&\|\mathfrak{u}\|_{\mathscr{X}} \\
\leqslant\ & \frac{|c_1|}{\Gamma(\nu)} + \frac{2\mathscr{F}^*\Gamma(\nu)}{\Gamma(\nu+\alpha)}\left(\psi(\mathfrak{b})-\psi(\mathfrak{a})\right)^{\alpha} + \frac{2(\mathrm{g}^*\eta_1+\mathrm{h}^*\eta_2)(\mathfrak{b}-\mathfrak{a})}{\Gamma(\alpha+1)}\left(\psi(\mathfrak{b})-\psi(\mathfrak{a})\right)^{1+\alpha-\nu} \\
&+2\left[\frac{\gamma\Gamma(\nu)}{\Gamma(\nu+\alpha)}\left(\psi(\mathfrak{b})-\psi(\mathfrak{a})\right)^{\alpha} + \frac{(\lambda_1\eta_1+\lambda_2\eta_2)}{\Gamma(\alpha+1)}\left(\psi(\mathfrak{b})-\psi(\mathfrak{a})\right)^{1+\alpha-\nu}\right]\|\mathfrak{u}\|_{\mathscr{X}}.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\|\mathfrak{u}\|_{\mathscr{X}} \leqslant\ & \frac{\frac{|c_1|}{\Gamma(\nu)} + \frac{2\mathscr{F}^*\Gamma(\nu)}{\Gamma(\nu+\alpha)}\left(\psi(\mathfrak{b})-\psi(\mathfrak{a})\right)^{\alpha} + \frac{2(\mathrm{g}^*\eta_1+\mathrm{h}^*\eta_2)(\mathfrak{b}-\mathfrak{a})}{\Gamma(\alpha+1)}\left(\psi(\mathfrak{b})-\psi(\mathfrak{a})\right)^{1+\alpha-\nu}}{1-2\left[\frac{\gamma\Gamma(\nu)}{\Gamma(\nu+\alpha)}\left(\psi(\mathfrak{b})-\psi(\mathfrak{a})\right)^{\alpha} + \frac{(\lambda_1\eta_1+\lambda_2\eta_2)}{\Gamma(\alpha+1)}\left(\psi(\mathfrak{b})-\psi(\mathfrak{a})\right)^{1+\alpha-\nu}\right]} \\
&:= \mathscr{A}.
\end{aligned}
$$

The demonstration is completed. $\square$

LEMMA 9. *If conditions* $(A1)$–$(A4)$ *and* (3.4) *are verified, then there exist a bounded open set* $\Omega \subset \mathscr{X}$ *with*

$$
\mathfrak{L}(\mathfrak{u}) - \mathscr{N}(\mathfrak{u}) \neq -\zeta[\mathfrak{L}(\mathfrak{u}) + \mathscr{N}(-\mathfrak{u})], \tag{3.5}
$$

for any $\mathfrak{u} \in \partial\Omega$ *and any* $\zeta \in (0,1]$.

Proof. Using Lemma 8, then there exists a positive constant $\mathscr{A}$ which is independent of ζ such that, if $\mathfrak{u}$ verify

$$
\mathfrak{L}(\mathfrak{u}) - \mathscr{N}(\mathfrak{u}) = -\zeta[\mathfrak{L}(\mathfrak{u}) + \mathscr{N}(-\mathfrak{u})],\ \zeta \in (0,1],
$$

thus $\|\mathfrak{u}\|_{\mathscr{X}} \leqslant \mathscr{A}$. So, if

$$
\Omega = \{\mathfrak{u} \in \mathscr{X}; \|\mathfrak{u}\|_{\mathscr{X}} < \vartheta\}, \tag{3.6}
$$

such that $\vartheta > \mathscr{A}$, we deduce that

$$
\mathfrak{L}(\mathfrak{u}) - \mathscr{N}(\mathfrak{u}) \neq -\zeta[\mathfrak{L}(\mathfrak{u}) - \mathscr{N}(-\mathfrak{u})],
$$

for all $\mathfrak{u} \in \partial\Omega = \{\mathfrak{u} \in \mathscr{X}; \|\mathfrak{u}\|_{\mathscr{X}} = \vartheta\}$ and $\zeta \in (0,1]$. $\square$

To prove the main result in this subsection, we need the following Lemma

LEMMA 10. *Assume that* $0 < \delta < 1$ *and* $0 < \mu \leqslant 1$. *Then, the following inequality holds*

$$
\frac{1}{\Gamma(\delta+1)} \leqslant \frac{\Gamma(\mu)}{\Gamma(\delta+\mu)}.
$$

Proof. By using Lemma 2 we have, for $\tau \in (\mathfrak{a}, \mathfrak{b}]$

$$\begin{aligned}
&\frac{1}{\Gamma(\delta+1)}(\psi(\tau)-\psi(\mathfrak{a}))^{\delta} \\
&= \frac{1}{\Gamma(\delta)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\delta-1}ds \\
&= \frac{1}{\Gamma(\delta)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\delta-1}(\psi(s)-\psi(\mathfrak{a}))^{\mu-1}(\psi(s)-\psi(\mathfrak{a}))^{1-\mu}ds \\
&\leqslant (\psi(\tau)-\psi(\mathfrak{a}))^{1-\mu}\mathfrak{J}_{\mathfrak{a}^+}^{\delta;\psi}(\psi(s)-\psi(\mathfrak{a}))^{\mu-1}(\tau) \\
&\leqslant \frac{\Gamma(\mu)}{\Gamma(\delta+\mu)}(\psi(\tau)-\psi(\mathfrak{a}))^{\delta+\mu-1}(\psi(\tau)-\psi(\mathfrak{a}))^{1-\mu} \\
&\leqslant \frac{\Gamma(\mu)}{\Gamma(\delta+\mu)}(\psi(\tau)-\psi(\mathfrak{a}))^{\delta},
\end{aligned}$$

which is the desired result. □

THEOREM 4. *Assume* $(A1)$–$(A4)$ *and* (3.4)*, then there exist at least one solution for the problem* (1.1)–(1.2).

Proof. It is clear that the set Ω defined in (3.6) is symmetric, $0 \in \Omega$ and $\mathscr{X} \cap \overline{\Omega} = \overline{\Omega} \neq \emptyset$. In addition, By Lemma 9, assume $(A1)$, $(A2)$, $(A3)$, $(A4)$ and (3.4), then

$$\mathfrak{L}(\mathfrak{u}) - \mathscr{N}(\mathfrak{u}) \neq -\zeta[\mathfrak{L}(\mathfrak{u}) - \mathscr{N}(-\mathfrak{u})],$$

for each $\mathfrak{u} \in \mathscr{X} \cap \partial\Omega = \partial\Omega$ and each $\zeta \in (0,1]$. By Lemma 3, problem (1.1)–(1.2) has at least one solution on $Dom\mathfrak{L} \cap \overline{\Omega}$. Which completes the demonstration. □

Now, we investigate the existence and uniqueness of periodic solutions for our problem (1.1)–(1.2).

THEOREM 5. *Let* $(A1)$, $(A2)$, $(A3)$ *and* $(A4)$ *satisfied. Moreover we assume that*

$(A5)$ *There exist constants* $\overline{\gamma} > 0$ *and* $\overline{\eta_1}, \overline{\eta_2} \geqslant 0$ *such that*

$$\begin{aligned}
&|\mathscr{F}(\tau,\mathfrak{u},\mathscr{G}(\mathfrak{u}),\mathscr{H}(\mathfrak{u})) - \mathscr{F}(\tau,\bar{\mathfrak{u}},\mathscr{G}(\bar{\mathfrak{u}}),\mathscr{H}(\bar{\mathfrak{u}}))| \\
&\geqslant \overline{\gamma}|\mathfrak{u}-\bar{\mathfrak{u}}| - \overline{\eta_1}|\mathscr{G}\mathfrak{u}-\mathscr{G}\bar{\mathfrak{u}}| - \overline{\eta_2}|\mathscr{H}\mathfrak{u}-\mathscr{H}\bar{\mathfrak{u}}|,
\end{aligned}$$

for every $\tau \in (\mathfrak{a}, \mathfrak{b}]$ *and* $\mathfrak{u}, \bar{\mathfrak{u}} \in C_{1-\nu;\psi}(\mathfrak{J}, \mathfrak{R})$.

If one has

$$\begin{aligned}
&\left[\frac{2\gamma\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha} + \frac{2\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\eta_1\lambda_1+\eta_2\lambda_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\alpha-\nu}\right. \\
&\qquad\left. + \frac{(\overline{\eta}_1\lambda_1+\overline{\eta}_2\lambda_2)}{\overline{\gamma}}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}\right] < 1, \quad (3.7)
\end{aligned}$$

then the problem (1.1)–(1.2) *has a unique solution in* $Dom\mathfrak{L} \cap \overline{\Omega}$.

Proof. By Lemma 10 we can see that the condition (3.7) is strong than condition (3.4). Then, by Theorem 4 we obtain that the problem (1.1)–(1.2) has at least one solution in $Dom\mathfrak{L}\cap\overline{\Omega}$.

Now, we prove the uniqueness result. Suppose that the problem (1.1)–(1.2) has two different solutions $\mathfrak{u}_1,\mathfrak{u}_2\in Dom\mathfrak{L}\cap\overline{\Omega}$. Then, we have for each $\tau\in(\mathfrak{a},\mathfrak{b}]$

$$\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}_1(\tau)=\mathscr{F}(\tau,\mathfrak{u}_1(\tau),\mathscr{G}(\mathfrak{u}_1)(\tau),\mathscr{H}(\mathfrak{u}_1)(\tau)),$$

$$\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}_2(\tau)=\mathscr{F}(\tau,\mathfrak{u}_2(\tau),\mathscr{G}(\mathfrak{u}_2)(\tau),\mathscr{H}(\mathfrak{u}_2)(\tau)),$$

where $\mathscr{G}$, $\mathscr{H}$ are defined as in (1.3) and

$$\mathfrak{u}_1(\mathfrak{a})=\mathfrak{u}_1(\mathfrak{b}),\ \mathfrak{u}_2(\mathfrak{a})=\mathfrak{u}_2(\mathfrak{b}).$$

Let $\mathfrak{U}(\tau)=\mathfrak{u}_1(\tau)-\mathfrak{u}_2(\tau),$ for all $\tau\in(\mathfrak{a},\mathfrak{b}]$.

Then

$$\begin{aligned}\mathfrak{L}\mathfrak{U}(\tau)&=\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}(\tau)\\&=\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}_1(\tau)-\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}_2(\tau)\\&=\mathscr{F}(\tau,\mathfrak{u}_1(\tau),\mathscr{G}(\mathfrak{u}_1)(\tau),\mathscr{H}(\mathfrak{u}_1)(\tau))-\mathscr{F}(\tau,\mathfrak{u}_2(\tau),\mathscr{G}(\mathfrak{u}_2)(\tau),\mathscr{H}(\mathfrak{u}_2)(\tau)).\end{aligned}\tag{3.8}$$

Using the fact that $\mathfrak{Img}\mathfrak{L}=\ker\mathscr{Q},$ we have

$$\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\beta(\alpha-1)}\\ [\mathscr{F}(s,\mathfrak{u}_1(s),\mathscr{G}(\mathfrak{u}_1)(s),\mathscr{H}(\mathfrak{u}_1)(s))-\mathscr{F}(s,\mathfrak{u}_2(s),\mathscr{G}(\mathfrak{u}_2)(s),\mathscr{H}(\mathfrak{u}_2)(s))]\,ds=0.$$

Since $\mathscr{F}\in C_{1-\nu;\psi}(\mathfrak{J},\mathfrak{R}),$ then there exist $\tau_0\in(\mathfrak{a},\mathfrak{b}]$ such that

$$\mathscr{F}(\tau_0,\mathfrak{u}_1(\tau_0),\mathscr{G}(\mathfrak{u}_1)(\tau_0),\mathscr{H}(\mathfrak{u}_1)(\tau_0))-\mathscr{F}(\tau,\mathfrak{u}_2(\tau_0),\mathscr{G}(\mathfrak{u}_2)(\tau_0),\mathscr{H}(\mathfrak{u}_2)(\tau_0))=0.$$

In view of $(A5)$ we have

$$\left|\mathfrak{u}_1(\tau_0)-\mathfrak{u}_2(\tau_0)\right|\leqslant\frac{(\overline{\eta}_1\lambda_1+\overline{\eta}_2\lambda_2)}{\overline{\gamma}}\left\|\mathfrak{u}_1-\mathfrak{u}_2\right\|_{\mathscr{X}},$$

then

$$|\mathfrak{U}(\tau_0)|\leqslant\frac{(\overline{\eta}_1\lambda_1+\overline{\eta}_2\lambda_2)}{\overline{\gamma}}\|\mathfrak{U}\|_{\mathscr{X}}.\tag{3.9}$$

On the other hand, by Theorem 1, we have

$$\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)=\mathfrak{u}(\tau)-\frac{c_1\,(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1}}{\Gamma(\nu)},$$

which implies that

$$c_1=\left[\mathfrak{u}(\tau_0)-\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau_0)\right]\Gamma(\nu)\,(\psi(\tau_0)-\psi(\mathfrak{a}))^{1-\nu},$$

and therefore

$$\begin{aligned}\mathfrak{U}(\tau) &= \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}(\tau) \\ &\quad + \left[\mathfrak{U}(\tau_0) - \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}(\tau_0)\right](\psi(\tau_0)-\psi(\mathfrak{a}))^{1-\nu}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1}.\end{aligned}$$

Using (3.9) we obtain, for every $\tau \in (\mathfrak{a}, \mathfrak{b}]$

$$\begin{aligned}|\mathfrak{U}(\tau)| &\leqslant \left[|\mathfrak{U}(\tau_0)| + \left|\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}(\tau_0)\right|\right](\psi(\tau_0)-\psi(\mathfrak{a}))^{1-\nu}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1} \\ &\quad + \left|\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}(\tau)\right| \\ &\leqslant \frac{\|\mathfrak{U}\|_{\mathscr{X}}(\overline{\eta}_1\lambda_1+\overline{\eta}_2\lambda_2)}{\overline{\gamma}}(\psi(\tau_0)-\psi(\mathfrak{a}))^{1-\nu}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1} \\ &\quad + \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}\left\|\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}\right\|_{\mathscr{X}}(\psi(\tau_0)-\psi(\mathfrak{a}))^{\alpha}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1} \\ &\quad + \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}\left\|\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}\right\|_{\mathscr{X}}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu+\alpha-1}. \qquad (3.10)\end{aligned}$$

By $(A2)$, $(A3)$, $(A4)$ and (3.8) we find that

$$\begin{aligned}&\left|\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}(\tau)\right| \\ &= |\mathscr{F}(\tau,\mathfrak{u}_1(\tau),\mathscr{G}(\mathfrak{u}_1)(\tau),\mathscr{H}(\mathfrak{u}_1)(\tau)) - \mathscr{F}(\tau,\mathfrak{u}_2(\tau),\mathscr{G}(\mathfrak{u}_2)(\tau),\mathscr{H}(\mathfrak{u}_2)(\tau))| \\ &\leqslant \left[\gamma(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1} + \eta_1\lambda_1 + \eta_2\lambda_2\right]\|\mathfrak{U}\|_{\mathscr{X}}.\end{aligned}$$

Then

$$\left\|\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{U}\right\|_{\mathscr{X}} \leqslant \left[\gamma + (\eta_1\lambda_1+\eta_2\lambda_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}\right]\|\mathfrak{U}\|_{\mathscr{X}}. \qquad (3.11)$$

Substituting (3.11) in the right side of (3.10) we get, for every $\tau \in (\mathfrak{a}, \mathfrak{b}]$

$$\begin{aligned}|\mathfrak{U}(\tau)| \leqslant \Bigg[&\frac{(\overline{\eta}_1\lambda_1+\overline{\eta}_2\lambda_2)}{\overline{\gamma}}(\psi(\tau_0)-\psi(\mathfrak{a}))^{1-\nu}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1} \\ &+ \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}\left(\gamma + (\eta_1\lambda_1+\eta_2\lambda_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}\right) \\ &\times (\psi(\tau_0)-\psi(\mathfrak{a}))^{\alpha}(\psi(\tau)-\psi(\mathfrak{a}))^{\nu-1} + \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)} \\ &\times \left(\gamma + (\eta_1\lambda_1+\eta_2\lambda_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}\right)(\psi(\tau)-\psi(\mathfrak{a}))^{\nu+\alpha-1}\Bigg]\|\mathfrak{U}\|_{\mathscr{X}}.\end{aligned}$$

Therefore

$$\begin{aligned}\|\mathfrak{U}\|_{\mathscr{X}} \leqslant \Bigg[&\frac{2\gamma\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha} + \frac{2\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\eta_1\lambda_1+\eta_2\lambda_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\alpha-\nu} \\ &+ \frac{(\overline{\eta}_1\lambda_1+\overline{\eta}_2\lambda_2)}{\overline{\gamma}}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}\Bigg]\|\mathfrak{U}\|_{\mathscr{X}}.\end{aligned}$$

Hence, by (3.7), we conclude that

$$\|\mathfrak{U}\|_{\mathscr{X}} = 0.$$

As a result, for any $\tau \in (\mathfrak{a}, \mathfrak{b}]$ we get

$$\mathfrak{U}(\tau) = 0 \Longrightarrow \mathfrak{u}_1(\tau) = \mathfrak{u}_2(\tau).$$

This completes the proof. $\square$

4. An example

We present an example of Volterra-Fredholm integro-differential equations to test our main results.

$$\mathfrak{D}_{1^+}^{\frac{1}{2},\frac{1}{3};\ln\tau}\mathfrak{u}(\tau) = \mathscr{F}\left(\tau, \mathfrak{u}(\tau), \mathscr{G}\mathfrak{u}(\tau), \mathscr{H}\mathfrak{u}(\tau)\right),\ \tau \in (1, e],$$

$$\mathfrak{u}(1) = \mathfrak{u}(e),$$

where for any $\tau \in (1, e]$, we have

$$\begin{aligned} \mathscr{F}\left(\tau, \mathfrak{u}(\tau), \mathscr{G}\mathfrak{u}(\tau), \mathscr{H}\mathfrak{u}(\tau)\right) = \frac{\ln^{\frac{-1}{3}}(\tau)}{(e^{2\tau}+3)} + \frac{1}{17\sqrt{\pi}}\left(\sin\mathfrak{u}(\tau) + \frac{3}{2}\mathfrak{u}(\tau)\right) \\ + \frac{1}{13e^3}\mathscr{G}\mathfrak{u}(\tau) + \frac{1}{19}\mathscr{H}\mathfrak{u}(\tau), \end{aligned}$$

with

$$\mathscr{G}\mathfrak{u}(\tau) = \int_1^\tau \mathfrak{g}(\tau, s, \mathfrak{u}(s))ds = \int_1^\tau \tau^5 e^{-7-\tau^2}\cos(\mathfrak{u}(s))\,ds,\ \tau \in \mathfrak{J}.$$

and

$$\mathscr{H}\mathfrak{u}(\tau) = \int_1^e \mathfrak{h}(\tau, s, \mathfrak{u}(s))ds = \int_1^e \frac{e^{-9-\tau^3}}{19(1+\mathfrak{u}(s))}ds,\ \tau \in \mathfrak{J}.$$

Here $\mathfrak{J} := [1, e]$, $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$ and $\psi(\tau) = \ln\tau$.

It is easy to see that $\mathscr{F} \in C_{\frac{1}{3};\psi}(\mathfrak{J}, \mathfrak{R})$. Hence condition $(A1)$ is verified.

Furthermore, for all $\tau \in (1, e]$ and $\mathfrak{u}, \bar{\mathfrak{u}} \in C_{\frac{1}{3};\psi}(\mathfrak{J}, \mathfrak{R})$, we obtain

$$\begin{aligned} &|\mathscr{F}(\tau, \mathfrak{u}, \mathscr{G}(\mathfrak{u}), \mathscr{H}(\mathfrak{u})) - \mathscr{F}(\tau, \bar{\mathfrak{u}}, \mathscr{G}(\bar{\mathfrak{u}}), \mathscr{H}(\bar{\mathfrak{u}}))| \\ &\leqslant \gamma|\mathfrak{u} - \bar{\mathfrak{u}}| + \eta_1|\mathscr{G}\mathfrak{u} - \mathscr{G}\bar{\mathfrak{u}}| + \eta_2|\mathscr{H}\mathfrak{u} - \mathscr{H}\bar{\mathfrak{u}}|, \end{aligned}$$

$$\begin{aligned} &|\mathfrak{g}(\tau, s, \mathfrak{u}) - \mathfrak{g}(\tau, s, \bar{\mathfrak{u}})| \leqslant \rho_1|\mathfrak{u} - \bar{\mathfrak{u}}|,\ (\tau, s) \in \Delta, \\ &|\mathfrak{h}(\tau, s, \mathfrak{u}) - \mathfrak{h}(\tau, s, \bar{\mathfrak{u}})| \leqslant \rho_2|\mathfrak{u} - \bar{\mathfrak{u}}|,\ (\tau, s) \in \Delta_0, \end{aligned}$$

and

$$\begin{aligned} &|\mathscr{F}(\tau, \mathfrak{u}, \mathscr{G}(\mathfrak{u}), \mathscr{H}(\mathfrak{u})) - \mathscr{F}(\tau, \bar{\mathfrak{u}}, \mathscr{G}(\bar{\mathfrak{u}}), \mathscr{H}(\bar{\mathfrak{u}}))| \\ &\geqslant \overline{\gamma}|\mathfrak{u} - \bar{\mathfrak{u}}| - \overline{\eta_1}|\mathscr{G}\mathfrak{u} - \mathscr{G}\bar{\mathfrak{u}}| - \overline{\eta_2}|\mathscr{H}\mathfrak{u} - \mathscr{H}\bar{\mathfrak{u}}|, \end{aligned}$$

with $\Delta = \{(\tau,s) : 1 \leqslant s \leqslant \tau \leqslant e\}$ and $\Delta_0 = \mathfrak{J}\times\mathfrak{J}$, which implies that $(A2)$, $(A3)$, $(A4)$ and $(A5)$ are satisfied with

$$\gamma = \frac{5}{34\sqrt{\pi}},\ \overline{\gamma} = \frac{1}{34\sqrt{\pi}},\ \eta_1 = \overline{\eta_1} = \frac{1}{13e^3},\ \eta_2 = \overline{\eta_2} = \frac{1}{19},\ \rho_1 = \frac{1}{e^6},\ \text{and}\ \ \rho_2 = \frac{1}{19e^8}$$

By simple calculations, we get $\lambda_1 = \frac{3}{e^5}$ and $\lambda_2 = \frac{3}{19e^7}$ and

$$\left[\frac{2\gamma\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha} + \frac{2\Gamma(\nu)}{\Gamma(\alpha+\nu)}(\eta_1\lambda_1+\eta_2\lambda_2)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1+\alpha-\nu}\right.$$
$$\left. + \frac{(\overline{\eta}_1\lambda_1+\overline{\eta}_2\lambda_2)}{\overline{\gamma}}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\nu}\right] \approx 0.39934 < 1.$$

So, by Theorem 5, our problem has a unique solution.

5. Conclusions

The main contribution of this research was to investigate some sufficient conditions ensuring the existence and uniqueness of periodic solutions to a great nonlinear class of Volterra-Fredholm integro-differential equations with fractional integral conditions, involving ψ-Hilfer fractional derivative, by using the coincidence degree theory of Mawhin [12]. To illustrate the efficiency of our findings, we have presented an important example.

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(Received July 24, 2021)

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