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EXISTENCE RESULTS FOR CAPUTO FRACTIONAL BOUNDARY VALUE PROBLEMS WITH UNRESTRICTED GROWTH CONDITIONS

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Abstract. This paper presents new results to fractional boundary value problems of the Caputo type with focal boundary conditions. This fractional derivative is used extensively in modelling real world applications. The main aim of this paper is to present results for the existence of solutions to ensure the usefulness in the context of modelling and providing *a priori* bounds on all possible solutions subject to a single versatile differential inequality. These results vastly expand the scope of problems which are applicable since it allows the fractional differential equation to have unrestricted growth and be nonlinear.

1. Introduction

The paper presents novel existence results and provides *a priori* bounds specifically for Caputo fractional boundary value problems with focal boundary conditions. Let $\alpha \in (1,2]$ and $a,b \in \mathbb{R}$, $a < b$. We consider the following fractional boundary value problem:

$$({}^C D_a^\alpha y)(t) = f(t,y(t)), \qquad t \in (a,b), \tag{1.1}$$

$$y'(a) = c_1, \quad y(b) = c_2. \tag{1.2}$$

The fractional Caputo derivative in the BVP is defined as

$$({}^C D_a^\alpha y)(t) := \frac{1}{\Gamma(k-\alpha)} \int_a^t (t-\tau)^{k-\alpha-1} D^{\lceil \alpha \rceil}(y)(\tau)\, d\tau$$

where $k = \lceil \alpha \rceil$. Denote the function space $A^k[a,b]$ as the set of functions with an absolutely continuous $(k-1)$st derviative. Furthermore, there is a direct link with the classical Riemann-Liouville fractional derivative which is the following

$$({}^C D_a^\alpha y)(t) = (D_a^\alpha [y - T_{k-1}[y;a]])(t)$$

almost everywhere where $y \in A^k[a,b]$ and $T_{k-1}[y;a]$ denotes the Taylor polynomial of degree $k-1$ for the function y centred at a. From here, it follows that $y \in A^k[a,b]$

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implies $y \in C^{k-1}[a,b]$, and the existence of the Taylor polynomial and its Riemann-Liouville derivative. The Riemann-Liouville derivative is defined as

$$(D_a^\alpha y)(t) = \frac{1}{\Gamma(k-\alpha)} \left(\frac{d}{dt}\right)^k \int_a^t (t-\tau)^{k-\alpha-1} y(\tau)\, d\tau.$$

This implies the fractional boundary value problem (1.1), (1.2) has a solution if and only if it is a solution to the fractional BVP

$$(D_a^\alpha[y - y(a) - (t-a)c_1])(t) = f(t,y(t)), \qquad t \in (a,b) \tag{1.3}$$

with (1.2) where $y'(a) = c_1$. This alternative representation will be applied throughout the paper to achieve the results since the theorems have been generally proved in the previous paper [8] for Riemann-Liouville fractional derivative. Also, hence a solution that solves (1.3) will be considered in $y \in C^1[a,b]$. Naturally, an important part is the fractional component of the order $1 < \alpha \leqslant 2$, this leads us to let $p = \alpha - 1$ from hereon. By using the equivalent reformulated problem (1.3) with a Riemann-Liouville derivative, the left boundary condition implies a natural and interesting property that is all q-fractional derivatives of order $0 < q \leqslant p < 1$ on the boundary are zero, that is $({}^C D_a^q y)(a) = 0$ since $y(a) - T_1[y;a](a) = 0$.

The study of fractional boundary value problems has been quite attractive with a lot of attention and research escalation in recent years due to the theory being a more effective and appropriate approach to modelling various real world phenomena [3], [4], [5], [11], [13], [15], [16], and their references therein. The reason is the fractional order assists in effectively capturing the past history of complex dynamical systems from past to present where we were limited to integer-order rates of change. Some real world applications include human life disorders, electrical engineering, chemistry, control theory and physics with examples found in Ibe [10], Kilbas et al. [11], Podlubny [16], Xuan et al. [21] and Zheng & Zhang [22].

The methods rely on using an equivalent integral representation, applying fractional differential inequalities and using topological methods. The first main aim is to determine an equivalent integral representation and qualitative information such as *a priori* bounds on all possible solutions, then to expand and present novel existence of solutions to the fractional BVP (1.1), (1.2) subject to the following differential inequality,

$$\|\mathbf{f}(t,\mathbf{u})\| \leqslant 2V\langle \mathbf{u}, \mathbf{f}(t,\mathbf{u})\rangle + W, \quad \textit{for all} \quad (t,\mathbf{u}) \in (a,b) \times \mathbb{R}^n \tag{1.4}$$

where V, W are non-negative constants. Also, the euclidean norm is used and defined as $\|\mathbf{u}\| := \langle \mathbf{u}, \mathbf{u}\rangle$ where $\mathbf{u} \in \mathbb{R}^n$. Notably, this inequality is satisfied for a variety of different functions f, including functions that are unbounded and allows the right hand side of the fractional BVP to be unrestricted for continuous systems of fractional equations. The results in the paper will be focussed on the scalar case, since the results naturally follow for systems by ensuring the natural adjustments in Banach spaces are made. This inequality has been used in Fewster-Young [8] in the fractional setting with the Riemann-Liouville derivative, and various other inequalities of a similar nature have been used by Tisdell et al [18], [19] for fractional initial value problems with Caputo derivative of order less than one. There are many results known when f is Lipschitz, satisfies certain Lypunaov conditions, monnotincity condition, Osgood condition or if f is a bounded function [2], [5], [16], [18, 19, 20]. However, this inequality gains its merit for allowing a vast increase in possible functions to be satisfied whereas the previous literature results cannot. We will present novel existence results with sufficient *a priori* bounds to all possible solutions to the fractional BVP under condition (1.4) and the use of topological methods such as Nonlinear Alternative Theorem. In addition, we will illustrate and motivate the applicability with examples of functions f which satisfy this condition but not the aforementioned conditions.

2. The alternative equivalent integral representation

This current section introduces some key lemmas and consequences in the general theory of fractional calculus which play an important role in proofs of the upcoming main results. The first step in proving the existence of solutions to the fractional BVP (1.1), (1.2) is to present equivalent integral representation. Furthermore, this representation for nonlinear problems allows us to determine possible *a priori* bounds for all possible solutions to the BVP (1.1), (1.2) given the inequality (1.4) is satisfied.

The following two lemmas apply to the Riemann-Liouville fractional derivative and can be found in [5] and [16].

LEMMA 1. *Let* $p \in \mathbb{R}^+$ *and* $0 < p < 1$. *Suppose* u *is a continuous function, its derivative is integrable in* $[a,b]$, *then*

$$u(a) = 0$$

if and only if

$$\lim_{t \to a^+} \left[(D_a^p u)(t) \right] = 0.$$

LEMMA 2. *If* $0 < p < 1$ *and Lemma* 1 *holds then*

$$D_a^p (D_a u)(t) = (D_a^{p+1} u)(t) = \frac{d}{dt} (D_a^p u)(t).$$

Furthermore, fractional integration improves the smoothness properties of functions, providing some interesting results relating to the behaviour of solutions at the lower terminal. If we suppose that y is continuous and has at least one continuous derivative in the closed interval $[a,t]$ then by using the fractional power series of order $q \geqslant 0$ and as $t \to a^+$ then

$$\lim_{t \to a^+} (J_a^q y)(t) = 0, \quad \text{for} \quad q > 0$$

and

$$\lim_{t \to a^+} (J_a^q y)(t) = y(a), \quad \text{for} \quad q = 0.$$

We now establish an equivalent integral representation for the fractional BVP (1.1) with (1.2).

THEOREM 1. *Suppose y is a continuous function and its derivative is integrable in $[a,b]$. A function y is a solution to the fractional BVP if and only if it is a solution to the equivalent integral representation given by*

$$y(t) = c_2 - c_1(b-t) - \frac{1}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} f(\tau, y(\tau))\, d\tau ds. \tag{2.1}$$

Proof. Suppose y is a solution to the fractional BVP (1.1), (1.2) and consider

$$(D_a^{p+1}[y - T_1[y;a]])(t) = f(t, y(t)), \qquad \text{for all} \quad t \in (a,b).$$

Let $u(t) := (y - T_1[y;a]])(t)$. Notice Lemma 1 implies $\lim_{t\to a^+}(D_a^p u)(t) = 0$, and since u is continuous and differentiable as well, then $\lim_{t\to a^+}(D_a^{p-1}u)(t) = 0$. By taking J_a^{p+1} of both sides and using the identity

$$J_a^{p+1}(D_a^{p+1}u)(t) = u(t) - \lim_{t\to a^+}\left[(D_a^p u)(t)\right]\frac{(t-a)^p}{\Gamma(p+1)} - \lim_{t\to a^+}\left[(D_a^{p-1}u)(t)\right]\frac{(t-a)^{p-1}}{\Gamma(p)}$$

then this yields

$$y(t) - y(a) - (t-a)y'(a) = J_a^{p+1} f(t, y(t)).$$

By imposing the initial condition (1.2) and letting $A := y(a)$ then we can simplify this to

$$y(t) = A + c_1(t-a) + (J_a^{p+1} f(\cdot, y(\cdot))(t).$$

If we substitute the other boundary condition and rearrange then

$$A = c_2 - c_1(b-a) - (J_a^{p+1} f(\cdot, y(\cdot),)(b).$$

Thus,

$$y(t) = c_2 - c_1(b-t) - (J_a^{p+1} f(\cdot, y(\cdot))(b) + (J_a^{p+1} f(\cdot, y(\cdot))(t)$$

for $t \in [a,b]$. To prove the equivalence, notice that the boundary condition $y(b) = c_2$ holds. If we differentiate then we obtain

$$y'(t) = c_1 + (J_a^p f(\cdot, y(\cdot))(t).$$

Hence, we see that the boundary condition $y'(a) = c_1$ holds. If we take the Caputo derivative of order p then this yields (1.1) and proves the equivalence. □

3. A priori bounds

This section begins by presenting two key fractional differential inequalities which has plenty of applications and variations in fractional calculus. They will play a key role in the upcoming proofs to prove the *a priori* bounds and existence results of solutions.

LEMMA 3. ([8]) *If $u : [a,b] \to \mathbb{R}$, $u \in C^1(a,b)$ and $p \in (0,1)$ then*

$$(D_a^p[u]^2)(t) \leqslant 2u(t)(D_a^p u)(t)$$

where $t \in (a,b)$.

In addition, a tighter inequality can be proven [8] which will be used in the next lemma:

$$D_a^p([u(t)]^2) \leqslant 2u(t)D_a^p([u(t)]) - [u(t)]^2\left[\frac{1}{\Gamma(1-p)}(t-a)^{-p}\right]$$

for $t \in (a,b)$. Notice that $u \in C^1(a,b)$ in the previous result and this aligns with the function space where we will prove possible solutions exist as aforementioned in the introduction. The main result relating to *a priori* bounds on all possible solutions requires additional inequality that stems from this result and is deduced by letting $u(t) := (D_a^p[y - T_1[y;a]])(t)$. The result is as follows:

LEMMA 4. *If* $y : [a,b] \to \mathbb{R}$ *is continuous and* ${}^CD_a^p y : (a,b) \to \mathbb{R}$ *is continuous with* $p \in (0,1)$ *then*

$$J_a^{p+1}((Dy)(\cdot)(D_a^p[y - T_1[y;a]])(\cdot))(t) \geqslant -\frac{c_1^2(b-a)^2}{2p(p+1)} \tag{3.1}$$

for $t \in [a,b]$.

Proof. The proof of Lemma 4 can be deduced from Lemma 3 by letting $u(t) = (D_a^p[y - T_1[y;a]])(t)$ and replacing p with $1-p$ to obtain the following inequality,

$$\begin{aligned}&(D_a^{1-p}[(D_a^p[y - T_1[y;a]])(\cdot)]^2)(t)\\ &\leqslant 2(D_a^p[y - T_1[y;a]])(t)(D_a^{1-p}((D_a^p[y - T_1[y;a]]))(\cdot))(t)\\ &\quad - [(D_a^p[y - T_1[y;a]])(t)]^2\left[\frac{(t-a)^{p-1}}{\Gamma(p)}\right]\end{aligned}$$

for $t \in (a,b)$. The law for the composition of fractional derivatives produces

$$(D_a^{1-p}(D_a^p[y - T_1[y;a]])(t) = (D[y - T_1[y;a]])(t) - \lim_{t\to a^+}(J_a^{1-p}[y - T_1[y;a]])(t)\frac{(t-a)^{p-2}}{\Gamma(p-1)}$$

for $t \in (a,b)$. Since the solutions y are continuous and differentiable then $\lim\limits_{t\to a^+}(D_a^p[y - T_1[y;a]])(t) = 0$, and they have at one continuous derivative implying $\lim\limits_{t\to a^+}(J_a^{1-p}[y - T_1[y;a]])(t) = 0$. Thus

$$(D_a^{1-p}(D_a^p[y - T_1[y;a]])(\cdot))(t) = (D[y - T_1[y;a]])(t) = (Dy)(t) - c_1$$

for $t \in (a,b)$. By using the Peter-Paul inequality with $\varepsilon > 0$,

$$2c_1(D_a^p[y - T_1[y;a]])(t) \leqslant \frac{c_1^2}{\varepsilon} + \varepsilon[(D_a^p[y - T_1[y;a]])(t)]^2, \quad \text{for} \quad t \in (a,b).$$

Let $\varepsilon = \frac{(b-a)^{p-1}}{\Gamma(p)}$, therefore,

$$\begin{aligned}
&D_a^{1-p}([(D_a^p[y-T_1[y;a]])(t)]^2)\\
&\leqslant 2(D_a^p[y-T_1[y;a]])(t)(D_a y)(t) + \frac{c_1^2\Gamma(p)}{(b-a)^{p-1}}\\
&\quad + [(D_a^p[y-T_1[y;a]])(t)]^2\Big(\frac{(b-a)^{p-1}}{\Gamma(p)} - \frac{(t-a)^{p-1}}{\Gamma(p)}\Big)\\
&\leqslant 2(D_a^p[y-T_1[y;a]])(t)(D_a y)(t) + \frac{c_1^2\Gamma(p)}{(b-a)^{p-1}}
\end{aligned}$$

for $t \in (a,b)$. By applying this inequality, we see that

$$\begin{aligned}
&J_a^{p+1}((Dy)(\cdot)(D_a^p[y-T_1[y;a]])(\cdot))(t)\\
&\geqslant \frac{1}{2}J_a^{p+1}(D_a^{1-p}[(D_a^p[y-T_1[y;a]])(\cdot)]^2)(t)\\
&\quad - \frac{1}{2}\Big(J_a^{p+1}\frac{c_1^2\Gamma(p)}{(b-a)^{p-1}}\Big)(t)\\
&= \frac{1}{2}J_a^{2p}([(D_a^p[y-T_1[y;a]])(\cdot)]^2)(t) - \frac{c_1^2(b-a)^2}{2p(p+1)}\\
&\quad - \frac{1}{2}\frac{(t-a)^p}{\Gamma(1+p)}\lim_{t\to a^+} J_a^p([(D_a^p[y-T_1[y;a]])(\cdot)]^2)(t))\\
&\geqslant -\frac{c_1^2(b-a)^2}{2p(p+1)}. \quad \square
\end{aligned}$$

We now present one of the main novel results proving that a solution to the fractional BVP (1.1), (1.2) satisfies the following *a priori* bounds.

THEOREM 2. *Let $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. If there exists non-negative constants V,W such that*

$$\|f(t,u)\| \leqslant 2Vuf(t,u)+W, \qquad \textit{for all} \quad (t,u)\in(a,b)\times\mathbb{R} \tag{3.2}$$

then all possible solutions to (1.1), (1.2) *satisfy*

$$\|y(t)\| \leqslant \|c_2\| + \|c_1\|(b-a) + \frac{V}{2}\left[[c_2 - c_1(b-a)]^2 + \frac{c_1^2(b-a)^2}{p(p+1)}\right] + \frac{W}{\Gamma(p+2)}(b-a)^{p+1},$$

for all $t\in[a,b]$.

Proof. By Theorem 1, we have the equivalent integral representation of all solutions to the BVP (1.1) and is given by

$$y(t) = c_2 - c_1(b-t) - \frac{1}{\Gamma(p)}\int_t^b\int_a^s (s-\tau)^{p-1} f(\tau,y(\tau))\, d\tau ds.$$

We now estimate $y(t)$ and by applying condition (3.2) yields

$$
\begin{aligned}
\|y(t)\| &\leqslant \|c_2\| + \|c_1\|(b-a) + \frac{1}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} \|f(\tau, y(\tau))\| \, d\tau ds \\
&\leqslant \|c_2\| + \|c_1\|(b-a) + \frac{1}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} \big[V y(\tau) f(\tau, y(\tau)) + W\big] d\tau ds.
\end{aligned}
$$

Recall that y is a solution to the differential equation implies

$$
\begin{aligned}
\|y(t)\| \leqslant \|c_2\| + \|c_1\|(b-a) &+ \frac{V}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} y(\tau) ({}^C D_a^{p+1} y)(\tau) d\tau ds \\
&+ \frac{W}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} \, d\tau ds.
\end{aligned}
$$

Integrating the last term yields

$$
\begin{aligned}
\|y(t)\| \leqslant \|c_2\| + \|c_1\|(b-a) &+ \frac{V}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} y(\tau) ({}^C D_a^{p+1} y)(\tau) d\tau ds \\
&+ \frac{W}{\Gamma(p+2)} (b-a)^{p+1}.
\end{aligned}
$$

We now focus on finding an estimate for the integral term and let

$$
H(t) := \int_t^b \int_a^s (s-\tau)^{p-1} y(\tau) ({}^C D_a^{p+1} y)(\tau) d\tau ds
$$

when $t = a$. See that $H(t)$ can be firstly written as follows

$$
\begin{aligned}
H(t) = &\int_t^b \int_a^s (s-\tau)^{p-1} \big[y(\tau)(D_a^{p+1}[y - T_1[y;a]])(\tau) + y'(\tau)(D_a^p[y - T_1[y;a]])(\tau)\big] d\tau ds \\
&- \int_t^b \int_a^s (s-\tau)^{p-1} \big[y'(\tau)(D_a^p[y - T_1[y;a]])(\tau)\big] d\tau ds.
\end{aligned}
$$

By Lemma 4, the last term of $H(a)$ being

$$
\Gamma(p) J_a^{p+1}((Dy)(\cdot)(D_a^p[y - T_1[y;a]])(\cdot))(a) \geqslant -\Gamma(p) \frac{c_1^2 (b-a)^2}{2p(p+1)}.
$$

Therefore,

$$
H(a) \leqslant \int_a^b \int_a^s (s-\tau)^{p-1} \big[y(\tau)(D_a^p[y - T_1[y;a]])(\tau)\big]' \, d\tau ds + \Gamma(p) \frac{c_1^2 (b-a)^2}{2p(p+1)}.
$$

Moreover, this is equivalent to the following bound

$$
\begin{aligned}
H(a) \leqslant &\int_a^b \int_a^s (s-\tau)^{p-1} \big[[y(\tau) - y(a)](D_a^p[y - T_1[y;a]])(\tau)\big]' \, d\tau ds + \frac{\Gamma(p) c_1^2 (b-a)^2}{2p(p+1)} \\
&+ \int_a^b \int_a^s (s-\tau)^{p-1} \big[y(a)(D_a^{p+1}[y - T_1[y;a]])(\tau)\big] \, d\tau ds.
\end{aligned}
$$

By completing the integral on the last term above gives

$$\int_a^b\int_a^s (s-\tau)^{p-1}y(a)(D_a^{p+1}[y-T_1[y;a]])(\tau)\,d\tau ds=\Gamma(p)y(a)(y(b)-y(a)-(b-a)c_1).$$

Thus,

$$\begin{aligned}H(a)\leqslant \int_a^b\int_a^s (s-\tau)^{p-1}\left[[y(\tau)-y(a)](D_a^p[y-T_1[y;a]])(\tau)\right]'\,d\tau ds+\frac{\Gamma(p)c_1^2(b-a)^2}{2p(p+1)}\\ +\Gamma(p)y(a)(y(b)-y(a)-(b-a)c_1).\end{aligned}$$

Furthermore, we can simplify the integral to produce

$$\begin{aligned}H(a)\leqslant \Gamma(p)\bigg[\int_a^b (D_a^{1-p}[y(\cdot)-y(a)](D_a^p[y-T_1[y;a]])(\cdot))(s)\,ds+\frac{c_1^2(b-a)^2}{2p(p+1)}\\ +y(a)(y(b)-y(a)-(b-a)c_1)\bigg].\end{aligned}$$

We apply the definition of a fractional derivative to just the integral part such that

$$\begin{aligned}H(a)\leqslant \frac{\Gamma(p)}{\Gamma(p-1)}\int_a^b\int_a^s (s-\tau)^{p-2}[[y(\tau)-y(a)](D_a^p[y-T_1[y;a]])(\tau)]\,d\tau\,ds\\ +\frac{\Gamma(p)c_1^2(b-a)^2}{2p(p+1)}+\Gamma(p)y(a)(y(b)-y(a)-(b-a)c_1).\end{aligned}$$

By employing Lemma 3 and since $\Gamma(p-1)<0$ then

$$\begin{aligned}H(a)\leqslant \frac{\Gamma(p)}{2\Gamma(p-1)}\int_a^b\int_a^s (s-\tau)^{p-2}(D_a^p[y-T_1[y;a]]^2)(\tau)\,d\tau\,ds+\frac{\Gamma(p)c_1^2(b-a)^2}{2p(p+1)}\\ +\Gamma(p)y(a)(y(b)-y(a)-(b-a)c_1).\end{aligned}$$

However, this is the composition of two fractional derivatives and this yields

$$\begin{aligned}H(a)&\leqslant \frac{\Gamma(p)}{2}\left[\int_a^b \frac{d[y-T_1[y;a]]^2}{ds}\,ds+2y(a)(y(b)-y(a)-(b-a)c_1)+\frac{c_1^2(b-a)^2}{p(p+1)}\right]\\ &=\frac{\Gamma(p)}{2}\bigg[[y(b)-y(a)-c_1(b-a)]^2+2y(a)(y(b)-y(a)-(b-a)c_1)\\ &\quad+\frac{c_1^2(b-a)^2}{p(p+1)}\bigg]\\ &=\frac{\Gamma(p)}{2}\left[[c_2-c_1(b-a)]^2-[y(a)]^2+\frac{c_1^2(b-a)^2}{p(p+1)}\right].\end{aligned}$$

Therefore, by putting everything together, we have

$$\|y(t)\|\leqslant \|c_2\|+\|c_1\|(b-a)+\frac{V}{2}\left[[c_2-c_1(b-a)]^2+\frac{c_1^2(b-a)^2}{p(p+1)}\right]+\frac{W(b-a)^{p+1}}{\Gamma(p+2)}$$

for $t\in[a,b]$. □

4. Existence results and examples

The upcoming result is the final key in proving the novel existence of solutions to the fractional BVP where all possible solutions satisfy the *a priori* bounds in Theorem 2. It is important to note for the reader's reference that the condition on the function $f : [a,b] \times \to \mathbb{R}$ implies the continuity of the solution y.

THEOREM 3. *If the conditions of Theorem 2 hold then there exists at least one solution to the fractional BVP* (1.1), (1.2).

Proof. Let

$$X := \{y \in C([a,b];\mathbb{R}) \mid \|y\| := \sup_{t\in[a,b]} |y(t)|\}.$$

Define the convex space

$$U := \{y \in C([a,b];\mathbb{R}) \mid \sup_{t\in[a,b]} |y(t)| \leqslant R+1\}$$

where

$$R := \|c_2\| + \|c_1\|(b-a) + \frac{V}{2}\left[[c_2 - c_1(b-a)]^2 + \frac{c_1^2(b-a)^2}{p(p+1)}\right] + \frac{W(b-a)^{p+1}}{\Gamma(p+2)}$$

and the continuous operator, $T : \bar{U} \to X$ by

$$Ty(t) := c_2 - c_1(b-t) - \frac{1}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} f(\tau, y(\tau))\, d\tau ds.$$

It follows that every fixed point of the operator T is a solution to the fractional BVP (1.1), (1.2) by the equivalence proved by Theorem 1. To show there exists at least one fixed point, we apply the Leray-Schauder Nonlinear Alternative theorem. By a standard argument and it is not too difficult to show that $T : \bar{U} \to X$ is a compact continuous map and this is mainly due to the continuity of f. It now suffices to show that $T(\bar{U}) \subset U$, this is equivalent to showing that for all $\lambda \in (0,1)$, there is no $y \in \partial U$ such that $y = \lambda T(y)$. If $y \in \partial U$ then $\|y\| = R+1$, however, by using the steps in the proof of Theorem 2 then

$$\|Ty(t)\| \leqslant R, \quad \textit{for all} \quad t \in [a,b].$$

Thus, this proves it is not possible to have $y \in \partial U$ such that $y = \lambda Ty$ for all $\lambda \in (0,1)$. In turn, the Nonlinear Alternative Theorem implies there exists at least one fixed point, $y \in C([a,b];\mathbb{R})$ and moreover, $y \in C^1([a,b];\mathbb{R})$ which satisfies the equivalent integral representation and thus is a solution to the fractional BVP (1.1), (1.2). □

Even though the results above technically deal with the scalar Caputo fractional derivative, they naturally hold for systems of fractional derivatives directly from the work herein with the appropriate changes in the Banach spaces and norms.

THEOREM 4. *If the fractional boundary value problem* (1.1)*,* (1.2) *is a system of equations and the conditions of Theorem* 2 *hold then there exists at least one solution to the fractional BVP system* (1.1)*,* (1.2)*.*

We now illustrate the applicability and usefulness of the results, and in particular where the function f is not Lipschitz or bounded.

EXAMPLE 1. Consider the fractional BVP given by

$$({}^{C}D_a^{p+1}y)(t) = y^3(t) + ty(t), \quad a < t < b \tag{4.1}$$

$$y'(a) = 0, \quad y(b) = 2 \tag{4.2}$$

where $0 < p < 1$ and $b > a \geqslant 0$. Here $f(t,u) := u^3 + tu$ and is a continuous function. It suffices to show that there exists non-negative constants V, W such that the inequality (3.2) is satisfied. If we choose $V = \frac{1}{2}, W = \frac{(2+b)}{4}$ then

$$\begin{aligned} \|f(t,u)\| = |u^3 + tu| &= |u|(u^2 + t) \\ &= |u|u^2 + t|u| \\ &\leqslant u^4 + \frac{1}{2} + (u^2 + 1/4)t \\ &\leqslant 2Vuf(t,u) + W \end{aligned}$$

Thus, all the conditions of Theorem 3 are satisfied and there exists at least one solution to the BVP (4.1), (4.2) with

$$\|y\| \leqslant 3 + \frac{2+b}{4\Gamma(p+2)}(b-a)^{p+1}.$$

The next example illustrates another BVP that has unrestricted growth and is not Lipschitz.

EXAMPLE 2. Consider the fractional BVP given by

$$({}^{C}D_a^{p+1}y)(t) = e^{y(t)} - 1, \quad 0 < t < 4 \tag{4.3}$$

$$y'(0) = 1, \quad y(3) = 10 \tag{4.4}$$

where $0 < p < 1$. Here $f(t,u) := e^u - 1$ and is a continuous function. It suffices to show that there exists non-negative constants V, W such that the inequality (3.2) is satisfied. If we choose $V = 1/2, W = 1$ then

$$\begin{aligned} \|f(t,u)\| = |(e^u - 1)| &= |e^u - 1| \\ &\leqslant u(e^u - 1) + 1 \\ &= 2Vuf(t,u) + W \end{aligned}$$

Thus, all the conditions of Theorem 3 are satisfied and there exists at least one solution to the BVP (4.3), (4.4) with

$$\|y\| \leqslant 23 + \frac{4}{p(p+1)} + \frac{4^{p+1}}{\Gamma(p+2)}.$$

EXAMPLE 3. Consider the fractional BVP given by

$$({}^C D_0^{3/2}\mathbf{y})(t) = (e^{y_1(t)} - 1 + y_1(t)y_2^2(t), y_2^5(t)) \quad 0 < t < 3 \tag{4.5}$$

$$\mathbf{y}'(0) = (1,1), \quad \mathbf{y}(3) = (2,1) \tag{4.6}$$

In this BVP, the function $\mathbf{f}(t,u_1,u_2) := (e^{u_1} - 1 + u_1u_2^2, u_2^5)$. It now suffices to show that $\mathbf{f}$ satisfies the inequality (3.2) we choose $V = 1/2$, $W = 2$ and see that

$$\begin{aligned}\|\mathbf{f}(t,\mathbf{u})\| = \|(e^{u_1} - 1 + u_1u_2^2, u_2^5)\| &= |e^{u_1} - 1 + u_1u_2^2| + |u_2^5| \\ &\leqslant u_1(e^{u_1} - 1) + 1 + u_1^2u_2^2 + u_2^2/2 + u_2^6 - u_2^2/2 + 1 \\ &= u_1((e^{u_1} - 1) + u_1u_2^2) + u_2(u_2^5) + 2 \\ &= 2V\langle \mathbf{u}, \mathbf{f}(t,\mathbf{u})\rangle + W\end{aligned}$$

for $(t,\mathbf{u}) \in (0,3) \times \mathbb{R}^2$. Therefore, since this is a continuous function and all the conditions of Theorem 4 are satisfied then there exists at least one solution to the fractional BVP (4.5), (4.6) where

$$\begin{aligned}\|\mathbf{y}\| &\leqslant \|\mathbf{c_2}\| + \|\mathbf{c_1}\|(b-a) + \frac{V}{2}\left[\|\mathbf{c_2} - \mathbf{c_1}(b-a)\|^2 + \frac{\|\mathbf{c_1}\|^2(b-a)^2}{p(p+1)}\right] + \frac{W(b-a)^{p+1}}{\Gamma(p+2)} \\ &\leqslant \sqrt{5} + \sqrt{2} + \frac{29}{4} + \frac{2 \times 3^{3/2}}{\Gamma(5/2)}.\end{aligned}$$

REFERENCES

[1] RAVI P. AGARWAL, MARIA MEEHAN AND DONAL O'REGAN, *Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics **141**, Cambridge University Press, Cambridge, 2001.

[2] ZHANBING BAI AND HAISHEN LÜ, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311**, 2, 2005, 495–505.

[3] K. DIETHELM, N. J. FORD, A. D. FREED AND YU. LUCHKO, *Algorithms for the fractional calculus: a selection of numerical methods*, Comput. Methods Appl. Mech. Engrg. **194**, 6–8, (2005), 743–773.

[4] KAI DIETHELM AND NEVILLE J. FORD, *Analysis of fractional differential equations*, J. Math. Anal. Appl. **265**, 2, (2002), 229–248.

[5] KAI DIETHELM, *The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type*, Lecture Notes in Mathematics **2004**, Springer-Verlag, Berlin, 2010.

[6] NICHOLAS FEWSTER-YOUNG AND CHRISTOPHER C. TISDELL, *The existence of solutions to second-order singular boundary value problems*, Nonlinear Anal. **75**, 13, (2012), 4798–4806.

[7] MARTIN BOHNER AND NICK FEWSTER-YOUNG, *Discrete Fractional Boundary Value Problems and Inequalities*, Fractional Calculus and Applied Analysis **24**, 6 (2021): 1777–1796, `https://doi.org/10.1515/fca-2021-0077`.

[8] NICHOLAS FEWSTER-YOUNG, *Existence results, inequalities and a priori bounds to fractional boundary value problems*, Journal of Fractional Differential Calculus **11**, 2 (2021), 175–191.

[9] PHILIP HARTMAN, *On boundary value problems for systems of ordinary, nonlinear, second order differential equations*, Trans. Amer. Math. Soc. **96**, (1960), 493–509.

[10] OLIVER C. IBE, *Markov processes for stochastic modeling, 2nd ed.*, Elsevier, Inc., Amsterdam, 2013.

[11] ANATOLY A. KILBAS, HARI M. SRIVASTAVA, JUAN J. TRUJILLO, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies **204**, Elsevier Science B.V., Amsterdam, 2006.

[12] V. LAKSHMIKANTHAM, A. S. VATSALA, *Theory of fractional differential inequalities and applications*, Commun. Appl. Anal. **11** (2007), 3–4, 395–402.

[13] V. LAKSHMIKANTHAM, A. S. VATSALA, *Basic theory of fractional differential equations*, Nonlinear Anal. **69**, (2008), 8, 2677–2682.

[14] AGNIESZKA B. MALINOWSKA, TATIANA ODZIJEWICZ AND DELFIM F. M. TORRES, *Advanced methods in the fractional calculus of variations*, Springer Briefs in Applied Sciences and Technology, Springer, Cham, 2015.

[15] KEITH B. OLDHAM AND JEROME SPANIER, *The fractional calculus*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974.

[16] IGOR PODLUBNY, *Fractional Differential Equations*, Mathematics in Science and Engineering **198**, Academic Press, Inc., San Diego, CA, 1999.

[17] STEFAN G. SAMKO, ANATOLY A. KILBAS AND OLEG I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications,* Gordon and Breach Publishers, Yverdon, 1993.

[18] CHRISTOPHER C. TISDELL, ZHENHAI LIU AND SHEV MACNAMARA, *Basic existence and uniqueness results for solutions to systems of nonlinear fractional differential equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **24**, 3, (2017), 181–193.

[19] CHRISTOPHER C. TISDELL, *Improved mathematical results and simplified pedagogical. approaches for Gronwall's inequality for fractional calculus*, Fract. Differ. Calc., **8**, 1 (2018), 33–41.

[20] CHRISTOPHER C. TISDELL, *Basic existence and a priori bound results for solutions to systems of boundary value problems for fractional differential equations*, Electron. J. Differential Equations, **84**, 9 2016.

[21] CHEN JIN CHANG FU-XUAN AND WEI HUANG, *Anomalous diffusion and fractional advection-diffusion equation*, Acta Physica Sinica., **53**, (2005), 1113–1117.

[22] LIANCUN ZHENG AND XINXIN ZHANG, *Modeling and analysis of modern fluid problems*, Mathematics in Science and Engineering, Elsevier/Academic Press, London, 2017.

[23] YONG ZHOU, *Fractional evolution equations and inclusions: analysis and control*, Elsevier/Academic Press, London, 2016.

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