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BIFURCATIONS OF LIMIT CYCLES IN PIECEWISE SMOOTH HAMILTONIAN SYSTEM WITH BOUNDARY PERTURBATION

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Abstract. In this paper, the general planar piecewise smooth Hamiltonian system with period annulus around the center at the origin is considered. We obtain the expressions for the first order and the second-order Melnikov functions of its general second-order perturbation, which can be used to find the number of limit cycles bifurcated from periodic orbits. Further, we have shown that the number of limit cycles of the system $\dot{X} = \begin{cases} (H_y^+, -H_x^+) & \text{if } y > \varepsilon f(x) \\ (H_y^-, -H_x^-) & \text{if } y < \varepsilon f(x) \end{cases}$ equal to the number of positive zeros of f when at $\varepsilon = 0$, the system has a period annulus around the origin.

1. Introduction

Motions of many non-smooth processes such as impact switching, sliding, and other discrete state transitions are modeled into piecewise smooth dynamical systems rather than the smooth dynamical systems. Recently piecewise smooth dynamical systems are of great interest. It has many applications in physical processes such as electrical circuits, impact oscillators, dry friction oscillators, relay control systems, modeling of irregular heartbeats etc.[2]. In many scientific applications, systems with self-sustained oscillations modeled where limit cycles play an important role. Limit cycle bifurcations in the case of smooth dynamical systems are very well studied [11], whereas the non-smooth systems have been studied recently.

Averaging theory, Melnikov theory, and normal form theory are well-known techniques used to study the limit cycle bifurcation of planar smooth differential systems [6, 12], whereas the techniques for piecewise smooth systems are in the process of development [5, 10].

In [4] authors considered a piecewise linear differential system (PLDS) having center-focus type singularity with switching manifold $y = 0$ in which limit cycle bifurcation of the system is studied when the switching manifold is $y = \varepsilon$. Also, in [19] authors studied PLDS with saddle-center type singularity at the origin and switching curve $y = b\sin x$ and shown that the number of limit cycles bifurcated from the period

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annulus of the system with $b=0$ is equal to the number of positive zeros of $\sin x$. Note that the system considered in [19] is symmetric about the y-axis and zeros of switching curve $y=b\sin x$ are also symmetric about the y-axis. In [14] first order Melnikov function is obtained for piecewise smooth near-Hamiltonian system and studied the Hopf bifurcation. In [18] authors studied the same system as in [19] by considering the switching curve $y=bx(x^2-x_1^2)(x^2-x_2^2)\cdots(x^2-x_m^2)$, wherein it is proved that the number of limit cycles bifurcated from the period annulus of the system at $b=0$ is equal to m, where m is a positive integer. In [3], authors studied the number of limit cycles bifurcated from the origin of the perturbation of a planar piecewise smooth system with center-center type singularity at the origin. Further, in [15], normal forms of some planar piecewise smooth systems with center-center type singularity of order (k,l) at the origin are considered and their limit cycles bifurcation from the origin under higher-order perturbations have been studied. It is natural to think about the limit cycles bifurcation of these normal forms when the separation boundary is an analytic function. In [9] authors considered the first-order perturbation of a planar piecewise smooth Hamiltonian system. If the unperturbed system has a period annulus centered at the origin, then using the first order Melnikov function, the number of limit cycles bifurcated from the periodic annulus is studied.

In [16] and [17], authors obtained a similar expression for the second order Melnikov function for planar Piecewise smooth differential systems with a straight line as a switching manifold. This expression of the second order Melnikov function contains the time of flight of the trajectories. In [16], authors used the second order Melnikov function to find the number of limit cycles of piecewise perturbation linear center in the class of second-degree polynomial functions. In [17], authors used the second order Melnikov function to find the number of limit cycles of piecewise linear differential systems with a nonregular line of separation. In paper [1], authors found first and second order averaging functions for the piecewise smooth differential systems system of the form

$$\dot{x}=\sum_{i=1}^{N}\varepsilon F^i(t,x)+\varepsilon^{N+1}R(t,x,\varepsilon), \tag{1.1}$$

where $F^i(t,x)$ are smooth periodic functions on open region defined by $\theta_{i-1}<t<\theta_i$. Note that the system (1.1) is called the standard form of the piecewise smooth system, where $\dot{x}$, i.e. oscillation, is of order ε. In [8] authors studied piecewise smooth near-integrable systems and developed the Melnikov function method and the averaging method for finding limit cycles and equivalence between them. In [13], the number of limit cycles is studied for a class of piecewise smooth near-Hamiltonian systems. Using the expression of the first order Melnikov function and results about Chebyshev systems, upper bounds are obtained for the number of limit cycles in Hopf bifurcation and Poincare bifurcation.

Averaging methods are used to study limit cycle bifurcation from the period annulus of such a system in standard form only. Note that the piecewise-smooth systems with a center at the origin and of the types saddle-saddle, saddle-center cannot be converted into standard form. Hence, we can not use averaging functions to study the limit cycle bifurcation of piecewise smooth differential systems with center and types saddle-saddle, saddle-center.

In this paper we have obtained the second-order Melnikov function for the piecewise Hamiltonian system with second-order perturbation and the separation boundary $y=0$. We also considered a general piecewise smooth Hamiltonian system with perturbed separation boundary $y=\varepsilon f(x)$ when f is a C^2 function.

We have obtained an expression of the second-order Melnikov function of a piecewise smooth planar differential system with a line of separation which does not contain the time of flights of the trajectories. Note that the expression of the second-order Melnikov function obtained only depends on integrals of 1-form over periodic orbits of the unperturbed system. Further, we have used this second-order Melnikov function to find an upper bound for the number of limit cycles of the piecewise smooth differential systems of types (i) k-l center, (ii) saddle-center, and (iii) saddle-saddle with any nonlinear smooth separation curve.

The paper is organized as follows: In *Section 2* we give some preliminary concepts about Melnikov theory, limit cycles, and stability of limit cycles. *Section 3* is devoted to investigating the first order and second order Melnikov functions for piecewise smooth Hamiltonian systems with second-order perturbation. *Section 4* deals with the general piecewise smooth Hamiltonian system with boundary perturbation. Finally, in *Section 5*, we give some application of piecewise smooth Hamiltonian systems with boundary perturbation.

2. Preliminaries

Consider a C^∞ smooth system of the form

$$\dot{X} = \begin{cases} H_y + \varepsilon f(x,y,\varepsilon,\delta) \\ -H_x + \varepsilon g(x,y,\varepsilon,\delta) \end{cases}, \tag{2.1}$$

where $X = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\dot{X} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$, H, f, g are C^∞ smooth functions for $\varepsilon \in \mathbb{R}$, $\delta \in K \subset \mathbb{R}^m$ with K compact and $H_x(x,y) = \dfrac{\partial H}{\partial x}(x,y)$, $H_y(x,y) = \dfrac{\partial H}{\partial y}(x,y)$. For $\varepsilon = 0$, the system (2.1) becomes Hamiltonian system

$$\dot{X} = \begin{cases} H_y \\ -H_x \end{cases}. \tag{2.2}$$

Suppose that the system (2.2) has a period annulus $\mathscr{A} = \{\Gamma_h : H(x,y) = h, h \in (\alpha,\beta) \subset \mathbb{R}\}$ such that $\Gamma_h \to \Gamma_\alpha$ as $h \to \alpha$, which is an elementary center point for the system and $\Gamma_h \to \Gamma_\beta$ as $h \to \beta$, which is usually a homoclinic loop consisting of a saddle point or heteroclinic loop containing two saddle points. For some $h_0 \in (\alpha,\beta)$, consider a periodic orbit Γ_{h_0} from the period annulus and a Poincare section $S = \{(a(h),0) : h \in (h_0-\varepsilon_0, h_0+\varepsilon_0)\}$, for some $\varepsilon_0 > 0$, at the point $A(a(h_0),0)$ to Γ_{h_0}. Let $\Gamma_{h_0\varepsilon}$ be the solution of (2.1) starting at $A(a(h_0),0)$ and let $B(b(h_0,\varepsilon,\delta),0)$ be its first point of intersection with the Poincare section. Then the Poincare map $\mathscr{P}$ maps $A(a(h_0),0)$ to $B(b(h_0,\varepsilon,\delta),0)$. Note that $H(A(a(h_0),0)) = H(B(b(h_0,\varepsilon,\delta),0))$ if and only if $A(a(h_0),0) = B(b(h_0,\varepsilon,\delta),0)$. Therefore we can use the difference map $H(B)-H(A)$ to investigate the number of limit cycles of (2.1) bifurcated from Γ_{h_0}. Thus,

$$\begin{aligned} H(B)-H(A) &= \int_{\widehat{AB}} dH = \int_{\widehat{AB}} (H_x dx + H_y dy) \\ &= \int_0^{\tau_{h_0}} [H_x(H_y+\varepsilon f) + H_y(-H_x+\varepsilon g)]dt = \varepsilon \int_0^{\tau_{h_0}} (H_x f + H_y g)dt \\ &= \varepsilon F(h_0,\varepsilon,\delta) = \sum_{k=1}^{\infty} M_k(h_0,\delta)\varepsilon^k, \end{aligned} \tag{2.3}$$

where τ_{h_0} is the time of flight along the trajectory $\widehat{AB}$ of (2.1) from A to B and

$$M_k(h_0,\delta) = \frac{1}{(k-1)!}\frac{\partial^{(k-1)}F}{\partial \varepsilon^{k-1}}(h_0,0,\delta).$$

Here, $M_k(h_0,\delta)$ is called as the kth order Melnikov function and F is called as a bifurcation function for the system (2.1).

Clearly, from equation (2.3) we have

$$\begin{aligned} F(h_0,0,\delta) = M_1(h_0,\delta) &= \int_{\Gamma_{h_0}} (H_x f + H_y g)dt = \int_{\Gamma_{h_0}} (g dx - f dy) \\ &= -\int\int_{\mathrm{Int}(\Gamma_{h_0})} (f_x + g_y) dx dy, \end{aligned}$$

where $\mathrm{Int}(\Gamma_{h_0})$ is the region bounded by Γ_{h_0}. Here, we say that the cyclicity of Γ_{h_0} is k if there exist ε_0 such that (2.1) has at most k limit cycles in some neighborhood of Γ_{h_0} for any $\delta \in K$ and for any $0 < \varepsilon < \varepsilon_0$ and that (2.1) has exactly k limit cycles in every neighbourhood of Γ_{h_0} for some (ε,δ).

The following proposition states that the number of periodic solutions of (2.1), called limit cycles, in the small neighborhood for Γ_{h_0} is less than or equal to the number of isolated zeros of the first order Melnikov function $M_1(h_0,\delta)$.

PROPOSITION 1. [6] *Let $\delta_0 \in K$ and let $h_0 \in (\alpha,\beta)$. Then we have the following:*

1. *There is no limit cycle near Γ_{h_0} for $|\varepsilon| + |\delta-\delta_0| \ll 1$, if $M_1(h_0,\delta_0) \neq 0$.*
2. *There is exactly one (at least one) limit cycle $\Gamma(\varepsilon,\delta)$ for $|\varepsilon| + |\delta-\delta_0| \ll 1$, which approaches Γ_{h_0} as $(\varepsilon,\delta) \to (0,\delta_0)$ if $M_1(h_0,\delta_0) = 0$, $\frac{\partial M_1}{\partial h}(h_0,\delta_0) \neq 0$ (if h_0 is a zero of $M(h,\delta_0)$ with odd multiplicity, respectively).*
3. *If there exist $0 \leqslant j \leqslant k$ such that $M_1(h_0,\delta_0) = 0$ and $\frac{\partial^j M_1}{\partial h^j}(h_0,\delta) \neq 0$ for every $\delta \in K$ then at most k limit cycles of (2.1) are bifurcated form Γ_{h_0}.*

Now we have the following result about the stability of limit cycles using the first order Melnikov function.

PROPOSITION 2. *The limit cycle of* (2.1) *bifurcated from the periodic orbit of* (2.2) *passing through* $(a(h_0),0)$ *of the Poincare section is stable if*

$$\frac{dM_1}{dh}(h_0,\delta_0) < 0,$$

where M_1 *is the first order Melnikov function for* (2.1) *and* $|\varepsilon|+|\delta-\delta_0| \ll 1$.

Proof. From equation (2.3), we have

$$H(b(h,\varepsilon,\delta),0) - H(a(h),0) = \varepsilon M_1(h,\delta) + o(\varepsilon^2).$$

By Taylor's expansion in powers of ε, we have

$$H(a(h),0) + \varepsilon H_x(a(h),0)\left(\frac{\partial b}{\partial \varepsilon}\right)_{\varepsilon=0} + o(\varepsilon^2) - H(a(h),0) = \varepsilon M_1(h,\delta) + o(\varepsilon^2). \tag{2.4}$$

Equating ε order terms in equation (2.4) on both sides we get

$$H_x(a(h),0)\left(\frac{\partial b}{\partial \varepsilon}\right)_{\varepsilon=0} = M_1(h,\delta).$$

Now if $\mathscr{P}_\varepsilon$ is the Poincare map of system (2.1) then we have $\mathscr{P}_\varepsilon(a(h)) = b(h,\varepsilon,\delta)$. Hence

$$\left(\frac{\partial \mathscr{P}_\varepsilon}{\partial \varepsilon}\right)_{\varepsilon=0} = \frac{M_1(h,\delta)}{H_x(a(h),0)}. \tag{2.5}$$

Now differentiating (2.5) with respect to h, we get

$$\frac{d}{dx}\left[\left(\frac{\partial \mathscr{P}_\varepsilon}{\partial \varepsilon}\right)_{\varepsilon=0}\right] a'(h) = \frac{1}{H_x(a(h),0)}\frac{dM_1}{dh} - M_1 \frac{H_{xx}(a(h),0)}{(H_x(a(h),0))^2} a'(h). \tag{2.6}$$

Since $H(a(h),0) = h$, we have $H_x(a(h),0)a'(h) = 1$. Therefore from (2.6) we get,

$$\frac{\partial}{\partial \varepsilon}\left[\left(\frac{d\mathscr{P}_\varepsilon}{dx}\right)\right]_{\varepsilon=0} = \frac{dM_1}{dh} - M_1 \frac{H_{xx}(a(h),0)}{H_x(a(h),0)^2}.$$

Now if $0 < \varepsilon \ll 1$, then we have

$$\frac{d\mathscr{P}_\varepsilon}{dx} - \frac{d\mathscr{P}_0}{dx} \approx \varepsilon\left(\frac{dM_1}{dh} - M_1 \frac{H_{xx}(a(h),0)}{H_x(a(h),0)^2}\right).$$

But $\mathscr{P}_0$ is Poincare return map for (2.2). Hence, $\mathscr{P}_0(a(h)) = a(h)$, which imply that

$$\frac{d\mathscr{P}_0}{dx}(a(h))a'(h) = a'(h).$$

Hence,

$$\frac{d\mathscr{P}_\varepsilon}{dx} - 1 \approx \varepsilon \left(\frac{dM_1}{dh} - M_1 \frac{H_{xx}(a(h),0)}{H_x(a(h),0)^2} \right).$$

Thus, the limit cycle passing through $A(a(h),0)$ is stable if

$$\frac{dM_1}{dh} - M_1 \frac{H_{xx}(a(h),0)}{H_x(a(h),0)^2} < 0.$$

At $h = h_0$, $\delta = \delta_0$, $M_1(h_0,\delta_0) = 0$, we have $\frac{dM_1}{dh}(h_0,\delta_0) < 0$ for $|\varepsilon| + |\delta - \delta_0| \ll 1$. $\square$

General planar piecewise smooth differential system with two zones and switching manifold $\Sigma = \varphi^{-1}(0)$ is given by

$$\dot{X} = \begin{cases} (X_1^+(x,y),\ X_2^+(x,y)), & (x,y) \in \Sigma^+ \\ (X_1^-(x,y),\ X_2^-(x,y)), & (x,y) \in \Sigma^- \end{cases}, \tag{2.7}$$

where $X_i^\pm$, $f^\pm$, $g^\pm$, $i = 1,2$ and φ are sufficiently smooth functions on some open region containing the origin with 0 as a regular value of φ and $\Sigma^+ = \varphi^{-1}(0,\infty)$, $\Sigma^- = \varphi^{-1}(-\infty,0)$ are two zones of (2.7). Now we denote $X^\pm := (X_1^\pm, X_2^\pm)$, $X^\pm\varphi := \langle X^\pm, \nabla\varphi\rangle$ and $(X^\pm)^k\varphi := \langle X^\pm, \nabla(X^\pm)^{k-1}\varphi\rangle$, where $\langle , \rangle$ is an Euclidean dot product.

We say that a point $p \in \Sigma$ is a kth contact point for the vector field X if $(X^k\varphi)(p) \neq 0$ and $(X^l\varphi)(p) = 0$ for $l = 1,\cdots,k-1$. A point $p \in \Sigma$ is a (k,l)-contact singularity of $X^\pm$ if X_0 is a kth contact point for X^+ and is a lth contact point for X^-.

3. Perturbation of piecewise smooth Hamiltonian system

Recently in [9], authors studied the number of limit cycles of the piecewise smooth Hamiltonian system

$$\dot{X} = (H_y^+(x,y), -H_x^+(x,y)) + \varepsilon(f^+(x,y,\varepsilon,\delta), g^+(x,y,\varepsilon,\delta)), \qquad (x,y) \in \Sigma^+ \tag{3.1}$$

$$\dot{X} = (H_y^-(x,y), -H_x^-(x,y)) + \varepsilon(f^-(x,y,\varepsilon,\delta), g^-(x,y,\varepsilon,\delta)), \qquad (x,y) \in \Sigma^- \tag{3.2}$$

i.e.,

$$\dot{X} = \begin{cases} (H_y^+(x,y), -H_x^+(x,y)) + \varepsilon(f^+(x,y,\varepsilon,\delta), g^+(x,y,\varepsilon,\delta)), & (x,y) \in \Sigma^+ \\ (H_y^-(x,y), -H_x^-(x,y)) + \varepsilon(f^-(x,y,\varepsilon,\delta), g^-(x,y,\varepsilon,\delta)), & (x,y) \in \Sigma^- \end{cases}. \tag{3.3}$$

System (3.3) is a perturbation of the Hamiltonian system

$$\dot{X} = (H_y^+(x,y), -H_x^+(x,y)), (x,y) \in \Sigma^+ \tag{3.4}$$

$$\dot{X} = (H_y^-(x,y), -H_x^-(x,y)), (x,y) \in \Sigma^- \tag{3.5}$$

or

$$\dot{X} = \begin{cases} (H_y^+(x,y), -H_x^+(x,y)), & (x,y) \in \Sigma^+ \\ (H_y^-(x,y), -H_x^-(x,y)), & (x,y) \in \Sigma^- \end{cases}. \tag{3.6}$$

Suppose that the switching manifold for (3.3) and (3.6) is $\Sigma = \varphi^{-1}(0)$, where $\varphi(x,y) = y$. Assume that the system (3.6) has a period annulus around the origin in some open region V. Let $L_+ = V \cap \{(x,0) : x > 0\}$ and $L_- = V \cap \{(x,0) : x < 0\}$. Let $\Gamma_r^+ : H^+(x,y) = r,\ y \geqslant 0$ be a trajectory of (3.4) which starts at $P(r) = (p(r),0)$ on L_+, ends at the point $P_1(r) = (p_1(r),0)$ on L_- with the time of flight $t^+(r)$. Then the Poincare half return map $\mathscr{P}^+ : L^+ \to L^-$ is given by

$$\mathscr{P}^+(p(r)) = p_1(r).$$

Let $\Gamma_r^- : H^-(x,y) = s,\ y \leqslant 0$ be the trajectory of (3.5) starting at $P_1(r)$ on L_- and ending at the point $P(r)$ with time of flight $t^-(r)$. Therefore the next half return map $\mathscr{P}^- : L^- \to L^+$ is given by

$$\mathscr{P}^-(p_1(r)) = p(r).$$

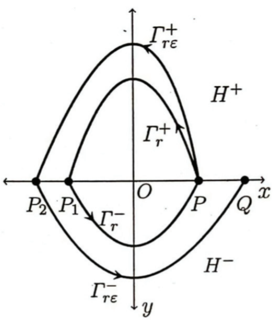


Figure 1: *Trajectories showing the difference map*

Let $\Gamma_{r\varepsilon}^+$ be a trajectory of (3.1) starting at $P(r)$ and meeting the first time on L_- at the point $P_2(p_2(r,\varepsilon),0)$ and let $\Gamma_{r\varepsilon}^-$ be the trajectory of the system (3.2) starting at P_2 and the meeting first time on L_+ at the point $Q(q(r,\varepsilon),0)$ (See Fig. 1).

Then the Poincare map $\mathscr{P}_\varepsilon$ for (3.3) defined on L_+ is given by

$$\mathscr{P}_\varepsilon(p(r)) = q(r,\varepsilon).$$

Observe that $\Gamma_{r\varepsilon}^+ \cup \Gamma_{r\varepsilon}^-$ forms a closed trajectory of the system (3.3) if and only if $p(r) = q(r,\varepsilon)$. But $p(r) = q(r,\varepsilon)$ is equivalent to $H^+(p(r),0) = H^+(q(r,\varepsilon),0)$. Hence, analogous to the case of a smooth differential system, we use the difference map

$$H^+(Q) - H^+(P) = \varepsilon F(r,\varepsilon,\delta) = \sum_{k=1}^{\infty} M_k(r,\delta)\varepsilon^k,$$

where $M_k(r,\delta)$ is called as the kth order Melnikov function for the system (3.3) and F is a bifurcation function.

Similar to the Proposition 1 and Proposition 2, we can state the conditions for cyclicity and stability of limit cycles for the system (3.3).

PROPOSITION 3. *Assume that the system (3.6) has a period annulus with a center at the origin. Let $M_1(r)$ be the first order Melnikov function for the system (3.3). Then we have the following:*

1. *If there exist $0 \leqslant j \leqslant k$ such that $M_1(r_0,\delta_0) = 0$ and $\frac{\partial^j M_1}{\partial r^j}(r_0,\delta_0) \neq 0$, then at most k limit cycles of (3.3) are bifurcated form Γ_{r_0}, where Γ_{r_0} is a periodic orbit of (3.6) through r_0.*
2. *Limit cycle of (3.3) bifurcated from the periodic orbit of (3.6) passing through $(p(r),0)$ of the Poincare section is stable if*
$$\frac{dM_1}{dr}(r_0,\delta_0) < 0$$
for $|\varepsilon| + |\delta - \delta_0| \ll 1$.

Proof. Proof is similar to that of Proposition 1 and Proposition 2. □

In [9], the first order Melnikov function for (3.3) when $f^{\pm}, g^{\pm}$ are independent of ε and δ, is given by the following proposition.

PROPOSITION 4. [9] *If the system (3.6) has a period annulus around the origin then the first order Melnikov function for the system (3.3) is given by*

$$M_1(r) = \frac{H_x^+(P)}{H_x^-(P)}\left[\frac{H_x^-(P_1)}{H_x^+(P_1)}\int_{\widehat{PP_1}}(g^+dx - f^+dy) + \int_{\widehat{P_1P}}(g^-dx - f^-dy)\right],$$

where $\widehat{PP_1}$ denotes the path along the trajectory Γ_r^+ and $\widehat{P_1P}$ denotes the path along the trajectory Γ_r^-.

In this section, we first derive the expressions for the first order and second order Melnikov functions for piecewise smooth perturbed Hamiltonian system

$$\dot{X} = (H_y^+(x,y) + \varepsilon f_1^+(x,y) + \varepsilon^2 f_2^+(x,y), -H_x^+(x,y) + \varepsilon g_1^+(x,y) + \varepsilon^2 g_2^+(x,y)),\ y > 0 \tag{3.7}$$

$$\dot{X} = (H_y^-(x,y) + \varepsilon f_1^-(x,y) + \varepsilon^2 f_2^-(x,y), -H_x^-(x,y) + \varepsilon g_1^-(x,y) + \varepsilon^2 g_2^-(x,y)),\ y < 0 \tag{3.8}$$

or

$$\dot{X} = \begin{cases} (H_y^+(x,y) + \varepsilon f_1^+(x,y) + \varepsilon^2 f_2^+(x,y), -H_x^+(x,y) + \varepsilon g_1^+(x,y) + \varepsilon^2 g_2^+(x,y)),\ y > 0 \\ (H_y^-(x,y) + \varepsilon f_1^-(x,y) + \varepsilon^2 f_2^-(x,y), -H_x^-(x,y) + \varepsilon g_1^-(x,y) + \varepsilon^2 g_2^-(x,y)),\ y < 0 \end{cases} \tag{3.9}$$

under the assumption that the unperturbed system (3.6) has a period annulus around the origin.

In [17] the expression for the second order Melnikov function for the system

$$dH + \varepsilon\omega_0 + \varepsilon^2\omega^2 = 0,$$

with two semi-straight lines as a switching line, is obtained and it contains integrals involving the time of flight. In general, it is difficult to obtain the time of flight of the trajectories. Here we obtain an expression of the second order Melnikov function for second order perturbed piecewise Hamiltonian system (3.9) in which integrals are free from the time of flights of the trajectories.

THEOREM 1. *If the system (3.6) has a period annulus around the origin, then the first order Melnikov function for the system* (3.9) *is given by*

$$M_1(r) = \frac{H_x^+(P)}{H_x^-(P)}\left(\frac{H_x^-(P_1)}{H_x^+(P_1)}\int_{\Gamma_r^+}\omega_1^+ + \int_{\Gamma_r^-}\omega_1^-\right). \tag{3.10}$$

Further, if $M_1 \equiv 0$ then the second order Melnikov function M_2 for the system (3.9) *is given by*

$$\begin{aligned} M_2(r)\frac{H_x^-(P)}{H_x^+(P)} = &\left(\int_{\Gamma_r^-}\omega_2^- + \frac{H_x^-(P_1)}{H_x^+(P_1)}\int_{\Gamma_r^+}\omega_2^+\right) \\ &+\left(K^-(P(r))\int_{\Gamma_r^-}\frac{\omega_1^-}{H_y^-} + \frac{H_x^-(P_1)K^+(P(r))}{H_x^+(P_1)}\int_{\Gamma_r^+}\frac{\omega_1^+}{H_y^+}\right) \\ &-\left(\int_{\Gamma_r^-}\frac{f_1^-\omega_1^-}{H_y^-} + \frac{H_x^-(P_1)}{H_x^+(P_1)}\int_{\Gamma_r^+}\frac{f_1^+\omega_1^+}{H_y^+}\right) \\ &+\frac{1}{2}\left(H_{xx}^-(P_1) - \frac{H_x^-(P_1)}{H_x^+(P_1)}H_{xx}^+(P_1)\right)\sigma^2, \end{aligned} \tag{3.11}$$

where

$$\omega_i^\pm = g_i^\pm dx - f_i^\pm dy \ \text{ for } \ i = 1,2; \ \sigma = \frac{1}{H_x^+(P_1)}\int_{\Gamma_r^+}\omega_1^+, \ \ K^\pm = \frac{H_x^\pm f_1^\pm + H_y^\pm g_1^\pm}{H_x^\pm}.$$

Expression (3.10) is proved in [9], we prove the expression (3.11) in sequence of following Lemmas.

LEMMA 1. *The difference map for the system* (3.9) *can be expressed as,*

$$H^+(Q(q(r,\varepsilon),0) - H^+(P(p(r),0)) = \varepsilon H_x^+(P)\rho + \frac{\varepsilon^2}{2}\left[H_{xx}^+(P)\rho^2 + H_x^+(P)\eta\right] + o(\varepsilon^3), \tag{3.12}$$

where $\rho = \left[\frac{\partial}{\partial\varepsilon}(q(r,\varepsilon))\right]_{\varepsilon=0}$ *and* $\eta = \left[\frac{\partial^2 q(r,\varepsilon)}{\partial\varepsilon^2}\right]_{\varepsilon=0}$.

Proof. Let $\sigma = \left[\dfrac{\partial p_2(r,\varepsilon)}{\partial \varepsilon}\right]_{\varepsilon=0}$, $\tau = \left[\dfrac{\partial^2 p_2(r,\varepsilon)}{\partial \varepsilon^2}\right]_{\varepsilon=0}$, $\rho = \left[\dfrac{\partial q(r,\varepsilon)}{\partial \varepsilon}\right]_{\varepsilon=0}$ and

$$\eta = \left[\frac{\partial^2 q(r,\varepsilon)}{\partial \varepsilon^2}\right]_{\varepsilon=0}.$$

The difference map for the system (3.9) is

$$H^+(Q(r,\varepsilon)) - H^+(P(r)) = L_1 + L_2 + L_3 + L_4, \tag{3.13}$$

where

$$L_1 = H^+(Q(r)) - H^-(Q(r)), \qquad L_2 = H^-(Q(r)) - H^-(P_2(r)),$$
$$L_3 = H^-(P_2(r)) - H^+(P_2(r)), \qquad L_4 = H^+(P_2(r)) - H^+(P(r)).$$

Now by Taylor's series expansion in powers of ε we have

$$\sum_{i=1}^{4} L_i(r,\varepsilon) = \sum_{i=1}^{4}\left[\varepsilon\left(\frac{\partial L_i}{\partial \varepsilon}\right)_{\varepsilon=0} + \frac{\varepsilon^2}{2}\left(\frac{\partial^2 L_i}{\partial \varepsilon^2}\right)_{\varepsilon=0} + o(\varepsilon^3)\right].$$

Since

$$L_1 = H^+(Q) - H^-(Q) = H^+(q(r,\varepsilon),0) - H^-(q(r,\varepsilon),0),$$

we get

$$\left[\frac{\partial L_1}{\partial \varepsilon}\right]_{\varepsilon=0} = (H_x^+(P) - H_x^-(P))\rho, \tag{3.14}$$

and

$$\left[\frac{\partial^2 L_1}{\partial \varepsilon^2}\right]_{\varepsilon=0} = (H_{xx}^+(P) - H_{xx}^-(P))\rho^2 + (H_x^+(P) - H_x^-(P))\eta. \tag{3.15}$$

Similarly,

$$L_3 = H^-(P_2) - H^+(P_2) = H^-(p_2(r,\varepsilon),0) - H^+(p_2(r,\varepsilon),0)$$

imply that

$$\left[\frac{\partial L_3}{\partial \varepsilon}\right]_{\varepsilon=0} = (H_x^-(P_1) - H_x^+(P_1))\sigma, \tag{3.16}$$

and

$$\left[\frac{\partial^2 L_3}{\partial \varepsilon^2}\right]_{\varepsilon=0} = (H_{xx}^-(P_1) - H_{xx}^+(P_1))\sigma^2 + (H_x^-(P_1) - H_x^+(P_1))\tau. \tag{3.17}$$

Also,

$$L_2 = H^-(Q) - H^-(P_2) = H^-(q(r,\varepsilon),0) - H^-(p_2(r,\varepsilon),0)$$

gives us

$$\left[\frac{\partial L_2}{\partial \varepsilon}\right]_{\varepsilon=0} = H_x^-(P)\rho - H_x^-(P_1)\sigma, \tag{3.18}$$

and

$$\left[\frac{\partial^2 L_2}{\partial \varepsilon^2}\right]_{\varepsilon=0} = H_{xx}^-(P)\rho^2 - H_{xx}^-(P_1)\sigma^2 + H_x^-(P)\eta - H_x^-(P_1)\tau. \tag{3.19}$$

Further,

$$L_4 = H^+(P_2) - H^+(P) = H^+((p_2(r,\varepsilon),0)) - H^+(p(r),0),$$

so that

$$\left[\frac{\partial L_4}{\partial \varepsilon}\right]_{\varepsilon=0} = H_x^+(P_1)\sigma, \tag{3.20}$$

and

$$\left[\frac{\partial^2 L_4}{\partial \varepsilon^2}\right]_{\varepsilon=0} = H_{xx}^+(P_1)\sigma^2 + H_x^+(P_1)\tau. \tag{3.21}$$

Hence, from equations (3.13)–(3.21) we get equation (3.12). □

LEMMA 2. *The expression for L_4 is given by*

$$L_4 = \varepsilon \int_{\Gamma_r^+} \omega_1^+ + \varepsilon^2 \left(\int_{\Gamma_r^+} \omega_2^+ + K^+(P(r)) \int_{\Gamma_r^+} \frac{\omega_1^+}{H_y^+} - \int_{\Gamma_r^+} \frac{f_1^+ \omega_1^+}{H_y^+} \right) + o(\varepsilon^3), \tag{3.22}$$

where $\omega_i^+ = g_i^+ dx - f_i^+ dy,\ i = 1,2.$

Proof. We have,

$$\begin{aligned} L_4 = H^+(P_2) - H^+(P) &= \int_{\widehat{PP_2}} dH^+ = \int_{\widehat{PP_2}} H_x^+ dx + H_y^+ dy \\ &= \int_{\widehat{PP_2}} \left[H_x^+(H_y^+ + \varepsilon f_1^+ + \varepsilon^2 f_2^+) + H_y^+(-H_x^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+) \right] dt \\ &= \varepsilon \int_{\widehat{PP_2}} (H_x^+ f_1^+ + H_y^+ g_1^+) dt + \varepsilon^2 \int_{\widehat{PP_2}} (H_x^+ f_2^+ + H_y^+ g_2^+) dt \\ &= \varepsilon \int_{\widehat{PP_2}} (H_x^+ f_1^+ + H_y^+ g_1^+) dt + \varepsilon^2 \int_{\widehat{PP_1}} (H_x^+ f_2^+ + H_y^+ g_2^+) dt + o(\varepsilon^3) \\ &= \varepsilon \int_{\widehat{PP_2}} (H_x^+ f_1^+ + H_y^+ g_1^+) dt + \varepsilon^2 \int_{\Gamma_r^+} \omega_2^+ + o(\varepsilon^3). \end{aligned} \tag{3.23}$$

Along the path $\widehat{PP_2}$, we have

$$\begin{aligned} dt = \frac{dy}{\dot{y}} &= \frac{dy}{-H_x^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+} \\ &= \frac{-1}{H_x^+}\left(1 + \varepsilon \frac{g_1^+}{H_x^+} + o(\varepsilon^2)\right) dy. \end{aligned}$$

Let us denote $K^+ = \dfrac{H_x^+ f_1^+ + H_y^+ g_1^+}{H_x^+}$, $K^+ dy = -g_1^+ dx + f_1^+ dy = -\omega_1^+$ on Γ_r^+ and R is the region bounded by $\Gamma_{r\varepsilon}^+$ and $\Gamma_r^+ \cup \overrightarrow{P_1P_2}$, where $\overrightarrow{P_1P_2}$ denote the line segment from P_1 to P_2. Then

$$\begin{aligned} &\int_{\widehat{PP_2}} (H^+{}_x f_1{}^+ + H_y{}^+ g_1{}^+) dt \\ &= \int_{\widehat{PP_2}} -\frac{H^+{}_x f_1{}^+ + H_y{}^+ g_1{}^+}{H_x{}^+}\left(1 + \varepsilon \frac{g_1{}^+}{H_x{}^+} + o(\varepsilon^2)\right) dy \\ &= -\int_{\widehat{PP_2}} \frac{H_x{}^+ f_1{}^+ + H_y{}^+ g_1{}^+}{H_x{}^+} dy - \varepsilon \int_{\widehat{PP_2}} \frac{g_1{}^+(H_x{}^+ f_1{}^+ + H_y{}^+ g_1{}^+)}{H_x{}^{+2}} dy + o(\varepsilon^2) \\ &= -\int_{\widehat{PP_2}} K^+ dy - \varepsilon\left(\int_{\widehat{PP_1}} \frac{g_1{}^+ K^+}{H_x{}^+} dy + o(\varepsilon)\right) + o(\varepsilon^2) \\ &= -\int_{\widehat{PP_1}} K^+ dy - \int_{\overrightarrow{P_1P_2}} K^+ dy - \iint_R \frac{\partial K^+}{\partial x} dxdy \\ &\quad - \varepsilon\left(\int_{\widehat{PP_1}} \frac{K^+ g_1{}^+}{H_x{}^+} dy + o(\varepsilon)\right) + o(\varepsilon^2) \\ &= \int_{\Gamma_r^+} \omega_1^+ - \int_{\overrightarrow{P_1P_2}} K^+ dy - \iint_R \frac{\partial K^+}{\partial x} dxdy - \varepsilon \int_{\Gamma_r^+} \frac{K^+ g_1{}^+}{H_x{}^+} dy + o(\varepsilon^2) \\ &= \int_{\Gamma_r^+} \omega_1^+ - \iint_R \frac{\partial K^+}{\partial x} dxdy + \varepsilon \int_{\Gamma_r^+} \frac{g_1{}^+ \omega_1^+}{H_x{}^+} + o(\varepsilon^2). \end{aligned} \tag{3.24}$$

Now suppose that $R = R_1 \cup R_2$, where R_1 is the region bounded by Γ_r^+, $\Gamma_{r\varepsilon}^+$, $x = p_1(r)$ and $x = p(r)$ whereas R_2 is bounded by $y = 0$, $\Gamma_{r\varepsilon}^+$, $x = p_2(r, \varepsilon)$ and $x = p_1(r)$. Note that, since the radial distance z from $P_1(r)$ to the point on $\Gamma_{r\varepsilon}^+$ in R_2 is of order ε, we have $\displaystyle\int\int_{R_2} dxdy = \int_0^{\frac{\pi}{2}} \int_0^z r dr d\theta = \frac{\pi}{4} z^2 = o(\varepsilon^2)$. Therefore,

$$\iint_{R_2} \frac{\partial K^+}{\partial x} dxdy = o(\varepsilon^2). \tag{3.25}$$

Let us represent $\Gamma_{r\varepsilon}^+$ and Γ_r^+ by $y_\varepsilon = y(x, \varepsilon)$, $y_0 = y(x, 0)$, respectively. Now put $y(x, s) = y_0(x) + s(y_\varepsilon(x) - y_0(x))$, $p_1(r) \leqslant x \leqslant p(r)$, $0 \leqslant s \leqslant 1$. Hence, area element for the region R_1 becomes $dydx = (y_\varepsilon(x) - y_0(x))dsdx$. Therefore,

$$\begin{aligned}
\iint_{R_1} \frac{\partial K^+}{\partial x} dxdy &= \int_{p_1(r)}^{p(r)} \left(\int_0^1 \frac{\partial K^+}{\partial x} (y_\varepsilon(x) - y_0(x)) ds \right) dx \\
&= \varepsilon \int_{p_1(r)}^{p(r)} \left(\int_0^1 \frac{\partial K^+}{\partial x} \left(\frac{\partial y_\varepsilon}{\partial \varepsilon} \right)_{\varepsilon=0} ds \right) dx + o(\varepsilon^2) \\
&= \varepsilon \int_{p_1(r)}^{p(r)} \left(\int_0^1 \frac{\partial K^+}{\partial x} (x, y_0(x)) \left(\frac{\partial y_\varepsilon}{\partial \varepsilon} \right)_{\varepsilon=0} ds \right) dx + o(\varepsilon^2) \\
&= \varepsilon \int_{p_1(r)}^{p(r)} \frac{\partial K^+}{\partial x} (x, y_0) \left(\frac{\partial y_\varepsilon}{\partial \varepsilon} \right)_{\varepsilon=0} dx + o(\varepsilon^2). \qquad (3.26)
\end{aligned}$$

Now along $\Gamma_{r\varepsilon}^+$, we have

$$\dot{y_\varepsilon} = \frac{\partial y_\varepsilon}{\partial x} \dot{x} = -H_x{}^+ + \varepsilon g_1{}^+ + \varepsilon^2 g_2{}^+ \Rightarrow \frac{\partial y_\varepsilon}{\partial x} = \frac{-H_x{}^+ + \varepsilon g_1{}^+ + \varepsilon^2 g_2{}^+}{H_y{}^+ + \varepsilon f_1{}^+ + \varepsilon^2 f_2{}^+}$$

$$\begin{aligned}
\Rightarrow y_\varepsilon &= \int_{p_1(r)}^{x} \frac{-H_x{}^+ + \varepsilon g_1{}^+ + \varepsilon^2 g_2{}^+}{H_y{}^+ + \varepsilon f_1{}^+ + \varepsilon^2 f_2{}^+} ds \\
&= \int_{p_1(r)}^{x} \frac{-H_x{}^+}{H_y{}^+} ds + \varepsilon \int_{p_1(r)}^{x} \frac{H_x{}^+ f_1{}^+ + H_y{}^+ g_1{}^+}{H_y{}^{+2}} ds + o(\varepsilon^2).
\end{aligned}$$

Note here that the last integral is taken along Γ_r^+. From the above expression we have

$$\left(\frac{\partial y_\varepsilon}{\partial \varepsilon} \right)_{\varepsilon=0} = \int_{p_1(r)}^{x} \frac{H_x{}^+ f_1{}^+ + H_y{}^+ g_1{}^+}{H_y{}^{+2}} ds = \int_{p_1(r)}^{x} \frac{K^+ H_x{}^+}{(H_y{}^+)^2} dx = I^+(x) \text{ (say)}. \quad (3.27)$$

Therefore, from equations (3.25), (3.26) and (3.27), we get

$$\iint_R \frac{\partial K^+}{\partial x} dxdy = \varepsilon \int_{p_1(r)}^{p(r)} \frac{\partial K^+}{\partial x} (x, y_0) I^+ dx + o(\varepsilon^2).$$

Now using integration by parts, we have

$$\begin{aligned}
\iint_R \frac{\partial K^+}{\partial x} dxdy &= \varepsilon \left(K^+(P(r)) \int_{p_1(r)}^{p(r)} \frac{K^+ H_x^+}{(H_y^+)^2} dx \right) - \varepsilon \int_{p_1(r)}^{p(r)} \frac{(K^+)^2 H_x^+}{(H_y^+)^2} dx + o(\varepsilon^2) \\
&= -\varepsilon K^+(P(r)) \int_{\Gamma_r^+} \frac{\omega_1^+}{H_y^+} + \varepsilon \int_{\Gamma_r^+} \frac{f_1^+ \omega_1^+}{H_y^+} + \varepsilon \int_{\Gamma_r^+} \frac{g_1^+ \omega_1^+}{H_x^+} + o(\varepsilon^2).
\end{aligned} \quad (3.28)$$

Hence, from (3.23), (3.24) and (3.28) we obtain the formula for L_4. □

LEMMA 3. *The expression for L_2 is given by*

$$L_2 = \varepsilon \int_{\Gamma_r^-} \omega_1^- + \varepsilon^2 \left(\int_{\Gamma_r^-} \omega_2^- + K^-(P(r)) \int_{\Gamma_r^-} \frac{\omega_1^-}{H_y^-} - \int_{\Gamma_r^-} \frac{f_1^- \omega_1^-}{H_y^-} \right) + o(\varepsilon^3), \quad (3.29)$$

where $\omega_i^- = g_i^- dx - f_i^- dy,\ i = 1, 2.$

Proof. The proof is similar to the proof of Lemma 2. □

Proof of Theorem 1. Comparing coefficients of ε and ε^2 in the expression for L_4 obtained in Lemma (2) and from expressions (3.20), (3.21), we have

$$\sigma = \frac{1}{H_x^+(P_1)} \int_{\Gamma_r^+} \omega_1^+, \tag{3.30}$$

and

$$\frac{1}{2}H_x^+(P_1)\tau = \int_{\Gamma_r^+} \omega_2^+ + K^+(P(r)) \int_{\Gamma_r^+} \frac{\omega_1^+}{H_y^+} - \int_{\Gamma_r^+} \frac{f_1^+\omega_1^+}{H_y^+} - \frac{1}{2}H_{xx}^+(P_1)\sigma^2. \tag{3.31}$$

Similarly, from Lemma (3) and expressions (3.18), (3.19) we get

$$H_x^-(P)\rho = H_x^-(P_1)\sigma + \int_{\Gamma_r^-} \omega_1^- = \frac{H_x^-(P_1)}{H_x^+(P_1)} \int_{\Gamma_r^+} \omega_1^+ + \int_{\Gamma_r^-} \omega_1^-, \tag{3.32}$$

and

$$\frac{1}{2}H_x^-(P)\eta = \int_{\Gamma_r^-} \omega_2^- + K^-(P(r)) \int_{\Gamma_r^-} \frac{\omega_1^-}{H_y^-} - \int_{\Gamma_r^-} \frac{f_1^-\omega_1^-}{H_y^-}$$
$$- \frac{1}{2}H_{xx}^-(P)\rho^2 + \frac{1}{2}H_{xx}^-(P_1)\sigma^2 + \frac{1}{2}H_x^-(P_1)\tau. \tag{3.33}$$

From (3.12) and (3.32) we get the first order Melnikov function.

Now if the first order Melnikov function is identically zero, then from equation (3.12) we have $\rho \equiv 0$. Hence, from (3.12) and (3.33), the second order Melnikov function is

$$M_2(r) = H_x{}^+(P)\eta$$
$$= \lambda \left(\int_{\Gamma_r^-} \omega_2{}^- - \int_{\Gamma_r^-} \frac{f_1{}^-\omega_1{}^-}{H_y{}^-} + K^-(P(r)) \int_{\Gamma_r^-} \frac{\omega_1{}^-}{H_y{}^-} + \frac{1}{2}H_{xx}{}^-(P_1)\sigma^2 + \frac{1}{2}H_x{}^-(P_1)\tau \right),$$

where $\lambda = \dfrac{H_x{}^+(P)}{H_x{}^-(P)}$. By substituting τ from (3.31) we get the required expression for M_2. □

If the Hamiltonian system (3.6) is extended smoothly on the boundary $y = 0$, it becomes a smooth Hamiltonian system. In this case, the first and second order Melnikov functions are simple line integrals of one forms. Further, if perturbation of this system is also smooth, then the first order and second order Melnikov functions obtained from Theorem 1 are well-known integrals of one forms as shown in the following corollary.

COROLLARY 1. *If in the system* (3.9),

$$\lim_{y\to 0^+} H_x{}^+(x,y) = \lim_{y\to 0^-} H_x{}^-(x,y) \text{ and } \lim_{y\to 0^+} H_y{}^+(x,y) = \lim_{y\to 0^-} H_y{}^-(x,y)$$

for all $x \in \mathbb{R}$, then the first order and the second order Melnikov functions are given by

$$M_1(r) = \int_{\Gamma_r^+} \omega_1^+ + \int_{\Gamma_r^-} \omega_1^- \quad \textit{and} \tag{3.34}$$

$$M_2(r) = \left(\int_{\Gamma_r^-} \omega_2^- + \int_{\Gamma_r^+} \omega_2^+\right) - \left(\int_{\Gamma_r^-} \frac{f_1^- \omega_1^-}{H_y^-} + \int_{\Gamma_r^+} \frac{f_1^+ \omega_1^+}{H_y^+}\right)$$
$$+ \left(K^-(P(r)) \int_{\Gamma_r^-} \frac{\omega_1^-}{H_y^-} + K^+(P(r)) \int_{\Gamma_r^+} \frac{\omega_1^+}{H_y^+}\right), \tag{3.35}$$

respectively.

Further, if $\lim_{y\to 0^+} f_i^+(x,y) = \lim_{y\to 0^-} f_i^-(x,y)$ and $\lim_{y\to 0^+} g_i^+(x,y) = \lim_{y\to 0^-} g_i^-(x,y)$ for $i = 1,2$ and for all $x \in \mathbb{R}$, then the first order and second order Melnikov functions are given by

$$M_1(r) = \oint_{\Gamma_r} \omega_1, \ \textit{and} \ M_2(r) = \oint_{\Gamma_r} \omega_2 - \oint_{\Gamma_r} \frac{f_1 \omega_1}{H_y} + K(P(r)) \oint_{\Gamma_r} \frac{\omega_1}{H_y}, \tag{3.36}$$

respectively,

where $H_x(x,y) = \begin{cases} H_x^+(x,y) & \textit{if } y > 0 \\ H_x^-(x,y) & \textit{if } y < 0 \\ \lim_{y\to 0^+} H_x^+(x,y) & \textit{if } y = 0 \end{cases}$, $H_y(x,y) = \begin{cases} H_y^+(x,y) & \textit{if } y > 0 \\ H_y^-(x,y) & \textit{if } y < 0 \\ \lim_{y\to 0^+} H_y^+(x,y) & \textit{if } y = 0 \end{cases}$,

$f_i(x,y) = \begin{cases} f_i^+(x,y) & \textit{if } y > 0 \\ f_i^-(x,y) & \textit{if } y < 0 \\ \lim_{y\to 0^+} f_i^+(x,y) & \textit{if } y = 0 \end{cases}$, $g_i(x,y) = \begin{cases} g_i^+(x,y) & \textit{if } y > 0 \\ g_i^-(x,y) & \textit{if } y < 0 \\ \lim_{y\to 0^+} g_i^+(x,y) & \textit{if } y = 0 \end{cases}$,

$\omega_i = g_i dx - f_i dy$, $K = \frac{H_x f_1 + H_y g_1}{H_x}$ *for $i = 1,2$ and $\Gamma_r = \Gamma_r^+ \cup \Gamma_r^-$, a closed trajectory of the unperturbed system.*

Proof. Proof follows from the equation (3.10) and (3.11). □

4. Piecewise Hamiltonian system with boundary perturbation

Consider a piecewise Hamiltonian system with boundary perturbation,

$$\dot{X} = \begin{cases} (H_y^+, -H_x^+), \ y > \varepsilon f(x) \\ (H_y^-, -H_x^-), \ y < \varepsilon f(x) \end{cases}, \tag{4.1}$$

where $H^+, H^- : \mathbb{R}^2 \to \mathbb{R}$ are C^2 functions and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function.

Here $\Sigma = \{(x,y) \in \mathbb{R}^2 : y = \varepsilon f(x)\}$ is a switching manifold and $\Sigma^\pm = \{(x,y) \in \mathbb{R}^2 : \pm(y - \varepsilon f(x)) > 0\}$ are two zones separated by Σ.

REMARK 1. According to the Filippov convention, [7], the switching manifold Σ is divided into the following regions:

Crossing region $\Sigma_c = \{(x,\varepsilon f(x)) : H_x^{\pm} + \varepsilon H_y^{\pm} f'(x) > 0 \text{ or } H_x^{\pm} + \varepsilon H_y^{\pm} f'(x) < 0\}$,
Sliding region $\Sigma_s = \{(x,\varepsilon f(x)) : H_x^{+} + \varepsilon H_y^{+} f'(x) < 0, H_x^{-} + \varepsilon H_y^{-} f'(x) > 0\}$, and
Escaping region $\Sigma_e = \{(x,\varepsilon f(x)) : H_x^{+} + \varepsilon H_y^{+} f'(x) > 0, H_x^{-} + \varepsilon H_y^{-} f'(x) < 0\}$.
Discontinuity-induced bifurcations are studied according to these regions.

Using the analytic invertible change of variables $u = x$, $v = y - \varepsilon f(x)$ and renaming the variables u by x and v by y, the system (4.1) becomes

$$\dot{X} = \begin{cases} (X^+, Y^+), \ y > 0 \\ (X^-, Y^-), \ y < 0 \end{cases}, \tag{4.2}$$

where

$$X^+ = H_y^+ + \varepsilon f(x) H_{yy}^+ + \varepsilon^2 \frac{1}{2} f(x)^2 H_{yyy}^+ + o(\varepsilon^3),$$
$$Y^+ = -H_x^+ - \varepsilon (f'(x) H_y^+ + f(x) H_{xy}^+) - \varepsilon^2 f(x) \left(\frac{1}{2} f(x) H_{xyy}^+ + f'(x) H_{yy}^+ \right) + o(\varepsilon^3),$$
$$X^- = H_y^- + \varepsilon f(x) H_{yy}^- + \varepsilon^2 \frac{1}{2} f(x)^2 H_{yyy}^- + o(\varepsilon^3), \text{ and}$$
$$Y^- = -H_x^- - \varepsilon (f'(x) H_y^- + f(x) H_{xy}^-) - \varepsilon^2 f(x) \left(\frac{1}{2} f(x) H_{xyy}^- + f'(x) H_{yy}^- \right) + o(\varepsilon^3).$$

REMARK 2. Note that if $\Phi(x,y) = (x, y - \varepsilon f(x))$, then Φ is a diffeomorphism. Therefore, systems (4.1) and (4.2) are topologically equivalent and their flows are C^1 conjugate.

Melnikov function for the system (4.2) is given by the following theorem.

THEOREM 2. *If the system (4.2) at $\varepsilon = 0$ has a period annulus around the origin and $H_y^{\pm}(P) = H_y^{\pm}(P_1)$, then the first order Melnikov function for (4.2) is*

$$\lambda M_1(r) = H_x^-(P_1) \left(\frac{H_y^+(P_1)}{H_x^+(P_1)} - \frac{H_y^-(P_1)}{H_x^-(P_1)} \right) (f(p) - f(p_1)), \tag{4.3}$$

and the second order Melnikov function is given by

$$\begin{aligned} \lambda M_2(r) = & \frac{H_x^-(P_1)}{2} \left(\frac{H_{yy}^+(P_1)}{H_x^+(P_1)} - \frac{H_{yy}^-(P_1)}{H_x^-(P_1)} \right) ((f(p(r)))^2 - (f(p_1(r)))^2) \\ & + \frac{H_x^-(P_1) K^+(P) - H_x^+(P_1) K^-(P)}{H_x^+(P_1)} (f(p(r)) - f(p_1(r))) \\ & - \left(K^-(P) \int_{\Gamma_r^-} \frac{f}{H_y^-} d(H_y^-) + \frac{H_x^-(P_1) K^+(P)}{H_x^+(P_1)} \int_{\Gamma_r^+} \frac{f}{H_y^+} d(H_y^+) \right) \\ & + \int_{\Gamma_r^-} H_{yy}^- \left[d\left(\frac{f^2}{2} \right) + \left(\frac{f^2}{H_y^-} \right) d(H_y^-) \right] \end{aligned}$$

$$+\frac{H_x^-(P_1)}{H_x^+(P_1)}\int_{\Gamma_r^+} H_{yy}^+\left[d\left(\frac{f^2}{2}\right)+\left(\frac{f^2}{H_y^+}\right)d(H_y^+)\right]$$
$$+\frac{1}{2}\left(H_{xx}^-(P_1)-\frac{H_x^-(P_1)}{H_x^+(P_1)}H_{xx}^+(P_1)\right)\sigma^2, \tag{4.4}$$

where $\lambda=\dfrac{H_x^-(P)}{H_x^+(P)}$.

Proof. From (4.2), we have

$$\int_{\Gamma_r^+}\omega_1^+=\int_{\Gamma_r^+}(g_1^+dx-f_1^+dy)=-\int_{\Gamma_r^+}d(fH_y^+)=f(p)H_y^+(P)-f(p_1)H_y^+(P_1),\ \text{and}$$
$$\int_{\Gamma_r^-}\omega_1^-=\int_{\Gamma^-(r)}(g_1^-dx-f_1^-dy)=-\int_{\Gamma_r^-}d(fH_y^-)=f(p_1)H_y^-(P_1)-f(p)H_y^-(P).$$

Hence, by Theorem 1,

$$M_1(r)$$
$$=\lambda\left(f(p)\left(\frac{H_x^-(P_1)H_y^+(P)}{H_x^+(P_1)}-H_y^-(P)\right)-f(p_1)\left(\frac{H_x^-(P_1)H_y^+(P_1)}{H_x^+(P_1)}-H_y^-(P_1)\right)\right), \tag{4.5}$$

where $\lambda=\dfrac{H_x^+(P)}{H_x^-(P)}$.

Now if $H_y^+(P)=H_y^+(P_1)$ and $H_y^-(P)=H_y^-(P_1)$, then we get (4.3).

Again, from (4.2) we have $\omega_1^\pm=-d(f(x)H_y^\pm),\omega_2^\pm=-d\left(\dfrac{f^2(x)H_{yy}^\pm}{2}\right)$. Hence, the formula for M_2 follows from (3.11). □

REMARK 3. If the system (4.2) at $\varepsilon=0$ is smooth, then $\dfrac{H_y^+(P_1)}{H_x^+(P_1)}-\dfrac{H_y^-(P_1)}{H_x^-(P_1)}=0$. Thus, $M_1(r)=0$ if and only if $f(p(r))=f(p_1(r))$ or the system is smooth. If $f(p(r))=f(p_1(r))$, the periodic orbit passing through $P(r)$ of (4.2) at $\varepsilon=0$ is also a periodic orbit of (4.2). Hence, the number of periodic orbits persists under perturbation equals to the number of roots of $f(p(r))-f(p_1(r))$.

If the system (4.1) at $\varepsilon=0$ is smoothly extended on $y=0$, then its period annulus persists under perturbation of the switching boundary. In the following corollary, we obtain the first order and second order Melnikov functions for such a system.

COROLLARY 2. *If the system (4.1) at $\varepsilon=0$ is smooth then its period annulus persists under smooth perturbation of switching manifold $y=0$.*

Proof. From (4.3) we have $M_1(r) = \oint_{\Gamma_r} \omega_1 = \oint_{\Gamma_r} -d(f(x)H_y) = 0$.

Now from (4.4) we have

$$\lambda M_2(r) = -K(P)\oint_{\Gamma_r} \frac{f}{H_y} dH_y + \oint_{\Gamma_r} H_{yy}\frac{f^2}{H_y} dH_y + \oint_{\Gamma_r} H_{yy} d\left(\frac{f^2}{2}\right), \tag{4.6}$$

where f, K, H_y, H_{yy} and Γ_r are as defined in Corollary 1.

Along Γ_r we have

$$\begin{aligned} \frac{f(x)}{H_y} dH_y &= \frac{f(x)}{H_y}(H_{yx}dx + H_{yy}dy) = \frac{f(x)}{H_y}\left(-H_{yx}\frac{H_y}{H_x} + H_{yy}\right)dy \\ &= f(x)\left(\frac{-H_{xy}}{H_x} + \frac{H_{yy}}{H_y}\right)dy = \frac{\partial}{\partial y}\left(f(x)\log\left(\frac{H_y}{H_x}\right)\right)dy, \end{aligned} \tag{4.7}$$

so that $\oint_{\Gamma_r} \frac{f}{H_y} dH_y = 0$. Also,

$$\begin{aligned} \oint H_{yy}\left(f^2\frac{dH_y}{H_y} + d\left(\frac{f^2}{2}\right)\right) &= \oint H_{yy}\left(f^2\frac{\partial}{\partial y}\left(\log\frac{H_y}{H_x}\right) - ff'\frac{H_y}{H_x}\right)dy \\ &= \oint H_{yy}\left(f^2 v_y - ff'u\right)dy, \end{aligned}$$

where $u = \frac{H_y}{H_x}$ and $v = \log(u)$. Integrating by parts twice, we get

$$\begin{aligned} \oint H_{yy}\left(f^2\frac{dH_y}{H_y} + d\left(\frac{f^2}{2}\right)\right) &= \oint H\left(f^2 v_{yyy} - ff'u_{yy}\right)dy \\ &= r\oint\left((f^2 v)_{yyy} - (ff'u)_{yy}\right)dy = 0. \end{aligned} \tag{4.8}$$

Hence from equation (4.6), (4.7) and (4.8), we get $M_2 \equiv 0$. Thus, we conclude that no limit cycle bifurcated from the period annulus of (4.1). □

REMARK 4. In the above proof we consider an extension of the natural logarithmic function on $\mathbb{R}\cup\{+\infty,-\infty\}$ as, $\log(x) = \begin{cases} \ln x, & x>0 \\ \ln(-x), & x<0 \\ -\infty, & x=0 \\ \infty, & x=\pm\infty \end{cases}$. This function is an antiderivative of the functions $f_1(x) = \begin{cases} \frac{1}{x}, & x>0 \\ \infty, & x=0 \end{cases}$ on $[0,\infty]$ and $f_2(x) = \begin{cases} \frac{1}{x}, & x<0 \\ -\infty, & x=0 \end{cases}$ on $[-\infty, 0]$.

5. Applications

There are various types of planar piecewise smooth systems according to the types of singularities in zones separated by the switching line with a center at the origin viz. center-center, saddle-center, and center-focus. Here we discuss the limit cycle bifurcation from period annulus due to perturbation of the switching manifold of center-center and saddle-center type using the first and second-order Melnikov functions.

5.1. Boundary perturbation of center-center type system

Consider the piecewise Hamiltonian system

$$\dot{X} = \begin{cases} (-1, 2x) & \text{if } y > 0 \\ (1, 2x) & \text{if } y < 0 \end{cases}. \tag{5.1}$$

System (5.1) has center at the origin (Fig. 2a). Hamiltonian of the system $\dot{X} = (-1, 2x)$ is $H^+(x,y) = -y - x^2$. Trajectories of this system at level $h = r^2$ are given by $y + x^2 = -r^2$. Hamiltonian for $\dot{X} = (1, 2x)$ is $H^-(x,y) = y - x^2$ and its trajectories at the levels $h = r^2$ are given by $y - x^2 = r^2$.

Now consider the perturbed piecewise smooth system

$$\dot{X} = \begin{cases} (-1 + \varepsilon f_1^+(x,y) + \varepsilon^2 f_2^+(x,y), 2x + \varepsilon g_1^+(x,y) + \varepsilon^2 g_2^+(x,y)) & \text{if } y > 0 \\ (1 + \varepsilon f_1^-(x,y) + \varepsilon^2 f_2^-(x,y), 2x + \varepsilon g_1^-(x,y) + \varepsilon^2 g_2^-(x,y)) & \text{if } y < 0, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} f_1^+(x,y) &= ax^2 + bxy + cy^2, & f_2^+(x,y) &= Ax^2 + Bxy + Cy^2, \\ g_1^+(x,y) &= dx^2 + exy + fy^2, & g_2^-(x,y) &= Dx^2 + Exy + Fy^2, \\ f_1^-(x,y) &= px^2 + qxy + sy^2, & f_2^-(x,y) &= Px^2 + Qxy + Sy^2, \\ g_1^-(x,y) &= lx^2 + mxy + ny^2, & g_2^-(x,y) &= Lx^2 + Mxy + Ny^2. \end{aligned}$$

and $a,b,c,d,e,f,p,q,s,l,m,n,A,B,C,D,E,F,P,Q,S,L,M$ and N are real constants.

From (3.10) we get

$$M_1(r) = r^3 \left(\frac{8}{15}(4b - 7f + 7n - 4q)r^2 + \frac{2}{3}(l - d) \right). \tag{5.3}$$

Note that, $M_1(r) \equiv 0$ if

$$4b - 7f + 7n - 4q = 0, \text{ and } d - l = 0. \tag{5.4}$$

Also, from (3.10) the second order Melnikov function is

$$\begin{aligned}M_2(r) &= \left(-\frac{2656\,sn}{315}+\frac{3712\,qs}{315}+\frac{3712\,bc}{315}-\frac{2656\,cf}{315}\right)r^9\\ &+\left(\frac{8}{15}(p(7n-4q)-a(4b-7f))-\frac{184}{105}(ls+qm+np+af+be+cd)+\frac{96}{35}(pq+ab)\right)r^7\\ &-\frac{4}{15}(l(7n-4q)+d(4b-7f))r^6+\frac{4}{15}(14(N-F)+8(B-Q)+lp+ad)r^5\\ &+\frac{1}{3}\left(d^2-l^2\right)r^4+\frac{2}{3}(L-D)r^3.\end{aligned} \tag{5.5}$$

In the view of the conditions (5.4), the expression (5.5) becomes $M_2(r) = r^3 M_2'(h)$, where $h = r^2$ and

$$\begin{aligned}M_2'(h) &= \frac{32}{315}(-29(7f-4b)(s+c)+120(sn+cf))h^3\\ &+\left(\frac{8}{15}(7f-4b)(p+a)-\frac{184}{105}(d(s+c)+qm+np+af+be)+\frac{96}{35}(pq+ab)\right)h^2\\ &+\frac{4}{15}(14(N-F)+8(B-Q)+d(p+a))h+\frac{2}{3}(L-D).\end{aligned} \tag{5.6}$$

Thus under the conditions (5.4), the system (5.2) can have at most three limit cycles.

In particular, if $a = p$, $b = q$, $c = s$, $m = -e$, $f = n = 2b$, $L = -D$, $F = N$, $B = Q$, then (5.6) becomes

$$M_2'(h) = -\frac{640}{63}bch^3+\left(\frac{80}{21}ab+\frac{16}{3}b^2-\frac{368}{105}cd\right)h^2+\frac{8}{15}adh-\frac{4}{3}D. \tag{5.7}$$

We can choose constants a,b,c,d and D such that (5.7) will have three distinct positive roots. In particular, if $a = 1$, $bc = 1$, $d = -\frac{8800}{42}$ and $D = -\frac{960}{21}$, then (5.7) becomes

$$M_2'(h) = -\frac{640}{63}(h^3-6.000000005h^2+11h-6). \tag{5.8}$$

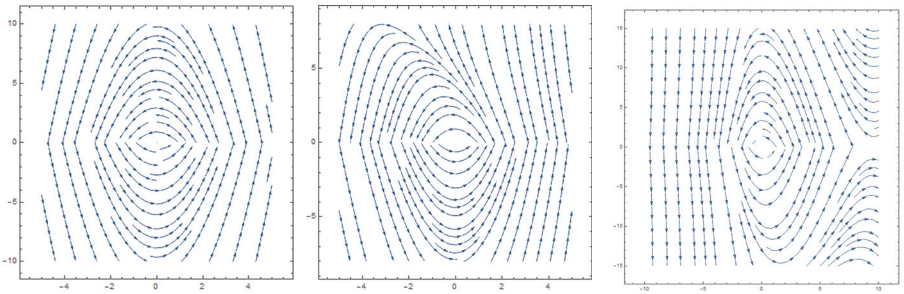
Polynomial (5.8) has three positive zeros; $h = 1.000000003, 1.999999980, 3.000000022$. Consequently, the corresponding system will have three limit cycles (Fig. 2c).

Further, in (5.6), if $4b = 7f$, $e = m = c = 0$, $a = p = s = n = 1$, $q = -b$, $d = 1/10$, $f = 98.9$, $L = D$, $14(N-F)+8(B-Q) = 0$, then

$$M_2'(h) = \frac{256}{21}h^3-\frac{3680}{21}h^2+\frac{4}{75}h. \tag{5.9}$$

Note that (5.9) has two real positive zeros, $h = \frac{115}{16}+\frac{3}{80}\sqrt{36733}, \frac{115}{16}-\frac{3}{80}\sqrt{36733}$. Therefore the corresponding system will have two limit cycles (Fig. 2b).

In [15], authors characterize all planar piecewise smooth differential systems having (k,l) center at the origin. Here we mention the result;



(a) *System* (5.1) (b) *Two limit cycles* (c) *Three limit cycles*

Figure 2: *Flow of the system* (5.2)

PROPOSITION 5. [15] *Let* k,l,r *be positive integers and* $\max\{k,l\} \leqslant r \in \mathbb{N} \cup \{\infty, \omega\}$. *Suppose that the system*

$$\dot{X} = \begin{cases} (F^+(x,y), G^+(x,y)) & \text{if } y > 0 \\ (F^-(x,y), G^-(x,y)) & \text{if } y < 0 \end{cases} \tag{5.10}$$

is piecewise smooth with $F^{\pm}, G^{\pm} \in C^r$ *and having* (k,l)*-*Σ*-center at the origin, where* Σ *is the* x*-axis. Then there exists a* C^r *diffeomorphism* h *from period annulus of* (5.10) *to a period annulus of* (5.1)*, which maps* x*-axis to the* x*-axis.*

We note that the system (5.1) is piecewise smooth Hamiltonian system with Hamiltonian $H^+(x,y) = -y - x^2$, $y > 0$ and $H^-(x,y) = y - x^2$, $y < 0$. The following proposition gives information about the limit cycles bifurcated from the period annulus of this system due to perturbation of the switching manifold.

PROPOSITION 6. *The number of limit cycles for the system*

$$\dot{X} = \begin{cases} (-1, 2x) & \text{if } \; y > \varepsilon f(x) \\ (1, 2x) & \text{if } \; y < \varepsilon f(x) \end{cases}$$

is equal to the number of isolated positive zeros of $f_o(x)$*, where* $f_o(x) = \dfrac{f(x) - f(-x)}{2}$.

Proof. From (4.3) and (4.4), we get

$$M_1(r) = 2[f(-r) - f(r)] = 4f_o(r), \quad \text{and}$$
$$M_2(r) = \frac{f'(r)}{r} M_1(r).$$

Therefore the number of limit cycles bifurcated from the period annulus of the unperturbed system is the same as the number of positive roots of $y = f_o(x)$. In particular, if f is an even function, then no limit cycles are bifurcated. $\square$

REMARK 5. From Proposition 3, it is clear that the limit cycle of system (5.1) through the point $(r,0)$ is stable if $\frac{dM_1}{dr} = \frac{-2}{r} f_e{}'(r) < 0$, where $f_e(x) = \frac{f(x)+f(-x)}{2}$.

Since the system (5.1) is the normal form of planar piecewise smooth systems of the center-center type, Proposition 6 also holds for the systems (5.10).

5.2. Boundary perturbation of saddle-center type system

In [18, 19], authors studied the number of limit cycles along with the stability and hyperbolicity of limit cycles of the system

$$\dot{X} = \begin{cases} (y-a,x) & \text{if } y > \varepsilon f(x) \\ (-y,x) & \text{if } y < \varepsilon f(x) \end{cases}. \tag{5.11}$$

Origin is singularity of this system of saddle-center type and $f(x) = \sin x$ or $f(x) = x(x^2-x_1^2)(x^2-x_2^2)\dots(x^2-x_m^2)$. Note that this system is a piecewise linear Hamiltonian system with two zones. We study the number of limit cycles bifurcated from the period annulus and stability of the above systems if we change the switching manifold $y=0$ to $y=\varepsilon f(x)$, under the assumption that f is sufficiently smooth. Using dilation $x=au$, $y=av$ and renaming the variables u and v by x and y respectively, system (5.10) becomes

$$\dot{X} = \begin{cases} (y-1,x), & y > \varepsilon f(ax) \\ (-y,x), & y < \varepsilon f(ax) \end{cases}. \tag{5.12}$$

At $\varepsilon = 0$, system (5.11) has a period annulus $\mathscr{A} = \bigcup_{r\in[0,1]} (\Gamma_r^+ \cup \Gamma_r^-)$, where

$$\Gamma_r^+ : H^+(x,y) = \frac{(y-1)^2}{2} - \frac{x^2}{2} = \frac{r}{2}, y \geqslant 0 \text{ and } \Gamma_r^- : H^-(x,y) = \frac{x^2}{2} + \frac{y^2}{2} = \frac{s}{2}, y \leqslant 0.$$

Now for any point $P(r) = (p(r),0)$, $r \in [0,1]$ we have

$$H^+(P) = \frac{r}{2} = H^-(P) = \frac{s}{2} \Rightarrow \frac{(0-1)^2}{2} - \frac{(p(r))^2}{2} = \frac{r}{2} = -\frac{(p(r))^2}{2} - \frac{0^2}{2} = \frac{s}{2}.$$

Hence, $p(r) = \sqrt{1-r} = \sqrt{s}$.

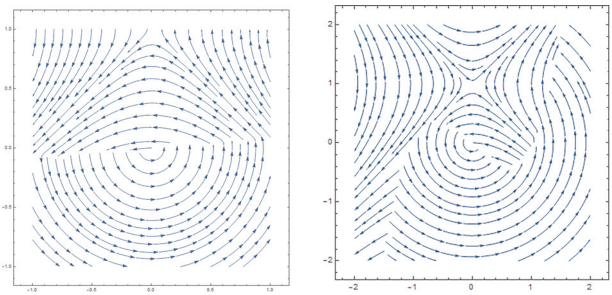
Since the trajectories are symmetric about the y-axis, we have $p_1(r) = -p(r) = -\sqrt{1-r}$. Melnikov functions for (5.11) are given by the following proposition.

PROPOSITION 7. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is C^1 smooth function. Then we have the following:*

1. *The first order Melnikov function for the system (5.11) is given by*

$$M_1(r) = 2f_o(a\sqrt{1-r}),$$

where $f_o(x) = \frac{f(x)-f(-x)}{2}$ for $0<r<1$. Further, if $M_1(r) \equiv 0$, then $M_2(r) \equiv 0$.



(a) $f(x) = 0.01\sin x$ (b) $f(x) = x(x^2-1)(x^2-4)$

Figure 3: *Saddle-center system with perturbation boundary* $y = \varepsilon f(x)$

2. *The number of limit cycles bifurcated from the period annulus around the origin inside the homoclinic orbit containing the saddle point $(0,a)$ is the number of isolated positive roots of $f_o(x)$. In particular, if f is an even function, then no limit cycle bifurcated from the period annulus.*

Proof. From (4.3), we have

$$M_1(r) = \frac{(-p(r))(-p_1(r))}{(-p(r))}\left(\frac{-1}{(-p_1(r))} - \frac{0}{(-p_1(r))}\right)(f(ap(r)) - f(ap_1(r)))$$
$$= f(ap(r)) - f(-ap(r)).$$

Therefore, $M_1(r) = 2f_0(ap(r)) = 2f_0(a\sqrt{1-r})$ for all $r \in [0,1]$.

Here, $H^{\pm}_{xx} = H^{\pm}_{yy} = H^{\pm}_{xy} = 0$. Therefore from (4.4), we get

$$M_2(r) = \frac{-f'(a\sqrt{1-r})}{\sqrt{1-r}} M_1(r), \quad r \in [0,1).$$

Hence the proof of (1).

Proof of (2) follows from the expression of M_1. □

REMARK 6. From Proposition 3, it is clear that the limit cycle of (5.12) through the point $(\sqrt{1-r},0)$ is stable if $\frac{dM_1}{dr} = \frac{-2a}{\sqrt{1-r}} f_e{}'(\sqrt{1-r}) < 0$, where

$$f_e(x) = \frac{f(x)+f(-x)}{2}.$$

Similarly, we can characterize all planar piecewise smooth differential systems having saddle-center type as stated in the following proposition.

PROPOSITION 8. *If the Filippov system*

$$\dot{X} = \begin{cases} (F^+, G^+), \ y > 0 \\ (F^-, G^-), \ y < 0 \end{cases} \tag{5.13}$$

has a period annulus around the origin inside the homoclinic orbit containing the saddle point $(0,a)$, $a > 0$, *then there is a homeomorphism which maps the period annulus of* (5.11) *to the period annulus of* (5.13) *and maps switching manifold to switching manifold.*

Proof. Let U^+ be an open region in the upper half plane lying inside the homoclinic connection containing the saddle point $(0,a)$ of

$$\dot{X} = (F^+, G^+). \tag{5.14}$$

Let $(b,0)$ and $(c,0)$ be the points of intersection of the homoclinic orbit of (5.14) with the x-axis and $b < 0 < c$. We may assume that the periodic orbits of (5.14) are convex (see [15]). Hence we use the polar co-ordinates (r,θ) to transform the system (5.14) into

$$\frac{dr}{d\theta} = P^+(r,\theta). \tag{5.15}$$

Let $0 < r_0 < c$ and $0 < \theta < \pi$. Consider the initial value problem $\frac{dr}{d\theta} = P^+(r,\theta)$, $r(0) = r_0$. Let $\xi = \xi(\theta, r_0)$ be its solution. Now define the function $\Phi^+ : [0,\pi] \times [0,c] \to U^+$ by

$$\Phi^+(\theta,r) = (\xi(\theta,r)\cos\theta, \xi(\theta,r)\sin\theta). \tag{5.16}$$

Then Φ^+ is a diffeomorphism and maps each horizontal line segment $r = r_0$ to the trajectory $\xi = \xi(\theta, r_0)$ of (5.14).

Similarly, if V^+ denotes the region in the upper half plane occupied by the periodic orbits of

$$\dot{X} = (y-1, x), \tag{5.17}$$

then we have the diffeomorphism $\Psi^+(\theta,r) : [0,\pi] \times [0,1] \to V^+$. Let $\chi^+ : [0,\pi] \times [0,c] \to [0,\pi] \times [0,1]$ be the map given by $\chi^+(x,y) = (x, y/c)$. Then χ^+ is also a diffeomorphism.

The composition $\mathscr{H}^+ := \Psi^+ \circ \chi^+ \circ (\Phi^+)^{-1} : U^+ \to V^+$ is a diffeomorphism.

Next, let U^- and V^- denote the open regions in lower half plane consisting of orbits of the systems

$$\dot{X} = (F^-, G^-) \tag{5.18}$$

and

$$\dot{X} = (-y, x), \tag{5.19}$$

respectively. Then we can construct a diffeomorphism $\mathscr{H}^- : U^- \to V^-$ which maps orbits of (5.18) to that of (5.19).

Since the system (5.13) is Filippov, every point on the switching manifold $y = 0$ is a singularity of order one. Hence, $\lim_{y\to 0^+} \mathscr{H}^+(x,y) = \lim_{y\to 0^-} \mathscr{H}^-(x,y)$.

Now we define the map

$$\mathscr{H} : U^+ \cup U^- \cup \{(x,0) : b < x < 0 \text{ or } 0 < x < c\} \to V^+ \cup V^- \cup \{(x,0) : 0 < |x| < 1\}$$

by

$$\mathscr{H}(x,y) = \begin{cases} \mathscr{H}^+(x,y), (x,y) \in U^+ \\ \mathscr{H}^-(x,y), (x,y) \in U^- \\ \lim_{y\to 0^+} \mathscr{H}^+(x,y) = \lim_{y\to 0^-} \mathscr{H}^-(x,y), y = 0 \end{cases} . \tag{5.20}$$

Note that due to the Filippov convention,

$$\lim_{y\to 0^+} F^+(x,y) = \lim_{y\to 0^-} F^-(x,y) \quad \text{and} \quad \lim_{y\to 0^+} G^+(x,y) = \lim_{y\to 0^-} G^-(x,y),$$

so that $\mathscr{H}$ is continuously differentiable on the switching manifold $y = 0$. Therefore $\mathscr{H}$ is a diffeomorphism. □

From Proposition (8) we conclude that the Proposition (7) holds for the system (5.13).

6. Concluding remark

In this article, we found expressions for first-order as well as second-order Melnikov functions for perturbed planar piecewise smooth Hamiltonian systems. Using Melnikov functions we study limit cycle bifurcations of piecewise smooth Hamiltonian systems due to the perturbation of the switching manifold.

This idea could be extended to study the limit cycle bifurcation of any piecewise smooth planar differential system.

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