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MULTIPLE SOLUTIONS FOR NONLOCAL FRACTIONAL KIRCHHOFF TYPE PROBLEMS

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Abstract. In this paper, using variational methods and critical point theory, we establish the existence of two and infinitely many solutions for a fractional Kirchhoff type problem driven by a nonlocal operator of elliptic type in a fractional Orlicz-Sobolev space with homogeneous Dirichlet boundary conditions. Some examples are presented to demonstrate the application of our main results.

1. Introduction

In this paper we consider the existence of multiple weak solutions for the following $a(.)$-Kirchhoff type problem

$$\begin{cases} M\left(\int\int_{\Omega\times\Omega}\Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}\right)(-\Delta)^s_{a(.)}u = f(x,u), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset in $\mathbb{R}^N$, $N \geqslant 1$, with Lipschitz boundary $\partial\Omega$, $0<s<1$, Φ is an N-function, the function $M:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and $\widehat{M}(t)=\int_0^t M(\xi)d\xi$. Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function and $(-\Delta)^s_{a(.)}$ be the nonlocal integro-differential operator of elliptic type defined as:

$$(-\Delta)^s_{a(.)}u(x) = 2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} a\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\cdot\frac{dy}{|x-y|^{N+s}},$$

for all $x\in\mathbb{R}^N$, where $a:\mathbb{R}\to\mathbb{R}$.

The Kirchhoff equation, proposed by Kirchhoff [19] in 1883 in the study of the oscillations of stretched strings and plates, is an extended version of the classical wave equation due D'Alembert by taking into account the effects of the changes in the length of the string during the vibrations. Nonlinear Kirchhoff models can be used for describing the dynamics of an axially moving string. Moreover, the Kirchhoff equation can be used for modeling, such as computer science, mechanical engineering, control systems,

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artificial or biological neural networks, economics, and many others. In recent years, the study of various mathematical problems with variable exponent growth condition has been received considerable attention. Existence results of the Kirchhoff equation have been established by several authors by applying different tools like fixed point theory, lower and upper solutions method, variational methods and critical point theory, and Morse theory. We refer the reader to [11, 12, 13, 15, 17] and the references therein for some recent results on this topic.

Fractional calculus is a broader concept, since it is a generalization of arbitrary order derivatives and integrals. In recent decades, fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance, and fractional dynamics. Thus fractional models are the natural substitutes of the classical integer-order model for such systems. Fractional calculus also provides an excellent tool to describe the hereditary properties of various materials and processes. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of fractional differential equations so far. Such as fixed-point theorems, the method of upper and lower solutions, the topological degree theory and the critical point theory, etc. (see [7, 8, 16, 18] and references therein).

Moreover, there are few results about the existence of solutions for fractional Kirchhoff problems involving the supercritical term. For the problems involving fractional Kirchhoff type, we refer the reader to the works [2, 3, 4, 6, 14]. Recently, Fiscella and Valdinoci [14] proposed a stationary Kirchhoff type variational model, which considered the nonlocal aspect of the tension arising from nonlocal measurements of fractional length of the string. More precisely, they studied the following fractional Kirchhoff type problem involving critical growth

$$\begin{cases} M\left(\int\int_{\mathbb{R}^{2N}} \Phi\left(\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\right) dxdy\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

by combining a truncated technique with the mountain pass theorem, they obtained the existence of nontrivial solutions for the above equation when λ is large enough. In [2], Ambrosio and Servadei first studied the existence of nontrivial solutions for factional Kirchhoff problems with supercritical growth by using a truncation argument, the mountain pass theorem and Moser iterative method.

In the present paper we are interested in ensuring the existence of at least two solutions and infinitely many solutions for the problem (1.1). The present paper is organized as follows. In Section 2, we recall some properties on fractional Orlicz-Sobolev spaces and our main tools. In Section 3, we state and prove the main Theorem of the paper and finally, we give two examples to show the application of our results.

2. Preliminaries and basic notation

In this section, we introduce some definitions and basic information about the fractional Orlicz-Sobolev space to find out the solution of problem (1.1).

Consider Ω be an open subset of $\mathbb{R}^N$ with $N > 1$. We let that $a : \mathbb{R} \to \mathbb{R}$ in (1.1) is such that $\varphi : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(t) = \begin{cases} a(|t|)t, & t \neq 0, \\ 0, & t = 0, \end{cases} \tag{2.1}$$

is increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau,$$

then, Φ is a N-function (see [1]), i.e. $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, convex and increasing function, with

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0, \lim_{t\to +\infty} \frac{\Phi(t)}{t} = +\infty.$$

For the function Φ introduced above, we define the Orlicz space:

$$L_\Phi(\Omega) = \{u : \Omega \to \mathbb{R} \text{ mesurable } : \int_\Omega \Phi(\lambda|u(x)|)dx < \infty \text{ for some } \lambda > 0\}.$$

The space $L_\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_\Phi = \inf\left\{\lambda > 0 \ : \int_\Omega \Phi\left(\frac{|u(x)|}{\lambda}\right) dx \leqslant 1\right\}.$$

The conjugate N-function of Φ is defined by $\overline{\Phi}(t) = \int_0^t \overline{\varphi}(\tau)d\tau$ where $\overline{\varphi} : \mathbb{R} \to \mathbb{R}$ is given by $\overline{\varphi}(t) := \sup\{s : \varphi(s) \leqslant t\}$. Furthermore, it is possible to prove a Hölder type inequality, that is,

$$|\int_\Omega uvdx| \leqslant 2\|u\|_\Phi \|v\|_{\overline{\Phi}} \ \forall u \in L_\Phi(\Omega) \text{ and } v \in L_{\overline{\Phi}}(\Omega).$$

In this paper, we let that

$$1 \leqslant \overline{\varphi} = \inf_{t\geqslant 0} \frac{t\varphi(t)}{\Phi(t)} \leqslant \varphi^+ = \sup_{t\geqslant 0} \frac{t\varphi(t)}{\Phi(t)} < \infty. \tag{2.2}$$

The above relation implies that $\Phi \in \Delta_2$ i.e., Φ satisfies the global Δ_2-condition (see [23]):

$$\Phi(2t) \leqslant k\Phi(t) \text{ for all } t \geqslant 0,$$

where K is a positive constant.

Furthermore, we assume that Φ satisfies the following condition: the function

$$t \to \Phi(\sqrt{t}) \ \ (\text{for } t \in [0,\infty)),$$

is convex.

The above relation assures that $L_\Phi(\Omega)$ is an uniformly convex space (see [23]).

DEFINITION 1. Assume A,B be two N-functions. A is said to be stronger (resp. essentially stronger) than B, $A \succ B$ (resp. $A \succ\succ B$) in symbols if

$$B(x) \leqslant A(ax), \ \ x \geqslant x_0 \geqslant 0,$$

for some (resp. for each) $a > 0$ and x_0 (depending on a).

Now, we defined the fractional Orlicz-Sobolev space $W^s L_\Phi(\Omega)$ as follows:

$$W^s L_\Phi(\Omega) = \left\{ u \in L_\Phi(\Omega) : \int_\Omega \int_\Omega \Phi\left(\frac{\lambda|u(x)-u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} < \infty \text{ for some } \lambda > 0 \right\}.$$

This space is equipped with the norm,

$$\|u\|_{s,\Phi} = \|u\|_\Phi + [u]_{s,\Phi},$$

where $[.]_{s,\Phi}$ is the Gagliardo seminorm, defined by

$$[u]_{s,\Phi} = \inf\left\{ \lambda > 0 : \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right) \frac{dxdy}{|x-y|^N} < 1 \right\}.$$

To deal with the problem under consideration, we choose

$$W_0^s L_\Phi(\Omega) = \{u \in W^s L_\Phi(\mathbb{R}^N) : u = 0 \text{ a.e. } \mathbb{R}^N \setminus \Omega\}$$

which can be equivalently renormed by setting $\|.\| = [.]_{s,\Phi}$. By [8], $W_0^s L_\Phi(\Omega)$ is a Banach space. Indeed, it is separable (resp. reflexive) space if and only if $\Phi \in \Delta_2$ (resp. $\Phi \in \Delta_2$ and $\overline{\Phi} \in \Delta_2$).

Furthermore if $\Phi \in \Delta_2$ and $\Phi(\sqrt{t})$ is convex, then the space $W^s L_\Phi(\Omega)$ is uniformly convex. The dual space of $(W^s L_\Phi(\Omega), \|.\|)$ is denoted by $((W^s L_\Phi(\Omega))^*, \|.\|_*)$.

PROPOSITION 1. ([5]) *Assume condition (2.2) is satisfied. Then the following relations hold true,*

$$[u]_{s,\Phi}^{\varphi^-} \leqslant \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} \leqslant [u]_{s,\Phi}^{\varphi^+}, \ \forall u \in W^s L_\Phi(\Omega) \text{ with } [u]_{s,\Phi} > 1,$$

and

$$[u]_{s,\Phi}^{\varphi^+} \leqslant \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} \leqslant [u]_{s,\Phi}^{\varphi^-}, \ \forall u \in W^s L_\Phi(\Omega) \text{ with } [u]_{s,\Phi} < 1.$$

In this paper, we assume the following conditions:

$$\int_0^1 \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau < \infty,$$

and

$$\int_1^\infty \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau = \infty. \tag{2.3}$$

We define the inverse Sobolev conjugate N-function of Φ as follows

$$\Phi_*^{-1} = \int_0^t \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau. \tag{2.4}$$

THEOREM 1. ([5]) *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. If (2.2), (2.3) and (2.4) hold, then*

$$W^s L_\Phi(\Omega) \hookrightarrow L_B(\Omega)$$

is compact for all $B \prec\prec \Phi_*$.

THEOREM 2. ([5]) *Let Ω be a bounded open subset of $\mathbb{R}^N$. Then,*

$$C_0^\infty \subset C_0^2(\Omega) \subset W^s L_\Phi(\Omega). \tag{2.5}$$

For every $u \in W_0^s L_\Phi$, set

$$I(u) = \Phi(u) - \Psi(u)$$

where

$$\Phi(u) = \widehat{M}\left(\int_\Omega \int_\Omega \Phi\left(\frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^N} \right), \quad \Psi(u) = \int_\Omega F(x,u)dx,$$

$$\widehat{M}(t) = \int_0^t M(\xi)d\xi, \ \forall\, t \in \mathbb{R}^+$$

and

$$F(x,t) = \int_0^t f(x,\tau)d\tau, \ \forall\, (x,t) \in \Omega \times \mathbb{R}.$$

We see that

$$\langle I'(u), v\rangle = M\left(\int_\Omega \int_\Omega \Phi\left(\frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^N} \right) \int_\Omega \int_\Omega a(\frac{|u(x) - u(y)|}{|x-y|^s}) \\ \times \frac{|u(x) - u(y)|}{|x-y|^s} \frac{|v(x) - v(y)|}{|x-y|^s} \frac{dxdy}{|x-y|^N} - \int_\Omega f(x,u)v dx$$

for every $u, v \in W_0^s L_\Phi$.

In the sequel, we will mention some auxiliary results used through the paper.

DEFINITION 2. Let X be a real reflexive Banach space. If any sequence $\{u_k\} \subset X$ for which $\{I(u_k)\}$ is bounded and $I'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence, then we say I satisfies Palais-Smale condition.

THEOREM 3. [22, Theorem 4.10] *Let $I \in C^1(X,\mathbb{R})$, and I satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and*

$$\inf_{v \in \partial\Omega} \varphi(v) > \max\{I(u_0), I(u_1)\},$$

then there exists a critical point u of I, i.e. $I'(u) = 0$ with $I(u) > \max\{I(u_0), \varphi(u_1)\}$.

THEOREM 4. [24, Theorem 9.12] *Let X be an infinite dimensional real Banach space. Let $I \in C^1(X,\mathbb{R})$ be an even functional which satisfies the (PS) condition, and $I(0) = 0$. Suppose that $X = V \bigoplus E$, where V is finite dimensional, and I satisfies that*

(i) *There exist $\alpha > 0$ and $\rho > 0$ such that $I(u) \geqslant \alpha$ for all $u \in E$ with $\| u \| = \rho$,*

(ii) *For any finite dimensional subspace $W \subset X$ there is $R = R(W)$ such that $I(u) \leqslant 0$ on $W \setminus B_R$.*

Then I possesses an unbounded sequence of critical values.

THEOREM 5. [25, Theorem 38] *For the functional $F : M \subseteq X \longrightarrow [-\infty, +\infty]$ with $M \neq \emptyset, \min_{u \in M} F(u) = \alpha$ has a solution in case the following conditions hold:*

(h_1) *X is a real reflexive Banach space,*

(h_2) *M is bounded and weak sequentially closed,*

(h_3) *F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $F(u) \leqslant \lim_{n\to\infty} \inf F(u_n)$ holds.*

We refer to the paper [9, 26, 20] in which Theorems 3 and 4 have been successfully employed to prove the existence of multiple solutions for some boundary value problems.

3. Main results

Throughout this paper, we utilize the following assumptions.

(M_0) There is a positive constant m_0, m_1 such that $m_0 \leqslant M(t) \leqslant m_1$ for all $t \geqslant 0$;

(A_1) there exists a constant $\nu > \frac{\varphi^+ m_1}{m_0}$ and $0 < \nu F(x,t) \leqslant t f(x,t)$, $|t| > L$;

(f_0) $f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies Carathéodory condition and

$$|f(x,t)|\leqslant c(1+|t|^{q-1}) \text{ for } t\leqslant L;$$

where $q>\frac{\varphi^+ m_1}{m_0}$,

(f_1) $f(x,t)=O(|t|^{\varphi^+-1})$, for $x\in\Omega$ uniformly.

Let

$$\rho(u)=\int_\Omega\int_\Omega\Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}.$$

To obtain main results is needed that we prove the following Lemma.

LEMMA 1. ([27], Lemma 2.2) *If condition (A_1) holds, then for every $x\in\Omega$, the following inequalities hold:*

$$F(x,t)\leqslant F\left(x,\frac{t}{|t|}\right)|t|^\nu,\ \text{ if } 0<|t|\leqslant 1,$$
$$F(x,t)\geqslant F\left(x,\frac{t}{|t|}\right)|t|^\nu,\ \text{ if } |t|\geqslant 1.$$

In view Lemma 1, (A_1) implies that for every $x\in\Omega$,

$$F(x,t)\leqslant a_3|t|^\nu,\ \text{ if } |t|\leqslant 1,$$

$$F(x,t)\geqslant a_1|t|^\nu,\ \text{ if } |t|\geqslant 1, \tag{3.1}$$

where $a_3=\max_{x\in\Omega,|t|=1}F(x,t)$ and $a_1=\min_{x\in\Omega,|t|=1}F(x,t)$. By assumption (f_0), we observe that $a_1,a_3>0$. In addition, since $F(x,t)-a_1|t|^\nu$ is continuous on $\Omega\times[0,T]$, there exists a constant $a_2>0$ such that

$$F(x,t)\geqslant a_1|t|^\nu-a_2 \text{ for all } (x,t)\in\Omega\times[0,T]. \tag{3.2}$$

so it follows from (3.1) and (3.2) that

$$F(x,t)\geqslant a_1|t|^\nu-a_2 \text{ for all } (x,t)\in\Omega\times\mathbb{R}. \tag{3.3}$$

The main result of this paper are the following theorems.

THEOREM 6. *Assume that the assumptions (M_0), (A_1), (f_0) and (f_1) hold. Then: if $f(x,t)\geqslant 0$ for all $(x,t)\in\Omega\times\mathbb{R}$, the problem (1.1) has at least two solutions.*

THEOREM 7. *Assume that the assumptions (M_0) and (A_1) hold. Then: if $f(x,t)$ is odd about t, the problem (1.1) has infinitely many solutions.*

We need the following lemma to prove our main results.

LEMMA 2. *Assume that* (A_1) *and* (M_0) *hold. Then* $I(u)$ *satisfies the* (PS)*-condition.*

Proof. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset W_0^s L_\Phi(\Omega)$ such that $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|I(u_n)| \leqslant c_0$ and $|I'(u_n)| \leqslant c_0$ for all $n \in \mathbb{N}$. Therefore, from the definition of I' and the assumption (A_1) and for some $c_1 > 0$ we have,

$$\begin{aligned} c_0 + c_1\|u_n\| &\geqslant \nu I(u_n) - I'(u_n)(u_n) \\ &\geqslant \nu\widehat{M}(\rho(u_n)) - \varphi^+ M(\rho(u_n))\rho(u_n) - (\nu \int_\Omega F(x,u_n)dx - \int_\Omega f(x,u_n)u_n dx) \\ &\geqslant \nu m_0 \rho(u_n) - \varphi^+ m_1 \rho(u_n) \\ &\geqslant (\nu m_0 - m_1\varphi^+)\rho(u_n) \geqslant (m_0\nu - m_1\varphi^+)\|u_n\|^{\varphi^+}. \end{aligned}$$

Since $\nu > \frac{m_1\varphi^+}{m_0}$ this implies that (u_n) is bounded. Now using the same argument as in [10, Lemma 2.4], we can prove $\{u_n\}$ converges strongly to u in $W_0^s L_\Phi(\Omega)$. Consequently, I satisfies the (PS)-condition. □

3.1. The proof of Theorem 6

Proof. In our case it is clear that $I(0) = 0$. Lemma 2 shows that I satisfies the (PS)-condition.

Step 1. We will show that there exists $M > 0$ such that the functional I has a local minimum $u_0 \in B_M = \{u \in W_0^s L_\Phi; \|u\| < M\}$. Let $\{u_n\} \subseteq \overline{B}_M$ and $u_n \rightharpoonup u$ as $n \to \infty$, by Mazur Theorem [21], there exists a sequence of convex combinations

$$v_n = \sum_{j=1}^n a_{n_j} u_j, \quad \sum_{j=1}^n a_{n_j} = 1, \qquad a_{n_j} \geqslant 0,\ j \in N$$

such that $v_n \to u$ in $W_0^s L_\Phi$. Since $\overline{B}_M$ is a closed convex set, we have $\{v_n\} \subseteq \overline{B}_M$ and $u \in \overline{B}_M$. Noting that I is weak sequentially lower semi-continuous on $\overline{B}_M$ and $W_0^s L_\Phi$ is a reflexive Banach space. Then by Theorem 5 we can know that I has a local minimum $u_0 \in \overline{B}_M$. We assume that $I(u_0) = \min_{u\in\overline{B}_M} I(u)$. Now we will show that

$$I(u_0) < inf_{u\in\partial B_M} I(u).$$

By Theorem 2, $c_2 > 0$ exists such that

$$\|u\|_\infty \leqslant c_2\|u\|, \quad \forall\ u \in W_0^s L_\Phi.$$

Let $\varepsilon > 0$ be small enough such that $mesur(\Omega)\varepsilon c_2 \leqslant \frac{m_0}{2}$, from (f_0) and (f_1) we have

$$F(x,t) \leqslant \varepsilon|t|^{\varphi^+} + c|t|^{\varphi^+}. \tag{3.4}$$

Then, from (M_0) and (3.4) one has

$$\begin{aligned} I(u) &\geqslant m_0\rho(u) - \varepsilon\int_\Omega |u|^{\varphi^+}dx - c\int_\Omega |u|^q dx \\ &\geqslant m_0\rho(u) - \varepsilon\int_\Omega \|u\|_\infty^{\varphi^+}dx - c\int_\Omega \|u\|_\infty^q dx \\ &\geqslant m_0\|u\|^{\varphi^+} - mesur(\Omega)\varepsilon c_2\|u\|^{\varphi^+} - mesur(\Omega)c_2c\|u\|^q \\ &\geqslant \frac{m_0}{2}\|u\|^{\varphi^+} - mesur(\Omega)c_2c\|u\|^q \end{aligned}$$

when $\|u\| < 1$. Since $q > \frac{m_1\varphi^+}{m_0} > \varphi^+$, there exist $r > 0$, $\delta > 0$ such that $I(u) \geqslant \delta > 0$ for every $\|u\| = r$, We choosing $M = r$, so $I(u) > 0 = I(0) \geqslant I(u_0)$ for $u \in \partial B_M$. Hence $u_0 \in B_M$ and $I'(u_0) = 0$.

Step 2. Since u_0 is a minimum point of I on $W_0^sL_\Phi$, we can consider $M > 0$ sufficiently large such that $I(u_0) \leqslant 0 < \inf_{u\in\partial B_M} I(u)$, where $B_M = \{u \in W_0^sL_\Phi; \|u\| < M\}$. Now we will illustrate that there exists u_1 with $\|u_1\| > M$ such that $I(u_1) < \inf_{\partial B_M} I(u)$. For this, consider the function $e_1(x) \in W_0^sL_\Phi$ such that $\|e_1\| = 1$. Let $u_1 = re_1, r > 0$. By [9, Remark 3.1] and (A_0) there exist constants $a_1, a_2 > 0$ such that $F(x,t) \geqslant a_1|t|^\nu - a_2$ for all $x \in \Omega$, $|t| \geqslant T$. Thus

$$\begin{aligned} I(u_1) &= \widehat{M}(\rho(re_1)) - \int_\Omega F(x, re_1)dx \\ &\leqslant m_1\rho(re_1) - \int_\Omega (a_1|re_1|^\nu - a_2)dx \\ &\leqslant m_1\|re_1\|^{\varphi^+} - r^\nu a_1\int_\Omega |e_1|^\nu dx + \int_\Omega a_2 dx. \end{aligned}$$

Since $\nu > \varphi^+$, there exists sufficiently large $r > M > 0$ so that $I(re_1) < 0$. Hence, $\max\{I(u_0), I(u_1)\} < \inf_{\partial B_M} I(u)$. Then, Theorem 3 gives the critical point u^*. Therefore, u_0 and u^* are two critical points of I, which are two solutions of the problem (1.1). $\square$

We now present the following example to illustrate Theorem 6.

EXAMPLE 1. Consider $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$, $M(t) = 2 + \cos t$, for $t \in \mathbb{R}^+$, $\varphi(t) = \log(1+|t|)|t|^2 t$ and $f(x,t) = 17t^{16}$ for $(x,t) \in \Omega\times\mathbb{R}$. By the expression of f, M and φ, we have

$$F(x,t) = t^{17},$$

$$\Phi(t) = \frac{1}{4}\log(1+|t|)|t|^2 - \frac{1}{4}\int_0^{|t|}\frac{\tau^2}{1+\tau}d\tau,$$

and

$$\widehat{M}(t) = 2t - \sin(t).$$

We observe that $\varphi^+ = \sup_{t\geqslant 0}\frac{t\varphi(t)}{\Phi(t)} = 5$, $\varphi^- = \inf_{t\geqslant 0}\frac{t\varphi(t)}{\Phi(t)} = 4$ and M satisfies the condition (M_0) with $m_0 = 1$, $m_1 = 3$. Also, M and f are continuous functions, $f(x,t) \geqslant 0$

for all $(x,t) \in \Omega \times \mathbb{R}$, and $\lim_{\xi \to 0} \frac{f(x,\xi)}{\xi^{\varphi^+-1}} = \lim_{\xi \to 0} \frac{17\xi^{16}}{\xi^4} = 0$. By taking $q = 16$ and $c = 2$ then $|f(x,t)| < 2(1+|t|^{15})$, for $|t| \leqslant 1$. By choosing $\nu = 17 > 15 = \frac{\varphi^+ m_1}{m_0}$, $17F(x,t) \leqslant tf(x,t)$, for $|t| > 1$. So we see that all conditions $(A_1), (f_0)$ and (f_1) are satisfied therefore, by applying Theorem 6, the problem

$$\begin{cases} (2+\cos(\rho(u)))(-\Delta)^s_{\log} u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has at least two weak solutions.

3.2. The proof of Theorem 7

Proof. According to definitions of the functional I, it is clear that $I(u)$ is even and $I(0) = 0$.

Step 1. We will depict that I satisfies condition (i) in Theorem 4. Since, I is coercive and also satisfies (PS)-condition, by the minimization theorem [22, Theorem 4.4] the functional I has a minimum critical point u with $I(u) \geqslant \alpha > 0$ and $\|u\| = \rho$ for $\rho > 0$ small enough.

Step 2. We will show that I satisfies condition (ii) in Theorem 4. Let $W \subset W_0^s L_\Phi$ be a finite dimensional subspace. By [9, Remark 3.1] there exist constants $a_1, a_2 > 0$ such that $F(x,u(x)) \geqslant a_1|u(x)|^\nu - a_2$ for all $x \in \Omega$. Now, For every $r > 0$ and $u \in W \setminus \{0\}$ with $\|u\| = 1$, one has

$$\begin{aligned} I(ru) &= \widehat{M}(\rho(ru)) - \int_\Omega F(x,ru)dx \\ &\leqslant m_1\|ru\|^{\varphi^+} - \int_\Omega (a_1|u(x)|^\nu - a_2)dx \\ &\leqslant m_1 r^{\varphi^+}\|u\|^{\varphi^+} - r^\nu a_1 \int_\Omega |u|^\nu dx + \int_\Omega a_2 dx. \end{aligned}$$

Since $\nu > \varphi^+$, the above inequality implies that there exists r_0 such that $\|ru\| > \rho$ and $I(ru) < 0$ for every $r \geqslant r_0 > 0$. Since W is a finite dimensional subspace, there exists $R = R(W) > 0$ such that $I(u) \leqslant 0$ on $W \setminus B_{R(W)}$. According to Theorem 4, the functional $I(u)$ possesses infinitely many critical points, i.e., the problem (1.1) has infinitely many solutions. □

Finally, we present the following example in which the hypotheses of Theorem 7 are satisfied.

EXAMPLE 2. Consider $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 < 2\}$, $M(t) = 2 + \sin t$, for $t \in \mathbb{R}^+$, $\varphi(t) = \log(1+|t|)|t|^3 t$ and $f(x,t) = 20t^{19}$ for $(x,t) \in \Omega \times \mathbb{R}$. By the expression of f, M and φ, we have

$$F(x,t) = t^{20},$$

$$\widehat{M}(t) = 3t + \cos(t)$$

and

$$\Phi(t) = \frac{1}{5}\log(1+|t|)|t|^5 - \frac{1}{5}\int_0^{|t|}\frac{\tau^4}{1+\tau}d\tau.$$

We observe that $\varphi^+ = \sup_{t\geqslant 0}\frac{t\varphi(t)}{\Phi(t)} = 6$, $\varphi^- = \inf_{t\geqslant 0}\frac{t\varphi(t)}{\Phi(t)} = 5$ and M satisfies the condition (M_0) with $m_0 = 1$, $m_1 = 3$. Also, M and f are continuous functions, $f(x,t)$ is odd about t. By choosing $\nu = 20 > 18 = \frac{\varphi^+ m_1}{m_0}$, $20F(x,t) \leqslant tf(x,t)$, for $|t| > 1$. So we see that all conditions (A_1) and (M_0) are satisfied, hence, by using Theorem 7, for every $\lambda > 0$ the problem

$$\begin{cases} (2+\sin(\rho(u)))(-\Delta)^s_{\log}u = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{3.5}$$

has infinitely many weak solutions.

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