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BOUNDED AND UNBOUNDED POSITIVE SOLUTIONS FOR SINGULAR ϕ–LAPLACIANS COUPLED SYSTEM ON THE HALF–LINE WITH FIRST–ORDER DERIVATIVE DEPENDENCE

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Abstract. In this paper we prove by means of expansion and compression of a cone principle, the existence of a positive solution to the second order boundary value problem

$$\begin{cases} -(\phi_1(u'))'(t) = a_1(t)f_1(t,u(t),v(t),u'(t),v'(t)) & t > 0, \\ -(\phi_2(v'))'(t) = a_2(t)f_2(t,u(t),v(t),u'(t),v'(t)) & t > 0, \\ u(0) = v(0) = \lim\limits_{t\to+\infty} u'(t) = 0, \ \lim\limits_{t\to+\infty} v'(t) = 0, \end{cases}$$

where for $i = 1,2$, $\phi_i\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi_i(0) = 0$, a_i is a measurable function with $a_i(t) > 0$ a.e. t in some interval of $(0,+\infty)$ and the nonlinearity $f_i\colon \mathbb{R}^+ \times (0,+\infty)^4 \to \mathbb{R}^+$ is continuous, and may exhibit singular at $u+v=0$ and $u'+v'=0$.

1. Introduction and main results

Over the past three decades, boundary value problems for ordinary differential equations have become a rapidly growing branch of applied mathematics. The study of these types of problems is driven by the theoretical interest as well as by the fact that several phenomena in engineering and the life sciences are modeled by such problems. Boundary value problems associated with second-order ordinary differential equations posed on the half line arise in many real world applications, such is the case when modeling nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, and concentration in chemical and biological problems, see [1], [2], [3], [9], [12] and [15].

The case of these kind of problems involving the second-order differential operator $(\phi(u'))'$ are commonly called ϕ-Laplacian boundary value problems and they have been the subject of many interesting papers, see [8], [16], [17], [21], [23] and [24]. Recall that the typical case where $\phi(x) = |x|^{p-2}x$ with $p > 1$ corresponds to the so-called one-dimensional p-Laplacian. Such a class of equations arise in different areas of physics, mechanics, and more generally in applied mathematics and the unknown variable u in (1.1) may refer to a density, temperature, etc.... This why the study of

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the existence and multiplicity of positive solutions for such problems have become an important area of investigation in recent years; see [4], [5], [6], [10], [11], [13], [14], [18], [19], [20], [22] and references therein.

This paper concerns the existence of positive solutions to the second order boundary value problem (bvp for short)

$$\begin{cases} -(\phi_1(u'))'(t) = a_1(t)f_1(t,u(t),v(t),u'(t),v'(t)) \ \ t>0, \\ -(\phi_2(v'))'(t) = a_2(t)f_2(t,u(t),v(t),u'(t),v'(t)) \ \ t>0, \\ u(0)=v(0)=\lim\limits_{t\to+\infty} u'(t)=0, \ \lim\limits_{t\to+\infty} v'(t)=0, \end{cases} \tag{1.1}$$

where for $i=1,2$, $\phi_i\colon \mathbb{R}\to\mathbb{R}$ is an increasing homeomorphism such that $\phi_i(0)=0$, a_i is a measurable function with $a_i(t)>0$ a.e. t in some interval of $(0,+\infty)$ and the nonlinearity $f_i\colon \mathbb{R}^+\times(0,+\infty)^4\to\mathbb{R}^+$ is continuous, and may exhibit singular at $u+v=0$ and $u'+v'=0$.

By a positive solution to the bvp (1.1), we mean a pair of functions (u,v) such that $u,v\in C^1([0,+\infty),\mathbb{R})$ $u>0$, $v>0$ in $(0,+\infty)$ and $\phi(u')$, $\phi(v')$ are absolutely continuous on compact intervals of $[0,+\infty)$, satisfying all equations in (1.1).

Throughout this paper, we set $\psi_i\colon = \phi_i^{-1}$ and we suppose that a_i, ϕ_i, and f_i satisfy the following conditions:

$$\begin{cases} \text{there exists } \alpha>0 \text{ such that for all } t\in[0,1] \\ \text{and } u\in\mathbb{R}^+, \quad \phi_1(tu)\geqslant t^{\alpha}\phi_1(u), \end{cases} \tag{1.2}$$

$$\begin{cases} \text{there exists } \beta>0 \text{ such that for all } t\in[0,1] \\ \text{and } u\in\mathbb{R}^+, \quad \phi_2(tu)\geqslant t^{\beta}\phi_2(u), \end{cases} \tag{1.3}$$

$$|a_i|_1 = \int_0^{+\infty} a_i(\tau)d\tau < \infty; \tag{1.4}$$

$$\begin{cases} \text{For all } R>0 \text{ there exists a decreasing function} \\ \Psi_{i,R}:(0,+\infty)\to(0,+\infty) \text{ such that} \\ f_i(t,(1+t)u,(1+t)v,w,z)\leqslant \Psi_{i,R}(u+v), \\ \text{for all } t,u,v,w,z\geqslant 0 \text{ with } u+v\leqslant R \text{ and } w+z>0, \\ \text{and } \int_0^{+\infty} a_i(t)\Psi_{i,R}(r\widetilde{\rho}(t))\,dt<\infty \text{ for all } r\in(0,R] \end{cases} \tag{1.5}$$

where

$$\widetilde{\rho}(t)=\frac{\rho(t)}{1+t} \ \text{ and } \ \rho(t)=\begin{cases} t \text{ if } t\in[0,1] \\ \frac{1}{t} \text{ if } t\geqslant 1, \end{cases}$$

$$\begin{cases} \lim\limits_{t\to+\infty} t\psi_i\left(\int_t^{+\infty} a_i(\tau)f_i(\tau,\lambda,\mu,\xi,\eta)d\tau\right)=+\infty \\ \text{uniformly for } \lambda,\ \mu\ \xi,\ \eta \text{ in compact intervals of } (0,+\infty). \end{cases} \tag{1.6}$$

$$\begin{cases} \text{For all } R > 0 \text{ there exists a function} \\ \chi_{i,R} : (0,+\infty) \to (0,+\infty) \text{ such that} \\ f_i(t,(1+t)u,(1+t)v,w,z) \leqslant \chi_{i,R}(t), \\ \text{for all } t,u,v,w,z \geqslant 0 \text{ with } u+v \leqslant R \text{ and } w+z > 0, \\ \text{and } \int_0^{+\infty} \psi_i\left(\int_s^{+\infty} a_i(t)\chi_{i,R}(t)\,dr\right) dt < \infty. \end{cases} \tag{1.7}$$

Notice that Hypothesis (1.2) is equivalent to

$$\begin{cases} \text{there exists } \alpha > 0 \text{ such that for all } t \in [0,1] \\ \text{and } u \in \mathbb{R}^+, \ \ t\psi_1(u) \geqslant \psi_1(t^\alpha u), \end{cases} \tag{1.8}$$

then to

$$\begin{cases} \text{there exists } \alpha > 0 \text{ such that for all } t \in [0,1] \\ \text{and } u \in \mathbb{R}^+, \ \ \psi_1(tu) \leqslant t^{\frac{1}{\alpha}} \psi_1(u). \end{cases} \tag{1.9}$$

Our approach in this work is based on a fixed point formulation and since the weight a and the nonlinearity f will supposed to be nonnegative functions, we will use in this work an adapted version of the Guo-Krasnoselskii's expansion and compression of a cone principle. Because of the singular nature of the nonlinearity f as well as its dependance on the first derivative and the boundary conditions in (1.1), we look for solutions in the cone of nonnegative and concave function belonging to the linear space E of all functions $u \in C^1([0,+\infty))$, satisfying $u(0) = \lim_{t\to+\infty} u'(t) = 0$.

Notice that functions u in E can be bounded, such is the case for $u_0(t) = \frac{t}{1+t}$, or unbounded as $u_1(t) = \ln(1+t)$. we provide in this study conditions which guarantee the boundedness or the unboundedness of the obtained solution. The main assumption giving existence of a positive solution to the bvp (1.1) looks like that in [6]. Under assumptions on the behavior of the ratio $f(t,u)/\phi(u)$ at 0 and $+\infty$, authors in [6] use the Guo-Krasnosel'skii's fixed point theorem to prove existence of at least one unbounded positive solution.

The statement of the main result in this paper needs to introduce the following notations. For $\theta > 1$, $\sigma \in (0,1)$ and $\eta \in [0,1]$ set $I_\theta = [1/\theta, \theta]$,

$$f_i^0 = \limsup_{|(u+v,w+z)|\to 0} \left(\sup_{t\geqslant 0} \frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v+w+z)}\right),$$

$$f_i^\infty = \limsup_{|(u+v,w+z)|\to +\infty} \left(\sup_{t\geqslant 0} \frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v+w+z)}\right),$$

$$f_{i,0}(\theta) = \liminf_{|(u+v,w+z)|\to 0} \left(\min_{t\in I_\theta} \frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v)}\right),$$

$$f_{i,\infty}(\theta) = \liminf_{|(u+v,w+z)|\to +\infty} \left(\min_{t\in I_\theta} \frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v)}\right),$$

$$\Theta_1(\theta,\eta) = \eta^{-\alpha}\theta^\alpha(1+\theta)^{2\alpha}\left(\int_{\frac{1}{\theta}}^{\theta} a_1(r)dr\right)^{-1}, \qquad \Gamma_1(\sigma) = \sigma^\alpha |a_1|_1^{-1},$$

$$\Theta_2(\theta,\eta) = (1-\eta)^{-\beta}\theta^\beta(1+\theta)^{2\beta}\left(\int_{\frac{1}{\theta}}^{\theta} a_2(r)dr\right)^{-1}, \quad \Gamma_2(\sigma) = (1-\sigma)^\beta |a_2|_1^{-1},$$

where $|(w,z)| = \sup(|w|, |z|)$.

THEOREM 1. *Assume that Hypotheses (1.2)–(1.5) hold and there exists* $\theta > 1$, $\sigma \in (0,1)$ *and* $\eta \in [0,1]$ *such that one of the following conditions*

$$f_i^0 < \Gamma_i(\sigma) \text{ and } \Theta_i(\theta,\eta) < f_{i,\infty}(\theta) \text{ for } i=1 \text{ and } i=2 \tag{1.10}$$

and

$$f_i^\infty < \Gamma_i(\sigma) \text{ and } \Theta_i(\theta,\eta) < f_{i,0}(\theta) \text{ for } i=1 \text{ and } i=2, \tag{1.11}$$

is satisfied. Then the bvp (1.1) has at least one positive solution (u,v). *Moreover, if Hypothesis (1.7) holds then the solution* (u,v) *is bounded and if Hypothesis (1.6) holds, then the solution* (u,v) *is unbounded (i.e.* $\lim_{t\to+\infty}(u,v)(t) = +\infty$*).*

Since for all $t,u,v,w,z > 0$

$$\frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v+w+z)} \leqslant \frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v)}$$

we have

$$f_i^0 \leqslant f_{i,+}^0 = \limsup_{u+v\to 0}\left(\sup_{t,w,z>0}\frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v+w+z)}\right),$$

$$f_{i,0}(\theta) \geqslant f_{i,0}^-(\theta) = \liminf_{u+v\to 0}\left(\inf_{t\in I_\theta,w,z>0}\frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v)}\right),$$

$$f_i^\infty \leqslant f_{i,+}^\infty = \limsup_{u+v\to +\infty}\left(\sup_{t,w,z>0}\frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v+w+z)}\right),$$

$$f_{i,\infty}(\theta) \geqslant f_{i,\infty}^-(\theta) = \liminf_{u+v\to +\infty}\left(\inf_{t\in I_\theta,w,z>0}\frac{f_i(t,(1+t)u,(1+t)v,w,z)}{\phi_i(u+v)}\right).$$

Moreover, notice that if the following hypothesis

$$\begin{cases} \text{for all } R>0 \text{ there exists a function } \omega_{i,R} : (0,+\infty) \to (0,+\infty) \\ \text{such that } f_i(t,u,v,w,z) \geqslant \omega_{i,R}(t) \text{ for all } t,u,v,w,z>0 \text{ with } u+v \leqslant R \\ \text{and } \lim_{t\to+\infty} t\psi_i\left(\int_t^{+\infty} a_i(\tau)\omega_{i,R}(\tau)\,d\tau\right) = +\infty, \end{cases} \tag{1.12}$$

holds, then the nonlinearity f_i satisfies (1.6).

The above remarks and Theorem 1 lead to the following corollary:

COROLLARY 1. *Assume that Hypotheses (1.2)–(1.5) hold and there exists* $\theta > 1$ *such that one of the following conditions*

$$f_{i,+}^0 < \Gamma_i(\sigma) \text{ and } \Theta_i(\theta,\eta) < f_{i,\infty}^-(\theta) \text{ for } i=1 \text{ and } i=2$$

and

$$f_{i,+}^\infty < \Gamma_i(\sigma) \text{ and } \Theta_i(\theta,\eta) < f_{i,0}^-(\theta) \text{ for } i=1 \text{ and } i=2,$$

is satisfied. Then the bvp (1.1) has at least one positive solution (u,v). *Moreover, if Hypothesis (1.7) holds then the solution* (u,v) *is bounded and if Hypothesis (1.6) holds, then the solution* (u,v) *is unbounded.*

2. Abstract background

Let X be a linear space and let $\|\cdot\|_N$ and p be respectively a norm and a semi-norm on X such that $(X, \|\cdot\|)$ is a Banach space, where for $x \in X$, $\|x\| = \max(\|x\|_N, p(x))$. Let K be a cone in X, that is: K is nonempty closed and covex such that $K \cap (-K) = \emptyset$ and $tK \subset K$ for all $t \geqslant 0$. The main result of this work will be proved by means of the following theorem:

THEOREM 2. ([21], Theorem 2.8) *Let r_1, r_2 be two positive real numbers such that $r_1 < r_2$ and let $T\colon\ K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a compact mapping where for $i = 1,2$, $\Omega_i = \{u \in E, \|u\|_N < r_i\}$. If one of the following conditions*

(a) $\|Tu\| \leqslant \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\|_N \geqslant \|u\|_N$ for $u \in K \cap \partial\Omega_2$,

(b) $\|Tu\|_N \geqslant \|u\|_N$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leqslant \|u\|$ for $u \in K \cap \partial\Omega_2$.

is satisfied, then T has at least a fixed point in $K \cap (\Omega_2 \setminus \bar{\Omega_1})$.

The above theorem is a new version of expansion and compression of a cone principal in a Banach space. Its improvement consists in the fact that it does not require bounded sets.

3. Fixed point formulation

Let F be the linear space defined by

$$F = \left\{ u \in C^1(\mathbb{R}^+, \mathbb{R}) : \lim_{t \to +\infty} u'(t) = 0 \right\}$$

By the mean value theorem for all $u \in F$, we have $\lim\limits_{t \to +\infty} \frac{u(t)}{1+t} = 0$. Therefore, equiped with the norm $\|.\|$, where for $u \in F$, $\|u\| = \max\{\|u\|_1, \|u\|_2\}$, where $\|u\|_1 = \sup\limits_{t \geqslant 0} \frac{|u(t)|}{1+t}$ and $\|u\|_2 = \sup\limits_{t \geqslant 0} |u'(t)|$, F becomes a Banach space.

Let E be the subspace of F defined by $E\colon = \{u \in F,\ u(0) = 0\}$. By the mean value theorem for all $u \in E$ and $t > 0$, we have

$$\frac{|u(t)|}{1+t} \leqslant \frac{|u(t)|}{t} = \frac{|u(t) - u(0)|}{t} = |u'(\eta)|$$

for some $\eta \in (0,t)$. This shows that for all $u \in E$, we have $\|u\|_1 \leqslant \|u\|_2$ and $\|u\| = \|u\|_2$. Hence, $(E, \|u\|_2)$ is a Banach space. Let the Banach space $Y = E \times E$ endowed with the sup-norm

$$\|(u,v)\| = \max(\|(u,v)\|_1, \|(u,v)\|_2),$$

where

$$\|(u,v)\|_1 = \|u\|_1 + \|v\|_1 \text{ and } \|(u,v)\|_2 = \|u\|_2 + \|v\|_2$$

Since $\|u\|_1 \leqslant \|u\|_2$, then

$$\|(u,v)\| = \|u\|_2 + \|v\|_2,$$

Thus

$$\|(u,v)\| = \|u\| + \|v\| = \|u\|_2 + \|v\|_2,$$

As usually the use of the fixed point theory needs a compactness criterion. The following lemma is an adapted version to the case of the space E of Corduneanu's compactness criterion ([7], p. 62). It will be used in this work to prove that the operator associated with the fixed point formulation of the bvp (1.1) is completely continuous.

LEMMA 1. ([16]) *A nonempty subset M of E is relatively compact if the following conditions hold:*

(a) M is bounded in E,

(b) the sets $\left\{u : u(t) = \frac{x(t)}{1+t},\ x \in M\right\}$ and $\{u : u(t) = x'(t),\ x \in M\}$ are almost equicontinuous on $\mathbb{R}^+$, that is, equicontinuous on every compact interval of $\mathbb{R}^+$,

(c) the sets $\left\{u : u(t) = \frac{x(t)}{1+t},\ x \in M\right\}$ and $\left\{u : u(t) = x^{'}(t),\ x \in M\right\}$ are equiconvergent at $+\infty$, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $\left|\frac{x(t)}{1+t} - \frac{x(t_\infty)}{1+t_\infty}\right| < \varepsilon$ and $|x'(t) - x'(t_\infty)| < \varepsilon$ for any $t \geqslant T(\varepsilon)$ and $x \in M$.

Throughout this paper P is the cone of Y defined by

$$P = \{(u,v) \in Y, u \geqslant 0, v \geqslant 0 \text{ on } (0,+\infty) \text{ and } u,\, v \text{ are concave in } (0,+\infty)\} \tag{3.1}$$

LEMMA 2. *For all $(u,v) \in P$ and $t > 0$, we have*

$$u(t) + v(t) \geqslant \rho(t)\, \|(u,v)\|_1 .$$

Proof. Let $(u,v) \in P$, $h(t) = \frac{u(t)}{1+t}$ and $\theta \geqslant 1$. Since $h(0) = \lim_{t\to+\infty} h(t) = 0$, h achieves its maximum at some $t_0 > 0$. Because that u is concave on $(0,+\infty)$, we have

$$\begin{aligned} u\left(\frac{1}{\theta}\right) &= u\left(\frac{\theta - 1 + \theta t_0}{\theta + \theta t_0}\frac{1}{\theta - 1 + \theta t_0} + \frac{t_0}{\theta + \theta t_0}\right) \\ &\geqslant \frac{\theta - 1 + \theta t_0}{\theta + \theta t_0} u\left(\frac{1}{\theta - 1 + \theta t_0}\right) + \frac{1}{\theta + \theta t_0} u(t_0) \\ &\geqslant \frac{1}{\theta}\frac{u(t_0)}{1 + t_0} = \frac{1}{\theta}\|u\|_1 . \end{aligned}$$

Similarly,

$$v\left(\frac{1}{\theta}\right) \geqslant \frac{1}{\theta}\|v\|_1 .$$

For $t \geqslant 0$ we distinguish the following cases:

1. $t = 0$, in this case we have $u(0) = 0 = \rho(0)\|u\|_1$ and $v(0) = 0 = \rho(0)\|v\|_1$,
2. $t \geqslant 1$, since u, v are nondecreasing we have in this case
$$u(t) \geqslant u\left(\frac{1}{t}\right) \geqslant \frac{1}{t}\|u\|_1;$$
and
$$v(t) \geqslant v\left(\frac{1}{t}\right) \geqslant \frac{1}{t}\|v\|_1;$$
Then
$$u(t)+v(t) \geqslant \frac{1}{t}\|(u,v)\|_1.$$
3. $0 < t < 1$, we have in this case
$$u(t) = u\left(\frac{1}{\frac{1}{t}}\right) \geqslant \frac{1}{\frac{1}{t}}\|u\|_1 = t\|u\|_1.$$
and
$$v(t) = v\left(\frac{1}{\frac{1}{t}}\right) \geqslant \frac{1}{\frac{1}{t}}\|v\|_1 = t\|v\|_1.$$
Then
$$u(t)+v(t) \geqslant t\|(u,v)\|_1.$$

The claim of the lemma is proved. □

LEMMA 3. *Assume that* (1.4) *and* (1.5) *hold, then there exists a continuous operator* $T : P \setminus \{(0,0)\} \to P$ *such that fixed points of* T *are positive solutions to the bvp* (1.1).

Proof. Let $(u,v) \in P \setminus \{(0,0)\}$ and $R = \|(u,v)\|_1$ For all $s > 0$, we have by hypothesis (1.5)

$$\begin{aligned}
&\int_s^{+\infty} a_i(\tau) f_i(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau))d\tau \\
&= \int_s^{+\infty} a_i(\tau) f_i\left(\tau, (1+\tau)\frac{u(\tau)}{1+\tau}, (1+\tau)\frac{v(\tau)}{1+\tau}, u'(\tau), v'(\tau)\right)d\tau \\
&\leqslant \int_0^{+\infty} a_i(\tau)\Psi_{i,R}(R\widetilde{\rho}(\tau))d\tau < \infty.
\end{aligned}$$

Therefore, for all $t \geqslant 0$

$$\begin{aligned}
&\int_0^t \psi_i\left(\int_s^{+\infty} a_i(\tau) f_i(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau))d\tau\right)ds \\
&\leqslant t\psi_i\left(\int_0^{+\infty} a_i(\tau)\Psi_{i,R}(R\widetilde{\rho}(\tau))d\tau\right) < \infty
\end{aligned}$$

and set

$$w_i(t)=\int_0^t \psi_i\left(\int_s^{+\infty} a_i(\tau)f_i(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right)ds.$$

Notice that w_i is continuously differentiable on $\mathbb{R}^+$ and

$$w_i'(t)=\psi_i\left(\int_t^{+\infty} a_i(\tau)f_i(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right). \tag{3.2}$$

Moreover, we have $w_i(0)=0$, $w_i(t)\geqslant 0$ for all $t\geqslant 0$ and from (3.2), we see that w_i' is nonincreasing, that is w_i is concave, and

$$w_i'(t)\leqslant \psi_i\left(\int_t^{+\infty} a_i(\tau)\Psi_{i,R}(R\widetilde{\rho}(\tau))d\tau\right)\to 0 \text{ as } t\to+\infty.$$

All the above show that $(w_1,w_2)\in P$ and the operator $T_i: P\setminus\{(0,0)\}\to P$ where for $(u,v)\in P\setminus\{(0,0)\}$

$$T_i(u,v)(t)=\int_0^t \psi_i\left(\int_s^{+\infty} a_i(\tau)f_i(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right)ds$$

is well defined.

In order to prove that T_i is continuous, let $(u_n,v_n)_n\subset P\setminus\{(0,0)\}$ be such that $\lim_{n\to+\infty}(u_n,v_n)=(u,v)$ in Y with $(u,v)\in P\setminus\{(0,0)\}$. Let $0<R<\bar{R}$ be such that $\underline{R}\leqslant\|(u_n,v_n)\|\leqslant R$, for all $n\geqslant 1$. We have

$$\sup_{s\geqslant 0}\left|\int_s^{+\infty} a_i(\tau)F_{i,n}\big(u_n(\tau),v_n(\tau)\big)d\tau-\int_s^{+\infty} a_i(\tau)F_i\big(u(\tau),v(\tau)\big)d\tau\right|\leqslant\int_0^{+\infty} g_{i,n}(\tau)d\tau$$

where

$$F_{i,n}\big(u_n(\tau),v_n(\tau)\big)=f_i(\tau,u_n(\tau),v_n(\tau),u_n'(\tau),v_n'(\tau)),$$
$$F_i\big(u(\tau),v(\tau)\big)=f_i(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))$$

and

$$g_{i,n}(\tau)=a_i(\tau)\left|F_{i,n}\big(u_n(\tau),v_n(\tau)\big)-F_i\big(u(\tau),v(\tau)\big)\right|.$$

Clearly, for a.e. $\tau>0$, $g_{i,n}(\tau)\to 0$ and

$$g_{i,n}(\tau)\leqslant 2a_i(\tau)\Psi_{i,\bar{R}}(R\widetilde{\rho}(\tau))$$

where $\Psi_{i,\bar{R}}$ is the function given by Hypothesis (1.5). Since $\int_0^{+\infty} a_i\Psi_{i,\bar{R}}(R\widetilde{\rho}(\tau))d\tau<\infty$, we coclude by Lebesgue dominated convergence theorem that

$$\int_s^{+\infty} a_i(\tau)f_i(\tau,u_n(\tau),v_n(\tau),u_n'(\tau),v_n'(\tau))d\tau$$

converge uniformly to $\int_s^{+\infty} a_i(\tau)f_i(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau$.

Therefore, the uniform continuity of ψ_i on compact intervals leads to

$$\begin{aligned}\lim_{n\to\infty}\|T_i(u_n,v_n)-T_i(u,v)\| &= \lim_{n\to\infty}\|T_i(u_n,v_n)-T_i(u,v)\|_2\\ &= \lim_{n\to\infty}\left(\sup_{s\geqslant 0}\left|(T_i(u_n,v_n))'(s)-(T_i(u,v))'(s)\right|\right)=0.\end{aligned}$$

It is easy to see that if $(u,v)\in P\setminus\{(0,0)\}$ is a fixed point of T, then (u,v) is a positive solution to the bvp (1.1), Ending the proof. $\square$

LEMMA 4. *Assume that* (1.4) *and* (1.5) *hold, then for all* R_1,R_2 *with* $0<R_1<R_2$, $T(B_{R_1,R_2})$ *is relatively compact in* Y, *where* $B_{R_1,R_2}=\{(u,v)\in P, R_1\leqslant\|(u,v)\|_1\leqslant R_2\}$.

Proof. Let R_1,R_2 be such that $0<R_1<R_2$ and let Φ_i be the function defined by $\Phi_i(t)=a_i(t)\Psi_{i,R_2}(R_1\widetilde{\rho}(t))$, where Ψ_{i,R_2} is the function given by Hypothesis (1.5) for $R=R_1$. Therefore, for all $(u,v)\in B_{R_1,R_2}$ and $t>0$, we have

$$a_i(t)f_i(t,u(t),v(t),u'(t),v'(t))\leqslant\Phi_i(t)$$

and

$$\|T_i(u,v)\|=\|T_i(u,v)\|_2\leqslant\psi_i\left(\int_0^{+\infty}\Phi_i(\tau)d\tau\right)<\infty.$$

Proving that $T_i(B_{R_1R_2})$ is bounded.

Let $A>0$ and $t_1,t_2\in[0,A]$ with $t_1<t_2$. We have

$$\begin{aligned}\left|\frac{T_i(u,v)(t_2)}{(1+t_2)}-\frac{T_i(u,v)(t_1)}{(1+t_1)}\right| &\leqslant\left|\frac{1}{(1+t_2)}-\frac{1}{(1+t_1)}\right|\int_0^{t_1}\psi_i\left(\int_s^{+\infty}\Phi_i(\tau)d\tau\right)ds\\ &\quad+\frac{1}{(1+t_2)}\int_{t_1}^{t_2}\psi_i\left(\int_s^{+\infty}\Phi_i(\tau)d\tau\right)ds\\ &\leqslant\overline{\Phi_i}\,|t_2-t_1|+|g_i(t_2)-g_i(t_1)|,\end{aligned}$$

and

$$\left|\phi_i\left((T_i(u,v))'\right)(t_1)-\phi_i\left((T_i(u,v))'\right)(t_2)\right|\leqslant|h_i(t_2)-h_i(t_1)|,$$

where

$$\begin{aligned}\overline{\Phi_i}&=\int_0^A\psi_i\left(\int_s^{+\infty}\Phi_i(\tau)ds\right)d\tau\\ g_i(t)&=\int_0^t\psi_i\left(\int_s^{+\infty}\Phi_i(\tau)d\tau\right)ds\quad\text{and}\\ h_i(t)&=\int_t^{+\infty}\Phi_i(\tau)d\tau.\end{aligned}$$

Let $\varepsilon>0$. Since the functions g, h and ψ are continuous, there exists $\delta_1>0$ such that

$$\begin{gathered}|g_i(s)-g_i(\tau)|<\varepsilon/2\ \text{ for }\ |s-\tau|<\delta_1,\\ |\psi_i(s)-\psi_i(\tau)|<\varepsilon\ \text{ for }\ |s-\tau|<\delta_1,\end{gathered}$$

and there exists $\delta_2 > 0$ such that

$$|h_i(s) - h_i(r)| < \delta_1 \text{ for } |s-r| < \delta_2.$$

Therefore, for $|t_2 - t_1| < \inf\left(\delta_1, \delta_2, \varepsilon/2\overline{\Phi_i}\right)$, we have

$$\left|\tfrac{T_i(u,v)(t_2)}{(1+t_2)} - \tfrac{T_i(u,v)(t_1)}{(1+t_1)}\right| < \varepsilon \text{ and } |(T_i(u,v))'(t_1) - (T_i(u,v))'(t_2)| < \varepsilon.$$

At this stage, for all $(u,v) \in B_{R_1R_2}$ we have

$$(T_i(u,v))'(t) \leqslant \psi_i\left(\int_t^{+\infty} \Phi_i(\tau)d\tau\right)$$

and L'Hopital's rule leads to

$$\lim_{t\to+\infty} \frac{T_i(u,v)(t)}{(1+t)} = \lim_{t\to+\infty} (T_i(u,v))'(t) \leqslant \lim_{t\to+\infty} \psi_i\left(\int_t^{+\infty} \Phi_i(\tau)d\tau\right) = 0.$$

In view of Lemma 1, $T_i(B_{R_1R_2})$ is relatively compact in Y, ending the proof. □

4. Proof of Theorem 1

Step 1. Existence in the case where (1.10) *holds*

Let $\varepsilon > 0$ be such that $(f_i^0 + \varepsilon) < \Gamma_i(\sigma)$. For such a positive real number ε, there exists $R_1 > 0$ such that

$$\begin{cases} f_i(t,(1+t)u,(1+t)v,w,z) \leqslant (f_i^0+\varepsilon)\phi_i(u+v+w+z) \\ \text{for all } u,v,w,z \text{ with } \sup(|u+v|,|w+z|) \leqslant R_1. \end{cases}$$

Thus, for all $(u,v) \in P \cap \partial\Omega_1$, where $\Omega_1 = \{(u,v) \in Y, \|(u,v)\|_1 < R_1\}$, the following estimates hold

$$\begin{aligned} \|T_1(u,v)\| &= \|T_1(u,v)\|_2 \\ &= \sup_{t\in\mathbb{R}^+}\left|\psi_1\left(\int_t^{+\infty} a_1(\tau)f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right)\right| \\ &\leqslant \psi_1\left(\int_0^{+\infty} a_1(\tau)(f_1^0+\varepsilon)\phi_1\left(\frac{u(\tau)+v(\tau)}{1+\tau}+u'(\tau)+v'(\tau)\right)d\tau\right) \\ &\leqslant \psi_1\left(|a|_1(f_1^0+\varepsilon)\phi_1(\|(u,v)\|_1+\|(u,v)\|_2)\right) \\ &\leqslant \psi_1(|a|_1\Gamma_1(\sigma)\phi_1(\|(u,v)\|)) \\ &\leqslant \psi_1\left((\sigma)^\alpha\phi_1(\|(u,v)\|)\right) \\ &\leqslant \psi_1(\phi_1(\sigma\|(u,v)\|)) \\ &= \sigma\|(u,v)\|. \end{aligned}$$

Similarly, we obtain for all $(u,v) \in P \cap \partial\Omega_1$,

$$\|T_2(u,v)\| = \|T_2(u,v)\|_2 \leqslant (1-\sigma)\|(u,v)\|.$$

Thus, for all $(u,v) \in P \cap \partial\Omega_1$ we have

$$\begin{aligned}\|T(u,v)\| &= \left\|\left(T_1(u,v), T_2(u,v)\right)\right\| \\ &= \|T_1(u,v)\| + \|T_2(u,v)\| \\ &\leqslant \sigma \|(u,v)\| + (1-\sigma)\|(u,v)\| \\ &= \|(u,v)\|.\end{aligned}$$

Now let $\varepsilon > 0$ be such that $(f_{i,\infty}(\theta) - \varepsilon) > \Theta_i(\theta,\eta)$. There exists $R_2 > R_1$ such that

$$\begin{cases} f_i(t,(1+t)u,(1+t)v,w,z) > (f_{i,\infty}(\theta)-\varepsilon)\phi_i(u+v) \\ \text{for all } u+v \geqslant R_2,\, t \in I_\theta \text{ and } w,z \geqslant 0 \end{cases}$$

Let

$$\Omega_2 = \left\{(u,v) \in Y : \|(u,v)\|_1 < \theta(1+\theta)R_2\right\}.$$

For all $(u,v) \in P \cap \partial\Omega_2$, we have from (1.8),

$$\begin{aligned}\|T_1(u,v)\|_1 &\geqslant \frac{T_1(u,v)(1/\theta)}{1+(1/\theta)} \geqslant \eta^2 \frac{\theta}{1+\theta} T_1(u,v)\left(\frac{1}{\theta}\right) \\ &= \eta^2 \frac{\theta}{1+\theta} \int_0^{1/\theta} \psi_1\left(\int_s^{+\infty} a_1(\tau) f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right) ds \\ &\geqslant \eta^2 \frac{\theta}{1+\theta} \int_0^{1/\theta} \psi_1\left(\int_{1/\theta}^{\theta} a_1(\tau) f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right) ds \\ &= \eta \frac{\eta}{1+\theta} \psi_1\left(\int_{1/\theta}^{\theta} a_1(\tau) f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right) \\ &\geqslant \eta \psi_1\left(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau) f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right) \\ &\geqslant \eta \psi_1\left(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau)(f_{1,\infty}(\theta)-\varepsilon)\phi_1\left(\frac{u(\tau)+v(\tau)}{1+\tau}\right)d\tau\right) \\ &\geqslant \eta \psi_1\left(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau)(f_{1,\infty}(\theta)-\varepsilon)\phi_1\left(\frac{\|(u,v)\|_1}{\theta(1+\theta)}\right)d\tau\right) \\ &> \eta \psi_1\left(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau)\Theta_1(\theta,\eta)\frac{1}{\theta^\alpha(1+\theta)^\alpha}\phi_1\left(\|(u,v)\|_1\right)d\tau\right) \\ &= \eta \psi_1\left(\phi_1(\|(u,v)\|_1)\Theta_1(\theta,\eta)\frac{\eta^\alpha}{\theta^\alpha(1+\theta)^{2\alpha}} \int_{1/\theta}^{\theta} a_1(\tau)d\tau\right) \\ &= \eta \psi_1\left(\phi_1(\|(u,v)\|_1)\eta^\alpha\right) \geqslant \eta \psi_1\left(\phi_1(\|(u,v)\|_1)\right) = \eta \|(u,v)\|_1.\end{aligned}$$

Similarly, we obtain for all $(u,v) \in P \cap \partial\Omega_2$,

$$\|T_2(u,v)\| \geqslant (1-\eta)\|(u,v)\|_1.$$

Thus, for all $(u,v) \in P \cap \partial\Omega_2$ we have

$$\begin{aligned}\|T(u,v)\| &= \left\|\left(T_1(u,v), T_2(u,v)\right)\right\| \\ &= \|T_1(u,v)\| + \|T_2(u,v)\| \\ &\geqslant \eta \|(u,v)\| + (1-\eta)\|(u,v)\| = \|(u,v)\|.\end{aligned}$$

Therfore, we deduce from Theorem 2, that T admits a fixed point $(u,v) \in P$ with $R_1 \leqslant \|(u,v)\|_1 \leqslant \theta(1+\theta)R_2$ which is, by Lemma 3 a positive solution to the bvp (1.1).

Step 2. Existence in the case where (1.11) *holds*

Let $\varepsilon > 0$ be such that $(f_{i,0}(\theta) - \varepsilon) > \Theta_i(\theta,\eta)$. There exists $\widetilde{R}_1 > 0$ such that

$$\begin{cases} f_i(t,(1+t)u,(1+t)v,w,z) > (f_{i,0}(\theta)-\varepsilon)\phi_i(u+v+w+z) \\ \text{for } |(u+v,w+z)| \leqslant \widetilde{R}_1 \text{ and } t \in I_\theta \end{cases}$$

Thus, for all $(u,v) \in P \cap \partial\Omega_1$, where

$$\Omega_1 = \left\{(u,v) \in Y : \|(u,v)\|_1 < \theta(1+\theta)\widetilde{R}_1\right\}$$

we have

$$\begin{aligned}
\|T_1(u,v)\|_1 &\geqslant \eta^2 \frac{T_1(u,v)(1/\theta)}{1+(1/\theta)} = \eta^2 \frac{\theta}{1+\theta} T_1(u,v)(\frac{1}{\theta}) \\
&= \eta^2 \frac{\theta}{1+\theta} \int_0^{1/\theta} \psi_1 \left(\int_s^{+\infty} a_1(\tau) f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau)) d\tau \right) ds \\
&\geqslant \eta^2 \frac{\theta}{1+\theta} \int_0^{1/\theta} \psi_1 \left(\int_{1/\theta}^{\theta} a_1(\tau) f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau)) d\tau \right) ds \\
&= \eta \frac{\eta}{1+\theta} \psi_1 \left(\int_{1/\theta}^{\theta} a_1(\tau) f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau)) d\tau \right) \\
&\geqslant \eta \psi_1 \left(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau) f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau)) d\tau \right) \\
&\geqslant \eta \psi_1 \Bigg(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau)(f_{1,0}(\theta)-\varepsilon) \\
&\qquad \times \phi_1 \Big(\frac{u(\tau)}{1+\tau} + \frac{v(\tau)}{1+\tau} + u'(\tau) + v'(\tau) \Big) d\tau \Bigg) \\
&\geqslant \psi_1 \left(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau)(f_{1,0}(\theta)-\varepsilon) \phi_1 \Big(\frac{u(\tau)+v(\tau)}{1+\tau} \Big) dr \right) \\
&\geqslant \eta \psi_1 \left(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau)(f_{1,0}(\theta)-\varepsilon) \phi_1 \left(\frac{\|(u,v)\|_1}{\theta(1+\theta)} \right) d\tau \right) \\
&> \eta \psi_1 \left(\frac{\eta^\alpha}{(1+\theta)^\alpha} \int_{1/\theta}^{\theta} a_1(\tau) \Theta_1(\theta,\eta) \frac{1}{\theta^\alpha (1+\theta)^\alpha} \phi_1(\|(u,v)\|_1) d\tau \right) \\
&= \eta \psi_1 \left(\phi_1(\|(u,v)\|_1) \Theta_1(\theta,\eta) \frac{\eta^\alpha}{\theta^\alpha (1+\theta)^{2\alpha}} \int_{1/\theta}^{\theta} a_1(\tau) d\tau \right) \\
&= \eta \psi_1(\phi_1(\|(u,v)\|_1) \Theta_1(\theta,\eta) \Theta_1(\theta,\eta)^{-1}) = \eta \|(u,v)\|_1 .
\end{aligned}$$

Similarly we obtain for all $(u,v) \in P \cap \partial\Omega_1$,

$$\|T_2(u,v)\| \geqslant (1-\eta) \|(u,v)\|_1 .$$

Thus, for all $(u,v) \in P \cap \partial\Omega_1$,

$$\begin{aligned}
\|T(u,v)\|_1 &= \left\|\left(T_1(u,v), T_2(u,v)\right)\right\|_1 \\
&= \|T_1(u,v)\|_1 + \|T_2(u,v)\|_1 \\
&\geqslant \eta \|(u,v)\|_1 + (1-\eta)\|(u,v)\|_1 \\
&= \|(u,v)\|_1 .
\end{aligned}$$

Let $\varepsilon > 0$ be such that $(f_i^\infty + \varepsilon) < \Gamma_i(\sigma)$, there exists $R_{i,\varepsilon} > 0$ such that

$f_i(t,(1+t)u,(1+t)v,w,z) \leqslant (f_i^\infty + \varepsilon)\phi_i(u+v+w+z) + \Psi_{i,R_\varepsilon}(u+v)$ for all $u,v,w,z > 0$,

where Ψ_{i,R_ε} is the function given by Hypothesis (1.5) for $R = R_{i,\varepsilon}$.

Let

$$\begin{aligned}
\Phi_{i,\varepsilon}(t) &= \Psi_{i,R_\varepsilon}\left(R_{i,\varepsilon}\widetilde{\rho}(t)\right) \\
\overline{\Phi}_{i,\varepsilon} &= \int_0^{+\infty} a_i(r)\Phi_{i,\varepsilon}(r)dr \\
R_{i,2} &= \psi_i\left(\frac{\Gamma_i(\sigma)\overline{\Phi}_{i,\varepsilon}}{\Gamma_i(\sigma) - (f_i^\infty + \varepsilon)}\right)
\end{aligned}$$

Notice that for all $R > 2R_{i,2}$,

$$(f_i^\infty + \varepsilon)\Gamma_i(\sigma)^{-1}\phi_i(R/2) + \overline{\Phi}_{i,\varepsilon} \leqslant \phi_i(R/2)$$

and let $\widetilde{R}_2 > \max(\theta(1+\theta)\widetilde{R}_1, 2R_{i,2}, R_{i,\varepsilon})$. Thus for all $(u,v) \in P \cap \partial\Omega_2$, where $\Omega_2 = \left\{(u,v) \in Y : \|(u,v)\|_1 < \widetilde{R}_2\right\}$, we have

$$\begin{aligned}
\|T_1(u,v)\| = \|T_1(u,v)\|_2 &\leqslant \psi_1\left(\int_0^{+\infty} a_1(\tau)f_1(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau))d\tau\right) \\
&\leqslant \psi_1\Bigg(\int_0^{+\infty} a_1(\tau)\Bigg((f_1^\infty + \varepsilon)\phi_1\left(\frac{u(\tau)+v(\tau)}{1+\tau} + u'(\tau) + v'(\tau)\right) \\
&\qquad + \Psi_{1,R_\varepsilon}\left(\frac{u(\tau)+v(\tau)}{1+\tau}\right)\Bigg)d\tau\Bigg) \\
&\leqslant \psi_1\left(\int_0^{+\infty} a_1(\tau)\left((f_1^\infty + \varepsilon)\phi_1(\|(u,v)\|) + \Phi_{1,\varepsilon}(\tau)\right)d\tau\right) \\
&\leqslant \psi_1\left(\left((f_1^\infty + \varepsilon)\phi_1(\|(u,v)\|)|a_1|_1 + \overline{\Phi}_{1,\varepsilon}\right)\right) \\
&= \psi_1\left(\left((f_1^\infty + \varepsilon)(\sigma)^\alpha\Gamma_1(\sigma)^{-1}\phi_1(\|(u,v)\|) + \overline{\Phi}_{1,\varepsilon}\right)\right) \\
&\leqslant \psi_1\left((f_1^\infty + \varepsilon)\Gamma_1(\sigma)^{-1}\phi_1(\sigma\|(u,v)\|) + \overline{\Phi}_{1,\varepsilon}\right) \\
&\leqslant \sigma\|(u,v)\|.
\end{aligned}$$

Similarly, we obtain for all $(u,v) \in P \cap \partial\Omega_2$,

$$\|T_2(u,v)\| = \|T_2(u,v)\|_2 \leqslant (1-\sigma)\|(u,v)\| .$$

Thus, for all $(u,v) \in P \cap \partial\Omega_2$

$$\begin{aligned}\|T(u,v)\| &= \left\|\left(T_1(u,v), T_2(u,v)\right)\right\| \\ &= \|T_1(u,v)\| + \|T_2(u,v)\| \\ &\leqslant \sigma \|(u,v)\| + (1-\sigma)\|(u,v)\| \\ &= \|(u,v)\|.\end{aligned}$$

We deduce from ii) of Theorem 2 that T admits a fixed point $(u,v) \in P$ with $\theta(1+\theta)\widetilde{R}_1 \leqslant \|(u,v)\|_1 \leqslant \widetilde{R}_2$ which is, by Lemma 3, a positive solution to the bvp (1.1).

Step 3. Boundedness of the solution

Let (u,v) be a positive solution of the bvp (1.1) and set $R_0 = \|(u,v)\|_1$. Then for all $t > 0$ we have from Hypothesis (1.7)

$$\begin{aligned}u(t) &= \int_0^t \psi_1\left(\int_s^{+\infty} a_1(\tau) f_1(\tau, (1+\tau)\frac{u(\tau)}{1+\tau}, (1+\tau)\frac{v(\tau)}{1+\tau}, u'(\tau), v'(\tau)) d\tau\right) ds \\ &\leqslant \int_0^t \psi_1\left(\int_s^{+\infty} a_1(\tau)\chi_{1,R_0}(\tau)\, d\tau\right) ds \\ &\leqslant \int_0^{+\infty} \psi_1\left(\int_s^{+\infty} a_1(\tau)\chi_{1,R_0}(\tau)\, d\tau\right) ds < \infty,\end{aligned}$$

and

$$\begin{aligned}v(t) &= \int_0^t \psi_2\left(\int_s^{+\infty} a_2(\tau) f_2(\tau, (1+\tau)\frac{u(\tau)}{1+\tau}, (1+\tau)\frac{v(\tau)}{1+\tau}, u'(\tau), v'(\tau)) d\tau\right) ds \\ &\leqslant \int_0^t \psi_2\left(\int_s^{+\infty} a_2(\tau)\chi_{2,\|(u,v)\|_1}(\tau)\, d\tau\right) ds \\ &\leqslant \int_0^{+\infty} \psi_2\left(\int_s^{+\infty} a_2(\tau)\chi_{2,\|(u,v)\|_1}(\tau)\, d\tau\right) ds < \infty.\end{aligned}$$

Step 4. Unboundedness of the solution

Let $(u,v) \in P$ be a positive solution of the bvp (1.1). We have from, Lemma 3 that

$$\begin{aligned}u(t) &= \int_0^t \psi_1\left(\int_s^{+\infty} a_1(\tau) f_1(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)) d\tau\right) ds \\ &\geqslant \int_0^t \psi_1\left(\int_t^{+\infty} a_1(\tau) f_1(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)) d\tau\right) ds \\ &= t\psi_1\left(\int_t^{+\infty} a_1(\tau) f_1(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)) d\tau\right)\end{aligned}$$

and

$$\begin{aligned} v(t) &= \int_0^t \psi_2\left(\int_s^{+\infty} a_2(\tau) f_2(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)) d\tau\right) ds \\ &\geqslant \int_0^t \psi_2\left(\int_t^{+\infty} a_2(\tau) f_2(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)) d\tau\right) ds \\ &= t\psi_2\left(\int_t^{+\infty} a_2(\tau) f_2(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)) d\tau\right). \end{aligned}$$

Suppose that (u,v) is bounded and let $(u_\infty, v_\infty) = \lim_{t\to+\infty}\big(u(t), v(t)\big) > (0,0)$; that is $\lim_{t\to+\infty} u(t) > 0$ and $\lim_{t\to+\infty} v(t) > 0$. Let $\varepsilon_0 > 0$ be such that $u_\infty - \varepsilon_0 > 0$ and $v_\infty - \varepsilon_0 > 0$. There exists $t_\infty > 0$ such that

$$u(t) \geqslant u_\infty - \varepsilon_0 \quad \text{and} \quad v(t) \geqslant v_\infty - \varepsilon_0, \quad \text{for all } t \geqslant t_\infty.$$

Therefore, we obtain from Hypothesis (1.6) and the above inequalities the contradiction

$$+\infty > u_\infty \geqslant \lim_{t\to+\infty} t\psi_1\left(\int_t^{+\infty} a_1(\tau) f_1(\tau, u(\tau), u'(\tau)) d\tau\right) = +\infty$$

and

$$+\infty > v_\infty \geqslant \lim_{t\to+\infty} t\psi_2\left(\int_t^{+\infty} a_2(\tau) f_2(\tau, u(\tau), u'(\tau)) d\tau\right) = +\infty.$$

The proof of the main theorem is complete.

5. Example

Consider the case of the bvp (1.1) where for $i \in \{1,2\}$ $\phi_i(x) = |x|^{p_i-2}x + |x|^{q_i-2}x$, $a_i(t) = t(1+t)^{-\xi_i}$ and

$$\begin{aligned} f_i(t,u,v,w,z) &= \left(\frac{A_i(1+t)}{u+v} + \frac{B_i(u+v)}{1+t}\right) \\ &\quad \times \left(1 + \frac{w}{1+w} + \frac{z}{1+z} + \sin\left(\frac{1+t}{u+v} + \frac{1}{w+z}\right)\right) \end{aligned}$$

where

$$2 < p_i < q_i, \;\; p_i - 1 > \xi_i > 4 \;\text{ and }\; A_i, B_i > 0.$$

For all $x \geqslant 0$, $t \in [0,1]$, we have

$$t^{q_i-1}\phi_i(x) \leqslant \phi_i(tx) \leqslant t^{p_i-1}\phi_i(x).$$

Then $\alpha = q_1 - 1$ and $\beta = q_2 - 1$. For all $x \geqslant 0$, $s \geqslant 1$, we have

$$\psi_i\left(s^{p_i-1}x\right) \leqslant s\psi_i(x) \leqslant \psi_i\left(s^{q_i-1}x\right). \tag{5.1}$$

We have

$$|a_i|_1 = \frac{1}{(\xi_i - 1)(\xi_i - 2)}, \quad \Gamma_i = \frac{(\xi_i - 1)(\xi_i - 2)}{4^{q_i - 1}},$$

for all $\theta > 1$

$$\Theta_i(\theta) = \theta^{q_i - 1}(1+\theta)^{2(q_i - 1)} \left(\int_{\frac{1}{\theta}}^{\theta} a_i(r)dr \right)^{-1},$$

where

$$\int_{1/\theta}^{\theta} a_i(s)ds = \frac{1}{\xi_i - 2} \frac{\theta^{\xi_i - 2} - 1}{(1+\theta)^{\xi_i - 2}} - \frac{1}{\xi_i - 1} \frac{\theta^{\xi_i - 1} - 1}{(1+\theta)^{\xi_i - 1}}.$$

For all $t, u, v, w, z > 0$ with $u + v < R$,

$$\begin{aligned}
& f_i(t, (1+t)u, (1+t)v, w, z) \\
&= \left(\frac{A_i}{u+v} + B_i(u+v) \right) \left(1 + \frac{w}{1+w} + \frac{z}{1+z} + \sin\left(\frac{1+t}{u+v} + \frac{1}{w+z} \right) \right) \\
&\leqslant 4 \left(\frac{A_i}{u+v} + B_i R \right) \\
&= \Psi_{i,R}(u+v)
\end{aligned}$$

Thus, for all $r \in (0, R]$, we have

$$\int_0^{+\infty} a_i(s) \Psi_{i,R}(r\widetilde{\rho}(s))\, ds = \frac{4A_i}{r} \int_0^{+\infty} \frac{sds}{(1+s)^{\xi_i} \widetilde{\rho}(s)} + B_i R \int_0^{+\infty} \frac{sds}{(1+s)^{\xi_i}} < \infty.$$

Now, we have

$$f_{i,+}^{\infty} = 0 \qquad f_{i,0}^{-}(\theta) = +\infty.$$

Let us prove that u and v are unbounded. By (5.1), we have

$$\begin{aligned}
& t\psi_i \left(\int_t^{+\infty} a_i(\tau) f_i\left(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)\right) d\tau \right) \\
&\geqslant \psi_i \left(t^{p_i - 1} \int_t^{+\infty} a_i(\tau) f_i\left(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)\right) d\tau \right) \\
&\geqslant \psi_i \left(t^{p_i - 1} \int_t^{+\infty} a_i(r) \frac{4B_i(u+v)(1)}{1+\tau} d\tau \right) \\
&= \psi_i \left(4B_i(u+v)(1) t^{p_i - 1} \int_t^{+\infty} \frac{d\tau}{(1+\tau)^{\xi_i + 1}} \right) \\
&= \psi_i \left(\frac{4B_i(u+v)(1)}{\xi_i} \frac{t^{p_i - 1}}{(1+t)^{\xi_i}} \right),
\end{aligned}$$

leading to

$$\lim_{t \to +\infty} t\psi_i \left(\int_t^{+\infty} a_i(\tau) f_i\left(\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)\right) d\tau \right) = +\infty.$$

We conclude from Corollary 1 that the bvp (1.1) admits at least one positive solution (u,v) where each of u and v is unbounded.

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