Differential
Equations
& Applications
Volume 15, Number 1 (2023), 29–60
doi:10.7153/dea-2023-15-03

ANALYSIS AND CONTROL OF PHYSIOLOGICALLY STRUCTURED MODELS WITH NONLOCAL DIFFUSION

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(*Communicated by C. Pignotti*)

Abstract. In the first part of this paper, a non-autonomous physiologically structured model with nonlocal diffusion is developed. Using the semigroup generated by the diffusion operator and characteristic method, the problem is reformulated as a fixed point problem in a suitable Banach space. We give conditions under which the model admits a unique positive solution. In the second part of this work, we give an application of the study done in the first part. We consider an optimal control problem. The optimal strategies are discussed using normal cone and dual techniques.

1. Introduction and setting of the problem

Lobesia botrana is one of the major pest that causes economic loss in vineyards around the world. Several control strategies are used to manage the insect: chemical tools, mating disruption, and integrated pest management, see [7, 36]. Insecticides remain the most control method of this pest. In most cases, dispersal ability has been ignored when formulating optimal control problem for Lobesia botrana. Nonlocal diffusion is recognized to describe movement of pests over long distances. Let d be a positive constant. The nonlocal logistic equation

$$\frac{\partial u}{\partial t} = dAu + u(1-u)$$

has been investigated in [11] where the diffusion process is described by the nonlocal dispersal operator A given by

$$A(u)(x) := \int_D J\big(x-y\big)\Big(u(y)-u(x)\Big)\,dy.$$

The function $J(x-y)$ is the rate at which individuals are dispersing from position y to x. The integro-differential equation

$$\frac{\partial u}{\partial t} = dAu + f(x,u)$$

Mathematics subject classification (2020): 35L02, 35L45, 35L50, 49K20, 92D25, 92D40.
Keywords and phrases: Physiologically structured model, nonlocal diffusion, semigroup, fixed point, optimal control.

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has been analyzed in [4, 13, 10]. Recently, age structured population models with nonlocal diffusion have been studied in [25, 24], the authors investigated the model

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}=-\mu(a,t)u+dAu.$$

They obtained a systematical treatment using integrated semigroup approach, and some comparison principles. Among other results, they determined the stability of zero steady state and establish the asymptotic behavior of the model under Dirichlet boundary conditions. Age structured models with Laplace diffusion have been extensively investigated in the literature, see [43] and the references therein. For basic theories involving nonlocal diffusion, we refer the reader to the monograph [4].

Another process that characterizes Lobesia botrana population is mutation. Mutation has received less attention for structured models. At low insecticide rate, individuals are highly stressed. The survivors mutate and develop insecticide resistance. We introduce a continuous phenotype $\omega\in\Omega$ describing the resistance level to insecticides where $\Omega\subset\mathbb{R}^n$ is a bounded domain. At time t, size or physiological age a, position x, and resistance level ω', the pest population with density $u(t,a,\omega',x)$ gives birth to population with resistance level ω at a rate

$$\beta\Big(P(t,x),t,a\Big)\gamma(\omega,\omega')\,u(t,a,\omega',x).$$

Here β is the birth rate, and

$$P(t,x):=\int_{\Omega}\int_0^L u(t,a,\omega,x)\,da\,d\omega,$$

is the total population at time t and position x. The quantity $\gamma(\omega,\omega')$, represents the probability that individuals with trait ω' emerge with a new trait ω. We refer the reader to [17, 19] and the references therein for other models with mutation.

Let $D\subset\mathbb{R}^m$ be a bounded domain with smooth boundary ∂D. Let $u(t,a,\omega,x)$ be the population density at time $t\in[0,T]$, size $a\in[0,L]$, position $x\in D$, an a continuous phenotype $\omega\in\Omega$, where $T>0$ is a given time, and L is the maximal size. In the present work, our attention is focused on the following model,

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,a,\omega,x)+\dfrac{\partial}{\partial a}\left[v(t,a,\omega)u(t,a,\omega,x)\right]\\ =-\mu\Big(P(t,x),t,a\Big)u\big(t,a,\omega,x\big)+dAu\big(t,a,\omega,x\big)\\ v\big(t,a=0,\omega\big)u\big(t,0,\omega,x\big)\\ =\displaystyle\int_0^L\int_{\Omega}\beta\Big(P(t,x),t,s\Big)\gamma(\omega,\omega')u\big(t,s,\omega',x\big)\,d\omega'\,ds\\ u(0,a,\omega,x)=u_0(a,\omega,x),\ \ (a,\omega,x)\in(0,L)\times\Omega\times D.\end{cases}\tag{1.1}$$

We assume that the death μ and the fertility β rates depend on the total population P. The growth rate v depends on time t, size a and ω. The boundary conditions are of nonlocal Neumann type, since in the definition of the operator A, the integral is defined only on D, the individuals may not enter or leave D see for instance [44], and the references therein. The dispersal rate d is constant and strictly positive. Almost all size-structured models have been studied under the assumption that $d=0$, and the newborn individuals do not mutate, see for instance [23,42,35,40,1,2,18,20,37,26,30,31,44,25] and the references therein.

At our knowledge, despite the large amount of the literature on size-structured models, very little results are known for the model (1.1). Well posedness and optimal control of (1.1) remain to be done, whereas the long time behavior of solutions of problem (1.1) is still open.

This work is organized as follows: Preliminaries are given in the next section. Our purpose in Section 3 is to discuss the existence and positivity of solutions of the system (1.1) in L^1. We solve the system by the characteristic method. By employing this approach, we obtain an integral representation of the solution. However, the uknown is still involved implicitely. Then, we define the solution as a fixed point of an integral operator K. By contraction mapping principle, we show that K has a unique fixed point which is the desired solution, see for instance [42] for similar approach. The results extend those obtained in [26]. In Section 4, we consider continuous dependence and comparison principles. As application, in Section 5, we investigate the optimal harvesting control for a size-structured Lobesia botrana population model with nonlocal diffusion. We obtain optimal strategies for managing the pest Lobesia-Botrana. The result extend partially that obtained in [39]. As a development of the study given in this paper, it could be interesting to discuss asymptotic behavior of the model (1.1) and numerical simulations. In a forthcoming work, we shall investigate a model with life stages.

2. Preliminaries

In this section, we introduce notations, assumptions and definitions which are used throughout this note.

Let $L^1 := L^1\Big((0,L)\times D\Big);\mathbb{R})$ be the Banach space of Lebesgue integrable functions with the norm

$$\|u\|_{L^1} := \int_D \int_0^L |u(a,x)|\,dadx.$$

Let $Q=[0,T]\times\Omega$ and let B be the Banach space

$$B = L^\infty\big(Q;\ L^1\big),$$

endowed with the norm

$$\|u\|_T = \sup_Q \|u\|_{L^1}.$$

We also define the Banach space B_0 by

$$B_0 = L^\infty\big(\Omega;\ L^1\big),$$

endowed with the norm

$$\|u\| = \sup_{\Omega} \|u\|_{L^1}.$$

Throughout this article, we require the following assumptions.

(A_J) The function $J(.) \in C\left(\overline{D}\right)$, J is bounded, $J \geqslant 0$, $J \neq 0$, $\int_{\mathbb{R}^N} J(x)dx = 1$, and $J(x) = J(-x)$.

(A_γ) The function $\gamma: \Omega \times \Omega \to \mathbb{R}^n$ is continuous, bounded, and $\gamma \geqslant 0$. In addition, for all $\omega, \omega' \in \Omega$, we assume that

$$\gamma(\omega, \omega') = \gamma(\omega', \omega).$$

(A_{u_0}) The function $u_0 \in L^\infty\Big(\big(0,L\big) \times \Omega \times D\Big)$ is everywhere positive.

(A_v) The function $\big(t,a,\omega\big) \to v\left(t,a,\omega\right)$ is bounded, continuous with respect to its arguments, strictly positive, continuously differentiable with respect to a. In addition, there exists a positive constant L_v such that

$$L_v = \sup_{t,a,w} \left| \frac{\partial v}{\partial a}\big(t,a,\omega\big) \right| < \infty.$$

Further, we assume that $v(.,L,.) = 0$ and

$$V_0 = \min_{0 \leqslant t \leqslant T, \omega \in \Omega} \Big\{ v\big(t,\, 0, \omega\big) \Big\} > 0.$$

(A_β) $\beta\left(P,t,a\right)$ is bounded, and nonnegative measurable function on $\mathbb{R} \times [0,T] \times [0,L]$. In addition, the function β is locally Lipschitz with respect to the first variables P.

(A_μ) $\mu\left(P,t,a\right)$ is bounded, and nonnegative measurable function on $\mathbb{R} \times [0,T] \times [0,L]$. In addition, the function μ is locally Lipschitz with respect to the first variable P.

REMARK 1. The assumptions (A_J) are biologically relevant. The assumption $\int_{\mathbb{R}^N} J(x)dx = 1$ means that individuals are neither created nor destroyed during the movement. The symmetry of J implies that an individual at position x has the same probability of dispersion to position y as an individual in location y has of jumping to x.

Let

$$\begin{cases} G\Big(u\big(t,.,\omega,.\big)\Big)(a,x) = -\mu\Big(P(t,x),t,a\Big)u\big(t,a,\omega,x\big), \\ F\Big(u\big(t,.,\omega,.\big)\Big)\big(x\big) = \displaystyle\int_0^L \int_\Omega \beta\Big(P\big(t,x\big),t,s\Big)\gamma\big(\omega,\omega'\big)u\big(t,s,\omega',x\big)d\omega'\,ds. \end{cases}$$

We have to develop the following lemmas.

LEMMA 1. (i) *There exists a constant* $c_1 > 0$ *such that for* $\varphi_1, \varphi_2 \in B$*, the function*

$$F : B \to L^1(D),$$

satisfies

$$\left|F(\varphi_1) - F(\varphi_2)\right|_{L^1(D)} \leqslant c_1 \|\varphi_1 - \varphi_2\|_T . \tag{2.1}$$

(ii) *There exists a positive constant* $c_2 > 0$ *such that for* $\varphi_1, \varphi_2 \in B$*, the function*

$$G : B \to L^1$$

satisfies

$$\left\|G(\varphi_1) - G(\varphi_2)\right\|_{L^1} \leqslant c_2 \|\varphi_1 - \varphi_2\|_T . \tag{2.2}$$

The proof is trivial, so we omit it.

The concept of a strong solution to system (1.1) requires the differentiability of the functions u and vu respectively with respect to the variables t, and a. This is quite restrictive. Along characteristics the system (1.1) behaves like a Cauchy problem. Based on the semigroup theory, we introduce a concept of mild solution. Let $T(t)$ be the semigroup generated by the bounded operator dA, where d is the dispersal rate. We have

$$T(t) = e^{tdA} := \sum_{n\geqslant 0} (tdA)^n.$$

We have the following properties.

LEMMA 2. (a) $\|T(t)\| \leqslant e^{2td}$,

(b) *the positive cone* $L^1_+(D))$ *is positively invariant by the semigroup defined by* $T(t),\ t \geqslant 0$.

Proof. a) Since A is an integral operator, it follows that

$$\|Au\|_{L^1(D)} \leqslant 2\|u\|_{L^1(D)} .$$

This implies that for all $n \geqslant 1$

$$\|A^n u\|_{L^1(D)} \leqslant 2^n \|u\|_{L^1(D)} ,$$

and

$$\|T(t)\| \leqslant e^{2td}.$$

b) We need to show that if $f \in L^1_+(D)$, then

$$T(t)f \in L^1_+(D),\ \forall t \geqslant 0.$$

Indeed, let $f \in L^1_+((D)$, then there exists a sequence of positive continuous functions with compact support

$$f_n \in C_c(D),$$

such that

$$f_n \to f \qquad \text{in } L^1(D).$$

We will show that $T(t)f_n \geqslant 0$ for $t \geqslant 0$. We follow the proof of proposition 4.2 in [22]. Let $u_n(t,x) = \Big(T(t)f_n\Big)(x)$, then $u_n(t,x)$ satisfies

$$\begin{cases} \frac{\partial u_n(t,x)}{\partial t} = dAu_n(t,x), \\ u_n(0,x) = f_n(x). \end{cases}$$

Note that $v_n(t,x) = e^{\lambda t}u_n(t,x)$ verifies

$$\frac{\partial v_n(t,x)}{\partial t} = d\int_D J(x-y)\Big(v_n(t,y) - v_n(t,x)\Big)\,dy + \lambda v_n(t,x). \tag{2.3}$$

For λ positive and large enough, we have $p_0 := \lambda - d > 0$. Let

$$J_0 = d\max_{x\in D}\int_D J(x-y)dy,$$

and

$$\tau = \frac{1}{p_0 + J_0}.$$

Suppose that for some $x \in D$, and $t \in [0,\tau]$,

$$v_n(t,x) < 0,$$

then there exis $x_1 \in D$, and $t_1 \in [0,\tau]$ such that

$$\min_{x\in D,\ t\in[0,\tau]} v_n(t,x) = v_n(t_1,x_1) < 0.$$

Integrating the equation (2.3) over $[0,t_1]$, we obtain

$$v_n(t_1,x_1) - v_n(0,x_1) \geqslant (p_0 + J_0)\,v_n(t_1,x_1)t_1.$$

Since $v_n(0,x_1) = f_n(x_1) \geqslant 0$ and $t_1 \leqslant \tau$, then

$$v_n(t_1,x_1) \geqslant 0,$$

which a contradiction. It follows that $v_n(t,x) \geqslant 0$ for $x \in D$ and $t \in [0,\tau]$. Hence $u_n(t,x) \geqslant 0$ for $x \in D$ and $t \in [0,\tau]$. Then we repeat the same arguments on the interval $[k\tau,(k+1)\tau]$ for $k = 1,2\ldots$. We conclude that

$$T(t)f_n \geqslant 0,\ \forall t \geqslant 0.$$

Since the linear operator $T(t)$ is bounded, then

$$T(t)f_n \to T(t)f \qquad \text{in } L^1((D),$$

and there exists a subsequence, such that

$$T(t)f_{n_j} \to T(t)f \quad \text{a.e in} \quad D.$$

It follows that

$$T(t)f \geqslant 0 \ \text{ a.e } \ \text{in}\ D\ .$$

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and there exists a subsequence, such that

$$T(t)f_{n_j} \to T(t)f \quad \text{a.e in} \quad D.$$

It follows that

$$T(t)f \geqslant 0 \ \text{ a.e } \ \text{in}\ D. \quad \square$$

We define a characteristic curve $\phi(t;\tau,\eta,\omega)$ through the point (τ,η) as the solution of the equation

$$\begin{cases} \frac{da}{dt} = v(t,a,\omega), \\ a(\tau) = \eta. \end{cases}$$

The function $a = \phi(t;\tau,\eta,\omega)$, is differentiable with respect to τ, and η, see for instance ([3], Chap 2., p. 116, Th. 9.2.). We have

$$\frac{da}{d\tau} = -v(\tau,\eta,\omega)\exp\left(\int_\tau^t \frac{\partial v}{\partial a}\Big(\sigma,\phi(\sigma;\tau,\eta,\omega),\omega\Big)d\sigma\right), \tag{P1}$$

and

$$\frac{da}{d\eta} = \exp\left(\int_\tau^t \frac{\partial v}{\partial a}\Big(\sigma,\phi(\sigma;\tau,\eta,\omega),\omega\Big)d\sigma\right). \tag{P2}$$

We define $z(t,\omega) = \phi(t;0,0,\omega)$, and $\tau = \tau(t,a,\omega)$, implicitely by the relation

$$\phi(t;\tau,0,\omega) = a. \tag{2.4}$$

As in [42], we define a mild solution of system (1.1) as follows

DEFINITION 1. By a *mild solution* to system (1.1), we mean a function $u \in B$ such that $u = K(u)$, where

$$K(u)(t,a,\omega,x) = \begin{cases} T(t-\tau)\dfrac{F\Big(u(\tau,.,\omega,.)\Big)(x)}{v(\tau,0,\omega)} \\ +\displaystyle\int_\tau^t T(t-s)\widetilde{G}\Big(s,u(s,.,.\omega,.)\Big)(\phi(s,\tau,0,\omega),x)ds & \text{if } a < z(t,\omega), \\ T(t)u_0\Big(\phi(0,t,a,\omega),\omega,x\Big) \\ +\displaystyle\int_0^t T(t-s)\widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s,t,a,\omega),x)ds & \text{if } a \geqslant z(t,\omega). \end{cases}$$

The definition of the mild solution is justified as follows. Let u be a solution of system (1.1), and define

$$U_{t_0,a_0}(t,\omega,x) := u\Big(t,\phi(t;t_0,a_0,\omega),\omega,x\Big),$$

then U_{t_0,a_0} satisfies

$$\frac{dU_{t_0,a_0}(t,\omega,x)}{dt} = \frac{\partial u\Big(t,\phi(t;t_0,a_0,\omega),\omega,x\Big)}{\partial t} + \frac{\partial u\Big(t,\phi(t;t_0,a_0,\omega),\omega,x\Big)}{\partial a} v\Big(t,\phi(t;t_0,a_0,\omega),\omega\Big),$$

and

$$\frac{dU_{t_0,a_0}(t,\omega,x)}{dt} = \widetilde{G}\Big(t,u(t,.,\omega,.)\Big)(\phi(t;t_0,a_0,\omega),x) + dAU_{t_0,a_0}(t,\omega,x)$$

where

$$\widetilde{G}\Big(t,u(t,.\omega,.)\Big)(a,x) = G\Big(u(t,.\omega,.)\Big)(a,x) - \frac{\partial v(t,a,\omega)}{\partial a} u(t,a,\omega,x).$$

Let $T(t)$ be the semigoup generated by the dispersal operator dA, then

$$\begin{aligned} U_{t_0,a_0}(t,\omega,x) &= T(t-\tau_0^*)U_{t_0,a_0}(\tau_0^*,\omega,x) \\ &\quad + \int_{\tau_0^*}^t T(t-s)\widetilde{G}\Big(s,u(s,.\omega,.)\Big)(\phi(s;t_0,a_0,\omega),x)\,ds, \end{aligned} \tag{2.5}$$

where $\tau_0^* \in [0,T]$ is an initial time defined by

$$\tau_0^* := \tau_0^*(t_0,a_0,\omega) = \begin{cases} \tau(t_0,a_0,\omega), & \text{if } a_0 < z(t_0,\omega), \\ 0 & \text{if } a_0 \geqslant z(t_0,\omega). \end{cases}$$

We distinguish two cases.

i) If

$$a_0 < z(t_0,\omega)$$

then

$$\tau_0 := \tau(t_0,a_0,\omega) > 0,$$

and we consider equation (2.5) with initial time $\tau_0^* = \tau_0$. Using equation (2.4), we obtain

$$\begin{aligned} &U_{t_0,a_0}(t,\omega,x) \\ &= U_{\tau_0,0}(t,\omega,x) \\ &= T(t-\tau_0)U_{\tau_0,0}(\tau_0,\omega,x) + \int_{\tau_0}^t T(t-s)\widetilde{G}\Big(s,u(s.,\omega,.)\Big)(\phi(s;\tau_0,0,\omega),x)\,ds. \end{aligned}$$

since $U_{\tau_0,0}(\tau_0,\omega,x) = u(\tau_0,\phi(\tau_0;\tau_0,0,\omega),\omega,x) = u(\tau_0,0,\omega,x)$, this gives that

$$\begin{aligned} &U_{\tau_0,0}(t,\omega,x) \\ &= T(t-\tau_0)u(\tau_0,0,\omega,x) + \int_{\tau_0}^t T(t-s)\widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s;\tau_0,0,\omega),x)\,ds. \end{aligned}$$

In particular, we have

$$\begin{aligned}U_{t,a}(t,\omega,x) &= u(t,\phi(t;t,a,\omega),\omega,x) = u(t,a,\omega,x)\\&= T(t-\tau)\,u(\tau,0,\omega,x)\\&\quad+\int_{\tau}^{t}T(t-s)\,\widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s;\tau,0,\omega),x)\,ds.\end{aligned}$$

ii) If

$$a_0 \geqslant z(t_0,\omega)$$

then $\tau_0 \leqslant 0$, and we consider the equation (2.5) with initial time $\tau_0^* = 0$. Hence

$$\begin{aligned}U_{t_0,a_0}(t,\omega,x) &= T(t)\,U_{t_0,a_0}(0,\omega,x)\\&\quad+\int_{0}^{t}T(t-s)\,\widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s;t_0,a_0,\omega),x)\,ds.\end{aligned}$$

This leads to

$$\begin{aligned}U_{t_0,a_0}(t,\omega,x) &= T(t)\,u_0(\phi(0,t_0,a_0,\omega),\omega,x)\\&\quad+\int_{0}^{t}T(t-s)\,\widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s;t_0,a_0,\omega),x)\,ds.\end{aligned}$$

In particular, we have

$$\begin{aligned}U_{t,a}(t,\omega,x) &= u(t,\phi(t;t,a,\omega),\omega,x) = u(t,a,\omega,x)\\&= T(t)\,u_0(\phi(0,t,a,\omega),\omega,x)\\&\quad+\int_{0}^{t}T(t-s)\,\widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s;t,a,\omega),x)\,ds.\end{aligned}$$

This justifies the definition of a mild solution via characterestics and semigroup.

3. Existence of positive solutions

It is important to ensure that the model (1.1) is well posed. As in [26], the following result shows that the mild solutions satisfy system (1.1) along characteristic curves. Let u be a mild solution of the system (1.1), i.e $u \in B$ and u=K(u).

LEMMA 3. *For fixed* $(t,a,\omega,x) \in (0,T)\times(0,L)\times\Omega\times D$, *the function*

$$U(s) = u\Big(s,\phi(s;t,a,\omega),\omega,x\Big),$$

is differentiable a.e. on (τ^*,T) *and satisfies*

$$\frac{dU(s)}{ds} = \widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s;t,a,\omega),x) + dAU(s),$$

where $\tau^* := \tau_0^*(t,a,\omega)$.

Proof. We distinguish two cases,

i) if $\phi(s,t,a,\omega) \in (0, z(t,\omega))$, then

$$\frac{1}{h}\left[Ku\Big(s+h, \phi(s+h,t,a,\omega), \omega, x\Big) - Ku\Big(s, \phi(s,t,a,\omega), \omega, x\Big)\right] = Q_1^h + Q_2^h,$$

where

$$\begin{aligned} Q_1^h &= \frac{1}{h}\left[T(s+h-\tau) - T(s-\tau)\right] \frac{F\Big(u(\tau,.,\omega,.)\Big)(x)}{v(\tau,\ 0,\omega)} \\ &= \frac{1}{h}\left[T(h) - 1\right] T(s-\tau) \frac{F\Big(u(\tau,.,\omega,.)\Big)(x)}{v(\tau,0,\omega)}. \end{aligned}$$

Since the semigroup $T(t)$ is differentiable, then

$$Q_1^h \to dAT(s-\tau)\frac{F\Big(u(\tau,.,\omega,.)\Big)(x)}{v(\tau,0,\omega)},$$

and

$$\begin{aligned} Q_2^h = \frac{1}{h}\Bigg[&\int_\tau^{s+h} T(s+h-\eta)\Big(\widetilde{G}\Big(\eta, u(\eta,.,\omega,.)\Big)(\phi(\eta,\tau,0,\omega),x)\Big)d\eta \\ &- \int_\tau^{s} T(s-\eta)\Big(\widetilde{G}\Big(\eta, u(\eta,.,\omega,.)\Big)(\phi(\eta,\tau,0,\omega),x)\Big)d\eta\Bigg], \end{aligned}$$

which implies that,

$$\begin{aligned} Q_2^h = \frac{1}{h}\left[T(h) - 1\right] &\int_\tau^{s} T(s-\eta)\Big(\widetilde{G}\Big(\eta, u(\eta,.,\omega,.)\Big)(\phi(\eta,\tau,0,\omega)),x\Big)d\eta \\ &+ \frac{T(h)}{h}\int_s^{s+h} T(s-\eta)\Big(\widetilde{G}\Big(\eta, u(\eta,.,\omega,.)\Big)(\phi(\eta,\tau,0,\omega)),x\Big)d\eta, \end{aligned}$$

then

$$\begin{aligned} Q_2^h \to\ &\widetilde{G}\Big(s, u(s,.,\omega,.)\Big)(\phi(s,\tau,0,\omega),x) \\ &+ dA\int_\tau^{s} T(s-\eta)\Big(\widetilde{G}\Big(\eta, u(\eta,.,\omega,.)\Big)(\phi(\eta,\tau,0,\omega),x)\Big)d\eta. \end{aligned}$$

Since $\phi(s,\tau,0,\omega) = \phi(s,t,a,\omega)$, it follows that when h goes to zero

$$\begin{aligned} Q_1^h + Q_2^h \to \widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s,t,a,\omega),x) \\ + dAKu\Big(s,\phi(s,t,a,\omega),\omega,x\Big) \\ = \widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s,t,a,\omega),x) + dAU(s). \end{aligned}$$

In a similar manner, we consider the case where $\phi(s,t,a,\omega) \in (z(t,\omega),L)$. $\square$

For $\alpha \in R$, and $u \in B$, we define the operators $K_\alpha(u)$ as follows

$$K_\alpha(u)(t,a,\omega,x) = \begin{cases} T(t-\tau)e^{-\alpha(t-\tau)}\dfrac{F\Big(u(\tau,.,.,)\Big)(x)}{v(\tau,0,\omega)} + \\ \displaystyle\int_\tau^t T(t-s)\left(\widetilde{G}\Big(s,u(s,.,\omega,.)+\alpha I\Big)(\phi(s,\tau,.0,\omega),x)e^{-\alpha(t-s)}\right)ds \\ \qquad \text{if } a < z(t,\omega), \\ T(t)e^{-\alpha t}u_0\Big(\phi(0,t,a,x),\omega,x\Big) + \\ \displaystyle\int_0^t T(t-s)\left(\widetilde{G}\Big(s,u(s,.,\omega,.)+\alpha I\Big)(\phi(s,t,a,\omega),x)e^{-\alpha(t-s)}\right)ds \\ \qquad \text{if } a \geqslant z(t,\omega). \end{cases}$$

LEMMA 4. *Let $u \in B$. For fixed $(t,a,\omega,x) \in (0,T)\times(0,L)\times\Omega\times D$, the function*

$$w_\alpha(s) := K_\alpha(u)(s,\phi(s,t,a,\omega),\omega,x),$$

is differentiable a.e. on (τ^*,T), and satisfies

$$\frac{d}{ds}w_\alpha(s) = -\alpha w_\alpha(s) + \left(\widetilde{G}\Big(s,u(s,.,\omega)+\alpha I\Big)\right)(\phi(s,t,a,x),x) + dAw_\alpha(s)\ .$$

The proof is similar to that given in Lemma 3, so we omit it.

LEMMA 5. *Let $\alpha,\beta \in \mathbb{R}$, and $u \in B$. Then*

$$\begin{aligned} &K_\beta u\Big(s,\phi(s,t,a,\omega),\omega,x\Big) \\ &= K_\alpha u\Big(s,\phi(s,t,a,\omega),\omega,x\Big) \\ &\quad + (\alpha-\beta)\int_{\tau^*}^t T(t-\eta)e^{-\beta(t-\eta)}(K_\alpha u - u)\Big(\eta,\phi(\eta,t,a,\omega),\omega,x\Big)d\eta. \end{aligned}$$

Proof. We have

$$\begin{aligned} \frac{d}{ds}(w_\beta - w_\alpha) = -\beta(w_\beta - w_\alpha) + (\alpha-\beta)\Big[w_\alpha - u\Big(s,\phi(s,t,a,x),\omega,x\Big)\Big] \\ + dA(w_\beta - w_\alpha). \end{aligned}$$

This gives that

$$\frac{d}{ds}\left(e^{\beta s}\left(w_\beta - w_\alpha\right)\right) = e^{\beta s}\left(\alpha-\beta\right)\left[w_\alpha - u\Big(s,\phi\left(s,t,a,\omega\right),\omega,\phi\Big)\right] + dAe^{\beta s}\left(w_\beta - w_\alpha\right),$$

and

$$\left(w_\beta - w_\alpha\right)(t) = \left(\alpha-\beta\right)\int_{\tau^*}^{t} T\left(s-\eta\right)e^{-\beta(t-\eta)}\left[w_\alpha(\eta) - u\Big(\eta,\phi\left(\eta,t,a,\omega\right),\omega,x\Big)\right]d\eta.$$

□

Let $\alpha,\beta \in \mathbb{R}$ and $u \in B$, then

COROLLARY 1. *$K_\alpha(u) = u$ implies that $K_\beta(u) = u$.*

Let L^1_+ be the positive cone of L^1, and let

$$B^+ = L^\infty\left(Q,\ L^1_+\right).$$

THEOREM 1. *Let $u_0 \in B^+_0$. Under conditions (A_J)–(A_μ), the problem (1.1) has a unique solution $u \in B^+$.*

Proof. Let

$$\alpha = \|\mu\|_\infty + L_\nu.$$

a) *First step*: We show that K_α is a map from B to itself. For simplicity of notations, we put

$$\widetilde{G}_\alpha = \widetilde{G} + \alpha.$$

Let $u \in B$. Following [2], we have

$$\int_D \int_0^L \left|K_\alpha(u)\left(t,a,\omega,x\right)\right| dadx \leqslant J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 = \int_D \int_0^{z(t,\omega)} \left| T\left(t-\tau\right)\frac{F\Big(u\left(\tau,.,\omega,.\right)\Big)(x)}{v\left(\tau,0,\omega\right)}\right| dadx,$$

$$J_2 = \int_D \int_0^{z(t,\omega)} \int_\tau^t \left| T\left(t-s\right)\widetilde{G}_\alpha\Big(u\left(s,.,\omega,.\right)\Big)\left(\phi\left(s,\tau,0,\omega\right),x\right)\right| dsdadx,$$

$$J_3 = \int_D \int_{z(t,\omega)}^L \left| T(t)u_0\Big(\phi\left(0,t,a,\omega\right),\omega,x\Big)\right| dadx,$$

$$J_4 = \int_D \int_{z(t,\omega)}^L \int_0^t \left| T\left(t-s\right)\widetilde{G}_\alpha\Big(u\left(s,.,\omega,.\right)\Big)\left(\phi\left(s,t,a,\omega\right),x\right)\right| dsdadx.$$

To estimate J_1, we make the change of variables from a to τ by the relation $\tau = \tau(t,a,\omega)$. It follows from (P1) that

$$\begin{aligned}
J_1 &= \int_D \int_0^{z(t,\omega)} \left| T(t-\tau) \frac{F\Big(u(\tau,.,\omega,.)\Big)(x)}{v(\tau,0,\omega)} \right| dadx \\
&= \int_0^{z(t,\omega)} \left\| T(t-\tau) F\Big(u(\tau,.,\omega,.)\Big) \right\|_{L^1(D)} \frac{1}{v(\tau,0,\omega)} da \\
&\leqslant \int_0^t \left\| T(t-\tau) F\Big(u(\tau,.,\omega,.)\Big) \right\|_{L^1(D)} \left(\exp \int_\tau^t \frac{\partial v}{\partial a}\Big(s, \phi(s,\tau,0,\omega), \omega\Big) ds \right) d\tau \\
&\leqslant e^{TL_v} \int_0^t \left\| T(t-\tau) F\Big(u(\tau,.,\omega,.)\Big) \right\|_{L^1(D)} d\tau \\
&\leqslant e^{T(2d+L_v)} \int_0^t \left\| F\Big(u(\tau,.,\omega,.)\Big) \right\|_{L^1(D)} d\tau.
\end{aligned}$$

By Lemma 1, since $F(0) = 0$ it follows that

$$\begin{aligned}
J_1 &\leqslant e^{T(2d+L_v)} \int_0^t \left\| F\Big(u(\tau,.,\omega,.)\Big) - F(0) \right\|_{L^1(D)} d\tau \\
&\leqslant e^{T(2d+L_v)} \int_0^t c_1 \|u\|_T \, d\tau \leqslant e^{T(2d+L_v)} c_1 \|u\|_T \, T.
\end{aligned}$$

Similarly to estimate $J_2 + J_4$, we make the change of variables

$$\eta = \phi(s,t,a,\omega) = \phi(s,\tau,0,\omega),$$

then

$$\begin{aligned}
J_2 + J_4 &\leqslant e^{TL_V} \Bigg\{ \int_D \int_0^t \int_\tau^{z(t,\omega)} \left| T(t-s) \widetilde{G}_\alpha \Big(u(s,.,\omega,.)\Big)(\eta,x) \right| d\eta ds dx \\
&\quad + \int_D \int_0^t \int_{z(t,\omega)}^L \left| T(t-s) \widetilde{G}_\alpha \Big(u(s,.,\omega,.)\Big)(\eta,x) \right| d\eta ds dx \Bigg\} \\
&\leqslant e^{T(2d+L_v)} \Bigg\{ \int_0^t \int_0^L \int_D \left| G\Big(u(s,.,\omega,.)\Big)(\eta,x) \right| dx d\eta ds \\
&\quad + \int_0^t \int_0^L \int_D \left[\left| \frac{\partial v}{\partial a}(s,\eta,\omega) + \alpha \right] u(s,\eta,\omega,x) \right| dx d\eta ds \Bigg\}.
\end{aligned}$$

From Lemma 1, it follows that

$$\int_D \int_0^L \left| G\Big(u(s,.,\omega,.)\Big)(\eta,x)\right| d\eta dx = \left\| G\Big(u(s,.,\omega,.)\Big)\right\|_{L^1}$$
$$= \|G(u(s,.,\omega,.)) - G(0)\|_{L^1}$$
$$\leqslant \|u\|_T c_2$$

and

$$\int_D \int_0^L \left| \frac{\partial v}{\partial a}(s,\eta,\omega)\, u(s,\eta,\omega,x)\right| d\eta dx \leqslant \|u\|_T L_v.$$

Therefore, we obtain that

$$J_2 + J_4 \leqslant e^{T(2d+L_v)}\left\{c_2 + \alpha + L_v\right\} T \|u\|_T .$$

To estimate J_3, we use the change of variables

$$\zeta = \phi(0,t,a,\omega),$$

this gives that

$$J_3 \leqslant e^{T(2d+L_v)} \int_0^L \int_D |u_0(\zeta,\omega,x)|\, dx d\zeta \leqslant \|u_0\|\, e^{T(2d+L_v)}.$$

Hence

$$J_1 + J_2 + J_3 + J_4 \leqslant \|u_0\|\, e^{T(2d+L_v)} + e^{T(2d+L_v)}\left\{\alpha + c_2 + L_v + c_1\right\} \|u\|_T T.$$

This shows that

$$K_\alpha(u) \in B.$$

b) *The second step*: We introduce an equivalent norm

$$\|u\|_\lambda = sup_{t\in[0,T]} e^{-\lambda t} \|u\|_T ,$$

where $\lambda > 0$ is a constant. Let $u_1, u_2 \in B$, then

$$\int_D \int_0^L \left|K_\alpha(u_1) - K_\alpha(u_2)\right| da dx \leqslant P_1 + P_2 + P_3,$$

where

$$P_1 = \int_0^{z(t,\omega)} \int_D \frac{\left|T(t-\tau)\left(F(u_1(\tau,.,\omega,.))(x) - F\big(u_2(\tau,.,\omega,.)\big)(x)\right)\right|}{v(\tau,0,\omega)} dx da,$$

and

$$P_2 = \int\limits_0^{z(t,\omega)} \int\limits_\tau^t \int\limits_D |S_1(s,t,x,u_1,u_2)|\, dx\, ds\, da,$$

$$P_3 = \int\limits_{z(t,\omega)}^{L} \int\limits_0^t \int\limits_D |S_2(s,t,x,u_1,u_2)|\, dx\, ds\, da.$$

Here

$$\begin{aligned} S_1(s,t,x,u_1,u_2) = {} & T(t-s)\left(\widetilde{G}_\alpha\left(s,u_1(s,.,\omega,.)\right)(\phi(s,\tau,0,\omega),x)\right) \\ & -T(t-s)\left(\widetilde{G}_\alpha\left(s,u_2(s,.,\omega,.)\right)(\phi(s,\tau,0,\omega),x)\right), \end{aligned}$$

and

$$\begin{aligned} S_2(s,t,x,u_1,u_2) = {} & T(t-s)\left(\widetilde{G}_\alpha(s,u_1(s,.,\omega,.))\left(\phi\left(s,t,a,\omega\right),x\right)\right) \\ & -T(t-s)\left(\widetilde{G}_\alpha(s,u_2(s,.,\omega,.))\left(\phi(s,t,a,\omega),x\right)\right). \end{aligned}$$

Note that

$$\begin{aligned} P_1 & \leqslant e^{TL_v} \int\limits_0^t \left\| T(t-\tau)\left(F\left(u_1(\tau,.,\omega,.)\right)(x) - F\left(u_2(\tau,.,\omega,.)\right)(x)\right)\right\|_{L^1(D)} d\tau \\ & \leqslant e^{T(2d+L_v)} \|\gamma\|_\infty \|\beta\|_\infty \int\limits_0^t \int\limits_\Omega \left\|u_1(\tau,.,.,\omega') - u_2(\tau,.,.,\omega')\right\|_{L^1} d\omega' d\tau. \end{aligned}$$

This gives that

$$\begin{aligned} e^{-\lambda t} P_1 \leqslant {} & e^{T(2d+L_v)} \|\gamma\|_\infty \|\beta\|_\infty e^{-\lambda t} \\ & \times \int\limits_\Omega \int\limits_0^t e^{-\lambda\tau} e^{\lambda\tau} \left\|u_1(\tau,.,.,\omega') - u_2(\tau,.,.,\omega')\right\|_{L^1} d\tau d\omega', \end{aligned}$$

and

$$e^{-\lambda t} P_1 \leqslant e^{T(2d+L_v)} |\Omega| \|\gamma\|_\infty \|\beta\|_\infty \frac{1}{\lambda} \|u_1 - u_2\|_\lambda,$$

where $|\Omega|$ denotes the measure of the set Ω. Using the change of variables

$$\eta = \phi(s,t,a,\omega) = \phi(s,\tau,0,\omega),$$

we obtain that

$$
\begin{aligned}
& P_2+P_3 \\
\leqslant & \int_0^L \int_0^t \int_D \left| T(t-s) G\Big(s, u_1(s,.,\omega,.)\Big)(\eta,x) - G\Big(s, u_2(s,.,\omega,.)\Big)(\eta,x) \right| dx\, d\eta\, da \\
& + \int_0^L \int_0^t \int_D \left| T(t-s)\left[\frac{\partial v}{\partial a}(t,\eta,\omega)+\alpha\right](u_1(s,\eta,\omega,x)-u_2(s,\eta,\omega,x)) \right| dx\, d\eta\, da.
\end{aligned}
$$

This implies that

$$
e^{-\lambda t}(P_2+P_3) \leqslant e^{T(2d+L_v)}\left(\alpha+L_v+\|\mu\|_\infty\right)\frac{1}{\lambda}\|u_1-u_2\|_\lambda .
$$

It follows that

$$
\|K_\alpha(u_1)-K_\alpha(u_2)\|_\lambda \leqslant C_T \frac{1}{\lambda}\|u_1-u_2\|_\lambda ,
$$

where

$$
C_T = e^{T(2d+L_v)}\left(\alpha+L_v+\|\mu\|_\infty+|\Omega|\,\|\gamma\|_\infty\|\beta\|_\infty\right).
$$

Choosing $\lambda > C_T$ gives that K_α is a contraction on the Banach space $(B, \|.\|_\lambda)$, and K_α has a unique fixed point $u \in B$.

c) *Third step*. It is clear that for $u \in B^+$, $K_\alpha u \in B^+$. By corollary 1, it follows that

$$
K_0(u) = u,
$$

and u is a positive solution of system (1.1). □

REMARK 2. Theorem 1 remains valid for function F and G having more general forms, provided that they satisfy some general properties such as Lipschitz conditions and assumptions to ensure that $K_\alpha(B^+) \subset B^+$.

4. Comparison principle and continuous dependence

We prove the following auxiliary theorems which are useful in proving our results of optimal control. Let

$$
\tilde{D_T} = [0,T]\times[0,L]\times\Omega\times D.
$$

Assume that $(\mathbf{A}_f)$ $f \in B$, and consider the following problem

$$
\begin{cases}
\dfrac{\partial}{\partial t}u(t,a,\omega,x)+\dfrac{\partial}{\partial a}\Big[v\big(t,a,\omega\big)u\big(t,a,\omega,x\big)\Big] \\
= G\Big(u\big(t,.,\omega,.\big)\Big)(a,x)+dAu, \\
v\big(t,a=0,\omega\big)u\big(t,0,\omega,x\big) = F\Big(u(t,.,\omega,.)\Big)\big(x\big), \\
u\big(0,a,\omega,x\big) = u_0\big(a,\omega,x\big),\ \ \big(a,\omega,x\big)\in\big(0,L\big)\times\Omega\times D,
\end{cases}
\tag{4.1}
$$

where

$$\begin{cases} G\Big(u(t,.,\omega,.)\Big)(a,x) = -\mu\Big(P(t,\omega),t,a\Big)u(t,a,\omega,x) + f(t,a,\omega,x), \\ F\Big(u(t,.,\omega,.)\Big)(x) = \int_\Omega \int_0^L \beta\Big(P(t,\omega),t,s\Big)\gamma(\omega,\omega')u(t,s,\omega',x)dsd\omega'. \end{cases}$$

THEOREM 2. *If* μ_i, β_i, γ_i, u_{i0} *and* f_i *satisfy* (A_μ), (A_β), (A_γ), (A_{u_0}), *and* (A_f) *respectively with* $f_1 \leqslant f_2$, $\beta_1 \leqslant \beta_2$, $\gamma_1 \leqslant \gamma_2$, $u_{10} \leqslant u_{20}$, *and* $\mu_1 \geqslant \mu_2$ *then*

$$u^1(t,a,\omega,x) \leqslant u^2(t,a,\omega,x) \text{ a.e on } \tilde{D_T},$$

where u^i *is the solution to* (4.1) *corresponding to* $\beta = \beta_i$, $\gamma = \gamma_i$ $\mu = \mu_i$, $f = f_i$, *and* $u_0 = u_{i0}$.

Proof. Let

$$\alpha = \|\mu_1\|_\infty + \|\mu_2\|_\infty + L_v,$$

and

$$K_{k,\alpha}(u)(t,a,\omega,x) = \begin{cases} T(t-\tau)e^{-\alpha(t-\tau)}\dfrac{F_k\Big(u(\tau,.,.,)\Big)(x)}{v(\tau,0,\omega)} + \\ \displaystyle\int_\tau^t T(t-s)\Big(\widetilde{G}_k\Big(s,u(s,.,\omega,.)+\alpha I\Big)(\phi(s,\tau,.0,\omega),x)e^{-\alpha(t-s)}\Big)ds \\ \qquad \text{if } a < z(t,\omega), \\ T(t)e^{-\alpha t}u_{k0}\Big(\phi(0,t,a,x),\omega,x\Big) + \\ \displaystyle\int_0^t T(t-s)\Big(\widetilde{G}_k\Big(s,u(s,.,\omega,.)+\alpha I\Big)(\phi(s,t,a,\omega),x)e^{-\alpha(t-s)}\Big)ds \\ \qquad \text{if } a \geqslant z(t,\omega), \end{cases}$$

Similarly as in the proof of Theorem 1, for $k = 1,2$, the opertor $K_{k,\alpha}$ has a fixed point u_k in B^+. Hence

$$K_{1,\alpha}(u_1) = u_1, \text{ and } K_{2,\alpha}(u_2) = u_2.$$

Since for $u \in B^+$, we have

$$\begin{aligned} G_1\Big(u(t,.,\omega,.)\Big)(a,x) &= -\mu_1\Big(P(t,\omega),t,a\Big)u(t,a,\omega,x) + f_1(t,a,\omega,x) \\ &\leqslant -\mu_2\Big(P(t,\omega),t,a\Big)u(t,a,\omega,x) + f_2(t,a,\omega,x) \\ &= G_2\Big(u(t,.,\omega,.)\Big)(a,x), \end{aligned}$$

and

$$F_1\Big(u(t,.,\omega,.)\Big)(x) \leqslant F_2\Big(u(t,.,\omega,.)\Big)(x),$$

then

$$u_1 = K_{1,\alpha}\big(u_1\big) \leqslant K_{2,\alpha}\big(u_1\big).$$

Moreover, the monotony of $K_{2,\alpha}$ with respect to $u \in B^+$ implies that

$$u_1 \leqslant K_{2,\alpha}\big(u_1\big) \leqslant K_{2,\alpha}\Big(K_{2,\alpha}\big(u_1\big)\Big),$$

this leads to

$$u_1 \leqslant K^2_{2,\alpha}\big(u_1\big).$$

By induction, we obtain for each $j \in \mathbb{N}$, and $j \geqslant 2$

$$u_1 \leqslant K^2_{2,\alpha}(u_1) \leqslant \ldots \leqslant K^j_{2,\alpha}(u_1).$$

Since B^+ is a normal cone, see [33], then

$$u_1 \leqslant \lim_{j\to\infty} K^j_{2,\alpha}\big(u_1\big) = u_2. \quad \square$$

We will analyze the continuous dependence of the solution of the system (4.1) with respect to the function f.

We assume that $(\mathbf{A}'_f)$ $f_n(t,a,\omega,x), f(t,a,\omega,x) \in B$ and

$$\|f_n - f\|_T = \sup_Q \|f_n - f\|_{L^1} \to 0, \text{ as} \, n \to \infty.$$

THEOREM 3. *Let u be a solution of system (4.1), and let u_n be a solution a system (4.1) with f replaced by f_n, If conditions (A'_f) holds, then*

$$u_n \to u \text{ as } n \to \infty \text{ in } B.$$

Proof. Let G_n be the function G with f replaced by f_n. We have

$$\|u_n(t,.,\omega,.) - u(t,.,\omega,.)\|_{L^1} \leqslant \Sigma_1 + \Sigma_2 + \Sigma_3$$

where

$$\Sigma_1 = \int_0^{z(t,\omega)} \int\limits_D \left| \frac{T\big(t-\tau\big)F\Big(u_n\big(\tau,.,\omega,.\big)\Big)\big(x\big)}{v\big(\tau,0,\omega\big)} - \frac{T\big(t-\tau\big)F\Big(u\big(\tau,.,\omega,.\big)\Big)\big(x\big)}{v\big(\tau,0,\omega\big)} \right| dxda,$$

$$\Sigma_2 = \int\limits_D \int_0^{z(t,\omega)} \int\limits_\tau^t \big|S_1\big(s,t,\omega,x\big)\big|\, dxdsda,$$

$$\Sigma_3 = \int\limits_D \int_{z(t,\omega)}^L \int\limits_0^t \big|S_2\big(s,t,\omega,x\big)\big|\, dxdsda,$$

with

$$S_1 = T(t-s)\Big(\widetilde{G}_n\big(s, u_n(s,.,\omega,.)\big)(\phi(s,\tau,0,\omega),x)\Big)$$
$$- T(t-s)\Big(\widetilde{G}\big(s, u(s,.,\omega,.)\big)(\phi(s,\tau,0,\omega),x)\Big),$$

and

$$S_2 = T(t-s)\Big(\widetilde{G}_n\big(s, u_n(s,.,\omega,.)\big)(\phi(s,t,a,\omega),x)\Big)$$
$$- T(t-s)\Big(\widetilde{G}\big(s, u(s,.,\omega,.)\big)(\phi(s,t,a,\omega),x)\Big).$$

By the change of variables $\tau = \tau(t,a,\omega)$, it follows that

$$\Sigma_1 \leqslant \frac{1}{V_0} e^{T(L_v+2d)} \|\beta\|_\infty \|\gamma\|_\infty |\Omega| \int_0^t \|u(\tau,.,\omega,.) - u_n(\tau,.,\omega,.)\|_{L^1} d\tau.$$

where $|\Omega|$ denotes the measure of the set Ω.

Similarly, we divide Σ_2 into two parts, we obtain

$$\Sigma_2 \leqslant \Sigma_2^1 + \Sigma_2^2,$$

where Σ_2^1 satisfies

$$\Sigma_2^1 \leqslant \int_D \int_0^{z(t,\omega)} \int_\tau^t \left|S_1^1(s,t,\omega,x)\right| dx ds da,$$

with

$$S_1^1(s,t,\omega,x) = T(t-s)\Big(\widetilde{G}\big(s, u(s,.,\omega,.)\big)(\phi(s,\tau,0,\omega),x)$$
$$- \widetilde{G}_n\big(s, u(s,.,\omega,.)\big)(\phi(s,\tau,0,\omega),x)\Big),$$

this implies that

$$\Sigma_2^1 \leqslant e^{2dT} \int_0^T \|f_n(s,.\omega,.) - f(s,.\omega,.)\|_{L^1} ds \leqslant e^{2dT} T \|f_n - f\|_T \to 0 \text{ as } n \to \infty.$$

The second term Σ_2^2 satisfies

$$\Sigma_2^2 \leqslant \int_D \int_0^{z(t,\omega)} \int_\tau^t \left|S_1^2(s,t,\omega,x)\right| dx ds da,$$

where

$$S_1^2(s,t,\omega,x) = T(t-s)\left(\widetilde{G}_n\Big(s,u(s,.,\omega,.)\Big)(\phi(s,\tau,0,\omega),x)\right.$$
$$\left. - \widetilde{G}_n\Big(s,u_n(s,.,\omega,.)\Big)(\phi(s,\tau,0,\omega),x)\right),$$

with the change of variables $\phi(s,\tau,0,\omega) = \eta$, we obtain that

$$\Sigma_2^2 \leqslant e^{2dT}(\|\mu\|_\infty + L_v)\int_0^t \|u_n(\eta,.,\omega,.) - u(\eta,.,\omega,.)\|_{L^1} d\eta,$$

hence

$$\Sigma_2 \leqslant \delta_n^1 + e^{2dT}(\|\mu\|_\infty + L_v)\int_0^t \|u_n(\eta,.,\omega,.) - u(\eta,.,\omega,.)\|_{L^1} d\eta.$$

We divide Σ_3 into two parts. We have

$$\Sigma_3 \leqslant \Sigma_3^1 + \Sigma_3^2,$$

where

$$\Sigma_3^1 \leqslant \int_D \int_{z(t,\omega)}^L \int_0^t \left|S_2^1(s,t,\omega,x)\right| dsdadx,$$

with

$$S_2^1(s,t,\omega,x) = T(t-s)\left(\widetilde{G}\Big(s,u(s,.,\omega,.)\Big)(\phi(s,t,a,\omega),x)\right)$$
$$- T(t-s)\left(\widetilde{G}_n\Big(s,u(s,.,\omega,.)\Big)(\phi(s,t,a,\omega),x)\right),$$

hence

$$\Sigma_3^1 \leqslant e^{2dT}\int_0^T \|f_n(s,.,\omega,.) - f(s,.,\omega,.)\|_{L^1} ds \leqslant Te^{2dT}\|f_n - f\|_T \to 0 \text{ as as } n \to \infty.$$

The quantity Σ_3^2 verifies

$$\Sigma_3^2 \leqslant \int_D \int_{z(t,\omega)}^L \int_0^t \left|S_2^2(s,t,\omega,x)\right| dsdadx,$$

where

$$S_2^2(s,t,\omega,x) = T(t-s)\left(\widetilde{G}_n\Big(s,u(s,.,\omega,.)\Big)(\phi(s,t,a,\omega),x)\right)$$
$$- T(t-s)\left(\widetilde{G}_n\Big(s,u_n(s,.,\omega,.)\Big)(\phi(s,t,a,\omega),x)\right).$$

By using the change of variables $\phi(s,t,a,\omega) = \eta$, we will have

$$\Sigma_3^2 \leqslant e^{2dT}\left(L_v + \|\mu\|_\infty\right)\int_0^t \|u_n(\eta,.,\omega,.) - u(\eta,.,\omega,.)\|_{L^1}\, d\eta.$$

We summarize the estimates as follows

$$\begin{aligned}\|u_n(t,.,\omega,.) - u(t,.,\omega,.)\|_{L^1} \leqslant\ & 2e^{2dT}T\,\|f_n - f\|_T \\ & + C\int_0^t \|u(\eta,.,\omega,.) - u_n(\eta,.,\omega,.)\|_{L^1}\, d\eta,\end{aligned}$$

for some positive constant C independent of n. Gronwall's Lemma implies that

$$\|u(t,.,\omega,.) - u_n(t,.,\omega,.)\|_{L^1} \leqslant (2e^{2dT}T\,\|f_n - f\|_T)e^{CT}.$$

The right-hand side of the previous inequality is independent of t and ω, this yields that

$$\|u - u_n\|_T \leqslant \left(2e^{2dT}T\,\|f_n - f\|_T\right)e^{CT},$$

and by letting n goes to ∞, we obtain the desired result. □

5. Application to optimal control

Optimal control for age structured model was first proposed in [15]. Then age structured models with single equation were investigated in [5, 9, 38, 14]. A multistage age structured model was studied in [16]. We refer the reader to [12, 6, 21, 32, 28, 34] for other works involving control of populations dynamics. Recently, some research have been conducted to understand the control, and the behavior of the pest Lobesia botrana, see for instance [2,41] and the references therein. Our study is inspired by [39], where the authors studied an optimal control for an age-dependent population dynamics without diffusion. Let $\Omega_0 \subset \Omega$, and let

$$D_T = [0,T]\times[0,L]\times\Omega_0\times D.$$

In this section, we are concerned with the optimal control problem **[P]**:

$$\text{maximize } J(I),$$

where

$$J(I) = \int_{D_T}\left[\eta(t,a,\omega,x)\,I(t,a,\omega,x))u^I(t,a,\omega,x)\right]dx\,d\omega\,da\,dt,$$

subject to the control

$$I \in A = \{I \in L^\infty(D_T) : 0 \leqslant I \leqslant I_{\max} \text{ a.e on } D_T\}.$$

Here u^I is the solution to

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,a,\omega,x)+\dfrac{\partial}{\partial a}\left[v(t,a,\omega)u(t,a,\omega,x)\right], \\ =-\mu(t,a)u(t,a,\omega,x)-\chi_{\Omega_0}(\omega)I(t,,a,\omega,x)u(t,a,\omega,x)+dAu, \\ v(t,0,\omega)u(t,0,\omega,x)=\displaystyle\int_{\Omega}\int_0^L \beta(t,s)\gamma\left(\omega,\omega'\right)u\left(t,s,\omega',x\right)dsd\omega', \\ u(0,a,\omega,x)=u_0(a,\omega,x), \quad (a,\omega,x)\in(0,L)\times\Omega\times D. \end{cases} \tag{5.1}$$

The control I represents the insecticide effort, and plays the role of additional mortality. Note that I is acting only on a nonempty open subset $\Omega_0\subset\Omega$. That means that individuals with level of resistance $\omega\in\Omega_0$ are more vulnerable. The quantity χ_{Ω_0} is the characterstic function. The function η is the net profit generated by the elimination of an individual with size a, and phenotype ω, at position x and time t. We assume that η is non-negative and bounded by η_{max}. Our aim is to maximize J, the benefits from eliminating the pest. By Mazur's Theorem, we establish the existence of the optimal solution, and by the concept of normal cone, we give conditions for optimality.

5.1. Existence of optimal control

We will prove the existence result for the optimal control problem [P].

DEFINITION 2. A pair $\left(I^*,u^{I^*}\right)$ is said to be *optimal* for the control problem if $I^*\in A$, maximizes the functional J, and the pair $\left(I^*,u^{I^*}\right)$ solves the problem (5.1).

In a similar way as in Theorem (1), we prove that for any $I\in A$, problem (5.1) has a unique nonnegative solution u^I. Let w be the nonegative solution of the problem (5.1) corresponding to to $I=0$, $\mu=0$, $\beta=\|\beta\|_\infty$, and $\gamma=\|\gamma\|_\infty$. We have

LEMMA 6. *There exists $C_w>0$ such that*

$$0\leqslant w(t,a,\omega,x)\leqslant C_w \quad a.e \text{ on } \; D_T.$$

Proof. Let P^w be the total population corresponding to w. Integrating the equation of w, we obtain

$$\frac{\partial P^w}{\partial t}+\int_\Omega v(t,L,\omega)u(t,L,\omega,x)\,d\omega-\int_\Omega v(t,0,\omega)u(t,0,\omega,x)\,d\omega=dAP^w,$$

this implies that

$$\frac{\partial P^w}{\partial t}=dAP^w+\|\beta\|_\infty\|\gamma\|_\infty|\Omega|P^w.$$

Define $M:L^\infty(D)\to L^\infty(D)$ by

$$Mu=dAu+\|\beta\|_\infty\|\gamma\|_\infty|\Omega|u.$$

Then

$$P^w(t,x) = e^{Mt} P^w(0,x),$$

and

$$\sup_{0 \leqslant t \leqslant T} \|P^w(t,.)\|_\infty \leqslant \|P^w(0,.)\|_\infty e^{[2d+\|\beta\|_\infty \|\gamma\|_\infty |\Omega|]T}.$$

For $\rho \in L^\infty([0,T] \times [0,L] \times \Omega)$, we define

$$\Pi_\rho(t,\tau,t_0,a_0,\omega) = e^{\int_\tau^t \rho\left(\sigma, X(\sigma,t_0,a_0,\omega),\omega\right) d\sigma}.$$

It is easy to see that the solution w is given by

$$w(t,a,\omega,x) = \begin{cases} T(t-\tau)\Pi_\rho(t,\tau,t,a,\omega) \dfrac{F\left(w(\tau,.,.,)\right)(x)}{v(\tau,0,\omega)} & \text{if } a < z(t,\omega), \\ T(t)\Pi_\rho(t,0,t,a,\omega) u_0\left(\phi(0,t,a,\omega),\omega,x\right), & \text{if } a \geqslant z(t,\omega), \end{cases}$$

with

$$\rho(t,a,w) = -\frac{\partial v}{\partial a}(t,a,\omega).$$

Since

$$F(w(\tau,.,.,))(x) = \|\beta\|_\infty \|\gamma\|_\infty P^w(t,x),$$

and u_0 are bounded, assumption (A_v) implies that there exists a positive constant C_w such that

$$0 \leqslant w(t,a,\omega,x) \leqslant C_w \text{ a.e on } D_T. \quad \square$$

THEOREM 4. *The problem **[P]** at least one optimal solution.*

Proof. Let $d = \max_{(I)\in A} J(I)$. Using comparaison Theorem 2, we obtain

$$0 \leqslant J(I) \leqslant \int_{D_T} [\eta_{max} I_{max} w(t,a,\omega,x)] \, dx d\omega da dt.$$

Hence $0 \leqslant d < \infty$. Let (I_n) be a maximizing sequence satisfying

$$d - \frac{1}{n} < J(I_n) \leqslant d.$$

The same comparaison theorem implies that

$$0 \leqslant u^{I_n}(t,a,\omega,x) \leqslant w(t,a,\omega,x) \text{ a.e in } D_T,$$

and so $\left\|u^{I_n}\right\|_{L^2(D_T)}$ is bounded. It follows that there exists a subsequence denoted again by u^{I_n} such that

$$u^{I_n} \to u^* \text{ weakly in } L^2(D_T).$$

By Mazur's Theorem, see [5] we obtain a sequence $\tilde{u_n}$ verifying

$$\tilde{u_n} \to u^* \text{ in } L^2(D_T),$$

where $\tilde{u_n}$ is given by the following convex combination

$$\tilde{u_n} = \sum_{i=n+1}^{k_n} \lambda_i u^{I_i}, \lambda_i \geqslant 0, \sum_{i=n+1}^{k_n} \lambda_i = 1, k_n \geqslant n+1.$$

Let the control $\tilde{I_n}$ be defined as follows

$$\tilde{I_n}(t,a,\omega,x) = \begin{cases} \dfrac{\sum_{i=n+1}^{k_n} \lambda^i u^{I_i}(t,a,\omega,x) I_i(t,a,\omega,x)}{\sum_{i=n+1}^{k_n} \lambda^i u^{I_i}(t,a,\omega,x)} & \text{if } \sum_{i=n+1}^{k_n} \lambda^i u^{I_i}(t,a,\omega,x) \neq 0, \\ 0 & \text{if } \sum_{i=n+1}^{k_n} \lambda^i u^{I_i}(t,a,\omega,x) = 0. \end{cases}$$

It is clear that $\tilde{I_n} \in A$. The sequence $\tilde{I_n}$ is bounded in $L^2(D_T)$ space, as a consequence there exists a subsequence, denoted again by $\tilde{I_n}$, such that $\tilde{I_n}$ converges weakly in L^2 to I^*. The system (5.1) is linear, then $\tilde{u_n} \in B^+$ is a solution of system (5.1) corresponding to $I = \tilde{I_n} \in A$, i.e

$$\tilde{u_n} = u^{\tilde{I_n}},$$

and

$$u^{\tilde{I_n}}(t,a,\omega,x) = \begin{cases} T(t-\tau)\dfrac{F\left(u^{\tilde{I_n}}(\tau,.,\omega,.)\right)(x)}{v(\tau,0,\omega)} \\ +\displaystyle\int_\tau^t T(t-s)\widetilde{G}\left(s,u^{\tilde{I_n}}(s,.,.\omega,.)\right)\left(\phi(s,\tau,0,\omega),x\right)ds & \text{if } a < z(t,\omega), \\ T(t)u_0\left(\phi(0,t,a,\omega),\omega,x\right) \\ +\displaystyle\int_0^t T(t-s)\widetilde{G}\left(s,u^{\tilde{I_n}}(s,.,\omega,.)\right)\left(\phi(s,t,a,\omega),x\right)ds & \text{if } a \geqslant z(t,\omega). \end{cases} \tag{5.2}$$

Since $u^{\tilde{I_n}}$ converges strongly to u^*, we find that $u^* \in B^+$, and passing to the limit in (5.2), we find that u^* is a mild solution of the probelm (5.1) corresponding to $I = I^*$. Next, we show that the control I^* is optimal. On the one hand, we have

$$\begin{aligned} J(\tilde{I_n}) &= \int_{D_T} \left[\eta(t,a,\omega,x)\tilde{I_n}(t,a,\omega,x))u^{\tilde{I_n}}(t,a,\omega,x)\right] dx d\omega da dt \\ &= \sum_{i=n+1}^{k_n} \lambda_i J(I_i). \end{aligned}$$

Since

$$d - \frac{1}{i} < J(I_i) \leqslant d,$$

and $i \geqslant n$, we have

$$d - \frac{1}{n} < J(I_i) \leqslant d.$$

Using $\lambda_i \geqslant 0$ and $\sum_{i=n+1}^{k_n} \lambda_i = 1$, we obtain

$$d - \frac{1}{n} < \sum_{i=n+1}^{k_n} \lambda_i J(I_i) \leqslant d.$$

This implies that

$$\sum_{i=n+1}^{k_n} \lambda_i J(I_i) \to d,$$

as $n \to +\infty$. We conclude that

$$J\left(\tilde{I}_n\right) \to d,$$

as $n \to +\infty$. This means that $\tilde{I}_n$ is a maximizing sequence. On the other hand, we have

$$J\left(\tilde{I}_n\right) \to \int_{D_T} \left[\eta(t,a,\omega,x) I^*(t,a,\omega,x)) u^{I^*}(t,a,\omega,x)\right] dx d\omega da dt,$$

as $n \to +\infty$. By uniqueness of the limit, we obtain

$$J(I^*) = \int_{D_T} \left[\eta(t,a,\omega,x) I^*(t,a,\omega,x)) u^{I^*}(t,a,\omega,x)\right] dx d\omega da dt = d,$$

and we conclude that $\left(I^*, u^{I^*}\right)$ is optimal. □

5.2. Optimality conditions

Characterization of solutions for optimal control problems are often stated in terms of first order necessary optimality conditions. If the dynamic system is described by ordinary differential equations, the conditions are given by maximum principle of Pontryagin, see ([27, 5]). For age-structured models, a maximum principle of Pontryagin type is obtained in ([15]). Note that the results obtained in [38] constitutes a reference for subsequent researches of the optimal control for age structured problems. Following [5] we give necessary optimality conditions for problem [P]. Before stating our main result, we establish the following useful lemmas. Let $\left(I^*, u^{I^*}\right)$ be an optimal pair for the probem. Then for any $\varepsilon > 0$ small enough, and for any $h \in L^\infty(D_T)$ such that $I^* + \varepsilon h \in A$, the solution u^{I^*} is differentiable with respect to the control I^* in the following sense

LEMMA 7.

$$\frac{u^{I^*+\varepsilon h} - u^{I^*}}{\varepsilon} \to z \ \textit{in } B, \quad \textit{as } \varepsilon \to 0$$

where $u^{I^*+\varepsilon h}$, and u^{I^*} are the solutions of system corresponding to controls $I^*+\varepsilon h$ and I^* respectively. The sensitivity function z, is a solution of the system

$$\begin{cases} \dfrac{\partial z}{\partial t}+\dfrac{\partial(v(t,a,\omega)z)}{\partial a}=dAz(t,a,\omega,x) \\ \qquad\qquad -\mu(t,a)z(t,a,\omega,x) \\ \qquad\qquad -h(t,a,\omega,x)\chi_{\Omega_0}(\omega)u^{I^*}(t,a,\omega,x) \\ \qquad\qquad -I^*(t,a,\omega,x)\chi_{\Omega_0}(\omega)z(t,a,\omega,x), \\ z(t,0,\omega,x)=\displaystyle\int_0^L\int_\Omega\beta(t,s)\gamma\left(\omega,\omega'\right)z\left(t,s,\omega',x\right)d\omega' ds, \\ z(0,a,\omega,x)=0. \end{cases} \tag{5.3}$$

Proof. The existence and uniqueness of solution to (5.3) can be proved by a similar way as that in Theorem 1. By Lemma 3, we have

$$u^{I^*+\varepsilon h}-u^{I^*}\to 0 \text{ in } B, \quad \text{as } \varepsilon\to 0.$$

Define

$$w_\varepsilon=\left[\frac{u^{I^*+\varepsilon h}-u^{I^*}}{\varepsilon}\right],$$

then w_ε is a solution of the system

$$\begin{aligned} Dw(t,a,\omega,x)&=dAw(t,a,\omega,x) \\ &\quad-\mu(t,a)w(t,a,\omega,x) \\ &\quad-h(t,a,\omega,x)\chi_{\Omega_0}(\omega)u^{I^*+\varepsilon h}-I^*(t,a,\omega,x)\chi_{\Omega_0}(\omega)w(t,a,\omega,x), \\ w(t,0,\omega,x)&=\int_\Omega\int_0^L\beta(t,s)\gamma\left(\omega,\omega'\right)w\left(t,s,\omega',x\right)ds\,d\omega', \\ w(0,a,\omega,x)&=0. \end{aligned}$$

Passing to the limit $\varepsilon\to 0$, and using Lemma 3, we obtain

$$w_\varepsilon(t,.,\omega,.)\to z \text{ in } B, \quad \text{as } \varepsilon\to 0. \qquad \square$$

Let $N_A(I^*)$ be the normal cone of A at I^* in X, see [8]. To characterize the optimal strategy, we define the dual problem

$$\begin{cases} \dfrac{\partial q}{\partial t}+v(t,a,\omega)\dfrac{\partial q}{\partial a}=-dAq(t,a,\omega,x)+\mu(t,a)q(t,a,\omega,x) \\ \qquad\qquad +\eta(t,a,\omega,x)I^*(t,a,\omega,x)\chi_{\Omega_0}(\omega) \\ \qquad\qquad +I^*(t,a,\omega,x)\chi_{\Omega_0}(\omega)q(t,a,\omega,x) \\ \qquad\qquad -\beta(t,a)\displaystyle\int_\Omega\gamma(w,w')q\left(t,0,\omega',x\right)d\omega', \\ q(t,L,\omega,x)=0, \\ q(T,a,\omega,x)=0. \end{cases} \tag{5.4}$$

Under the change of variables $\nu := T - t$, $s := L - a$, and $\tilde{q}(\nu, s, \omega, x) := q(T-\nu, L-s, \omega, x)$, the above problem becomes

$$\begin{cases} \dfrac{\partial \tilde{q}}{\partial \nu} + v(T-\nu, L-s, \omega)\dfrac{\partial \tilde{q}}{\partial s} = dA\tilde{q}(\nu, s, \omega, x) \\ \qquad\qquad -\mu(T-\nu, L-s)\tilde{q}(\nu, s, \omega, x) \\ \qquad\qquad -\eta(t, a, \omega, x) I^*(t, a, \omega, x)\chi_{\Omega_0}(\omega) \\ \qquad\qquad -I^*\chi_{\Omega_0}(\omega)\tilde{q}(\nu, s, \omega, x) \\ \qquad\qquad +\beta(T-\nu, L-s)\int_\Omega \gamma(w, w')\tilde{q}(\nu, 0, \omega', x)\, d\omega', \\ \tilde{q}(\nu, 0, \omega, x) = 0, \\ \tilde{q}(0, s, \omega, x) = 0. \end{cases} \tag{5.5}$$

Treating the system (5.5) in the same manner as in Theorem 1, we get existence and uniqueness. The main result of this section is the following result

THEOREM 5. *Assume that $\left(I^*, u^{I^*}\right)$ is optimal, q is the solution of the dual problem, and z is a solution of system (5.3), then*

$$I^* = \begin{cases} 0 & \text{if } \eta(t, a, \omega, x) + q(t, a, \omega, x) < 0, \\ I_{max} & \text{if } \eta(t, a, \omega, x) + q(t, a, \omega, x) > 0. \end{cases}$$

Proof. Let $T_A(I^*)$ be the tangent cone to A at I^*. For any element $h \in T_A(I^*)$, and for any $\varepsilon > 0$ small enough, we have $I^* + \varepsilon h \in A$. Since I^* is optimal, we obtain

$$J(I^*) \geqslant \int_{D_T} \left[\eta u^{I^*+\varepsilon h}(I^* + \varepsilon h)\right](t, a, \omega, x)\, dt da d\omega dx,$$

this gives that

$$\int_{D_T} \left[\eta I^* w_\varepsilon + \eta h u^{I^*+\varepsilon h}\right](t, a, \omega, x)\, dt da d\omega dx \leqslant 0.$$

By lemma 7, passing to the limit, we obtain

$$\int_{D_T} \left[\eta I^* z + \eta h u^{I^*}\right](t, a, \omega, x)\, dt da d\omega dx \leqslant 0.$$

Multiplying the dual problem by z and integrating over D_T, we get

$$\begin{aligned} &\int_{D_T} \left[\frac{\partial q}{\partial t} + v(t, a, \omega)\frac{\partial q}{\partial a} + dAq - \mu(t, a) q(t, a, \omega, x)\right] z(t, a, \omega, x)\, dt da d\omega dx \\ &= \int_{D_T} \left[I^* q(t, a, \omega, x) + \eta I^* - \beta(t, a)\int_\Omega \gamma(w, w') q(t, 0, \omega', x)\, d\omega'\right] z\, dt da d\omega dx. \end{aligned}$$

Note that the operator A is self-adjoint on $L^2(D)$, see appendix for a proof. Using the equation of z, we obtain

$$\begin{aligned}
&-\int_{D_T}\left[\frac{\partial z}{\partial t}+\frac{\partial(v(t,a,\omega)z)}{\partial a}-dAz+\mu(t,a)z(t,a,\omega,x)\right]qdtdad\omega dx\\
&-\int_{D_T}q(t,0,\omega,x)\beta(t,a)\int_\Omega\gamma(\omega,\omega')z(t,a,\omega',x)\,d\omega' dtdad\omega dx\\
=&\int_{D_T}\left[I^*q(t,a,\omega,x)+\eta I^*-\beta(t,a)\int_\Omega\gamma(w,w')q\left(t,0,\omega',x\right)d\omega'\right]zdtdad\omega dx,
\end{aligned}$$

this implies that

$$\begin{aligned}
&\int_{D_T}\left[hu^{I^*}+I^*z\right]qdtdad\omega dx\\
&-\int_{D_T}q(t,0,\omega,x)\beta(t,a)\int_\Omega\gamma(\omega,\omega')z(t,a,\omega',x)\,d\omega' dtdad\omega dx\\
=&\int_{D_T}\left[I^*q(t,a,\omega,x)+\eta I^*-\beta(t,a)\int_\Omega\gamma(w,w')q\left(t,0,\omega',x\right)d\omega'\right]zdtdad\omega dx.
\end{aligned}$$

Changing ω' by ω and using assumption (A_γ), we have

$$\begin{aligned}
&\int_{D_T}q(t,0,\omega,x)\beta(t,a)\int_\Omega\gamma(\omega,\omega')z(t,a,\omega',x)\,d\omega' dtdad\omega dx\\
=&\int_{D_T}\int_\Omega q(t,0,\omega',x)\beta(t,a)\gamma(\omega,\omega')z(t,a,\omega,x)\,d\omega' dtdad\omega dx,
\end{aligned}$$

hence

$$\int_{D_T}hu^{I^*}qdtdad\omega dx=\int_{D_T}\eta I^*zdtdad\omega dx.$$

It follows that

$$\int_{D_T}hu^{I^*}(\eta+q)\,dtdad\omega dx\geqslant 0.$$

for any element of tangent cone $h\in T_A(I^*)$, that is

$$u^{I^*}(\eta+q)\in N_A(I^*)\,. \tag{5.6}$$

which implies that

$$I^*=\begin{cases}0 & \text{if } \eta(t,a,\omega,x)+q(t,a,\omega,x)<0,\\ I_{max} & \text{if } \eta(t,a,\omega,x)+q(t,a,\omega,x)>0.\end{cases}\quad\square$$

REMARK 3. a) The first equations of (5.1) and (5.4) with (5.6) represent Pontryagin's principle. The first equation of (5.4) and (5.6) constitute the first order necessary conditions of optimality, see for instance [5, 15].

b) The optimal policity depends on $(\eta+q)$. If $(\eta+q)<0$, that is η is small enough then there is no need to apply insecticides; otherwise, one needs to apply the insecticides at the maximum control I_{max}.

6. Appendix

LEMMA 8. *The operator A is self adjoint on $L^2(D)$.*

Proof. We have

$$\int_D Au(x)v(x)\,dx = \int_D \left[\int_D J(x-y)u(y)\,dy - u(x)\right] v(x)\,dx$$
$$= \int_D \int_D J(x-y)u(y)v(x)\,dydx - \int_D u(x)v(x)\,dx.$$

Since J is symmetric, then

$$\int_D Au(x)v(x)\,dx = \int_D \int_D J(y-x)v(x)u(y)\,dydx - \int_D u(x)v(x)\,dx$$
$$= \int_D \left[\int_D J(y-x)v(x)\,dx\right] u(y)\,dy - \int_D u(y)v(y)\,dy$$
$$= \int_D Av(y)u(y)\,dy. \quad \square$$

Acknowledgement. The authors are grateful to the anonymous referees, who provided valuable suggestions and insight that greatly improved the paper.

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(Received August 24, 2022)

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