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# EXTREMAL SOLUTIONS AT INFINITY FOR SYMPLECTIC SYSTEMS ON TIME SCALES II — EXISTENCE THEORY AND LIMIT PROPERTIES

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*Abstract.* In this paper we continue with our investigation of principal and antiprincipal solutions at infinity solutions of a dynamic symplectic system. The paper is a continuation of part I appeared in Differential Equations and Applications in 2022, where we have presenteded a theory of genera of conjoined bases for symplectic dynamic systems on time scales and its connections with principal solutions at infinity and antiprincipal solutions at infinity for these systems together with some basic properties of this new concept on time scales. Here we provide a characterization of all principal solutions of dynamic symplectic system at infinity in the given genus in terms of the initial conditions and a fixed principal solution at infinity from this genus. Further, we provide a characterization of all antiprincipal solutions of dynamic symplectic system at infinity in the given genus in terms of the initial conditions and a fixed principal solution at infinity from this genus. We also establish mutual limit properties of principal and antiprincipal solutions at infinity.

## 1. Introduction

We recall that in our approach we deal with the symplectic dynamic system

$$x^{\Delta} = \mathcal{A}(t)x + \mathcal{B}(t)u, \quad u^{\Delta} = \mathcal{C}(t)x + \mathcal{D}(t)u, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{$\mathbb{S}$}$$

on time scales. We deal with a time scale $\mathbb{T}$, that is, $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ with the standard topology inherited from $\mathbb{R}$. We assume that $\mathbb{T}$ is unbounded from above and bounded from below with $a := \min \mathbb{T}$ and set $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$ as the time scale interval. The coefficients $\mathcal{A}(t)$, $\mathcal{B}(t)$, $\mathcal{C}(t)$, $\mathcal{D}(t)$ of system ($\mathbb{S}$) are real piecewise rd-continuous $n \times n$ matrices on $[a,\infty)_{\mathbb{T}}$ such that the $2n \times 2n$ matrices

$$\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{1.1}$$

satisfy the identity

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^T(t)\,\mathcal{J}\mathcal{S}(t) = 0, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{1.2}$$

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Here $\mu(t)$ is the graininess function of $\mathbb{T}$. We consider the solutions of system ($\mathbb{S}$) as piecewise rd-continuously $\Delta$-differentiable functions, i.e., they are continuous functions on $[a,\infty)_{\mathbb{T}}$ and their $\Delta$-derivative is piecewise rd-continuous on $[a,\infty)_{\mathbb{T}}$.

A solution $(X,U)$ of system ($\mathbb{S}$) is called a *conjoined basis*, if the matrix $X^T(t)\,U(t)$ is a symmetric matrix and $\operatorname{rank}(X^T(t),\,U^T(t))^T = n$ at some and hence at any point $t \in [a,\infty)_{\mathbb{T}}$. According to [13, Definition 3], a conjoined basis $(X,U)$ of ($\mathbb{S}$) is called *nonoscillatory*, if there exists point $\alpha \in [a,\infty)_{\mathbb{T}}$ such that $(X,U)$ has no focal points in the real interval $(\alpha,\infty)$, i.e., if

$$\operatorname{Ker} X(s) \subseteq \operatorname{Ker} X(t) \quad \text{for all } t,s \in [\alpha,\infty)_{\mathbb{T}} \text{ with } t \leqslant s, \tag{1.3}$$

$$X(t)\,[X^{\sigma}(t)]^{\dagger}\,\mathcal{B}(t) \geqslant 0 \quad \text{for all } t \in [\alpha,\infty)_{\mathbb{T}}. \tag{1.4}$$

System ($\mathbb{S}$) is called nonoscillatory if every conjoined basis of ($\mathbb{S}$) is nonoscillatory. We will say that the conjoined basis $(X,U)$ has constant kernel (or constant rank) on the interval $[\alpha,\infty)_{\mathbb{T}}$, if the kernel (or rank) of the matrix $X(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$. As a consequence of (1.3), such properties are always satisfied on intervals $[\beta,\infty)_{\mathbb{T}}$ for sufficiently large $\beta \in [a,\infty)_{\mathbb{T}}$, when the conjoined basis $(X,U)$ is nonoscillatory.

## 2. Known results – recapitulation

In this section we provide a brief list of the results from [14], which are inevitable to make the main part of the article understandable. This section can be skipped by the reader who is familiar with [14]. We divide this section into several subsections and we recommend to see [14] for all the details. The aim of this section is to make the article readable without the need of detailed study of [14].

### 2.1. Matrices

For a matrix $M \in \mathbb{R}^{m\times n}$ we will use the orthogonal decomposition

$$\mathbb{R}^n = (\operatorname{Im} M) \oplus (\operatorname{Ker} M^T), \quad \text{i.e.,} \quad (\operatorname{Im} M)^{\perp} = \operatorname{Ker} M^T. \tag{2.1}$$

For a linear subspace $V \subseteq \mathbb{R}^n$ we denote by $\mathcal{P}_V$ the orthogonal projector onto $V$. Since we work with possibly abnormal symplectic dynamic systems, the Moore-Penrose pseudoinverses occur in our theory. A real $n\times m$ matrix $M^{\dagger}$ satisfying

$$MM^{\dagger}M = M, \quad M^{\dagger}MM^{\dagger} = M^{\dagger}, \quad M^{\dagger}M = (M^{\dagger}M)^T, \quad MM^{\dagger} = (MM^{\dagger})^T, \tag{2.2}$$

is called the *Moore–Penrose pseudoinverse* of the matrix $M$. From [14] we highlight the following properties of the Moore–Penrose pseudoinverse, which we will use later in the proofs of our new results.

REMARK 2.1. (Remark 2.1, [14]) For any real matrix $M \in \mathbb{R}^{m\times n}$ there exists a unique matrix $M^{\dagger} \in \mathbb{R}^{n\times m}$ satisfying the identities in (2.2). Moreover, the following properties hold.

(i) $(M^{\dagger})^T = (M^T)^{\dagger}$, $(M^{\dagger})^{\dagger} = M$, and $\operatorname{Im} M^{\dagger} = \operatorname{Im} M^T$, $\operatorname{Ker} M^{\dagger} = \operatorname{Ker} M^T$.

(ii) The matrix $MM^\dagger$ is the orthogonal projector onto $\operatorname{Im} M$, and the matrix $M^\dagger M$ is the orthogonal projector onto $\operatorname{Im} M^T$. Moreover, $\operatorname{rank} M = \operatorname{rank}(MM^\dagger) = \operatorname{rank}(M^\dagger M)$.

(iii) Let $M(t)$ be an $m \times n$ matrix function defined on the interval $[a,\infty)_{\mathbb{T}}$ such that $\lim_{t\to\infty} M(t) = M$. Then the limit of $M^\dagger(t)$ for $t \to \infty$ exists if and only if there exists a point $t_0 \in [a,\infty)_{\mathbb{T}}$ such that $\operatorname{rank} M(t) = \operatorname{rank} M$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. In this case we have $\lim_{t\to\infty} M^\dagger(t) = M^\dagger$.

(iv) For any matrices $M$ and $N$ with suitable dimensions we have

$$(MN)^\dagger = (\mathcal{P}_{\operatorname{Im} M^T} N)^\dagger \, (M \mathcal{P}_{\operatorname{Im} N})^\dagger = (M^\dagger MN)^\dagger \, (MNN^\dagger)^\dagger. \tag{2.3}$$

### 2.2. Ortogonal projectors

Next, for a matrix function $X : [a,\infty)_{\mathbb{T}} \to \mathbb{R}^{n\times n}$ we define the orthogonal projectors onto the image of $X^T(t)$ or onto the image of $X(t)$ on $[\alpha,\infty)_{\mathbb{T}}$ as follows. For $t \in [a,\infty)_{\mathbb{T}}$ we put

$$P(t) := \mathcal{P}_{\operatorname{Im} X^T(t)} = X^\dagger(t) X(t), \quad R(t) := \mathcal{P}_{\operatorname{Im} X(t)} = X(t) X^\dagger(t), \tag{2.4}$$

i.e., matrices $P(t)$ and $R(t)$ are symmetric on $[a,\infty)_{\mathbb{T}}$ and

$$\operatorname{Im} X^T(t) = \operatorname{Im} P(t), \quad \operatorname{Im} X(t) = \operatorname{Im} R(t), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{2.5}$$

Then for $t \in [a,\infty)_{\mathbb{T}}$ we have

$$P(t)X^\dagger(t) = X^\dagger(t), \quad X^\dagger(t)R(t) = X^\dagger(t), \quad X(t)P(t) = X(t), \quad R(t)X(t) = X(t). \tag{2.6}$$

The orthogonal projectors $P(t)$ and $R(t)$ on $[a,\infty)_{\mathbb{T}}$ are idempotent, i.e.,

$$P(t)P(t) = P(t), \quad R(t)R(t) = R(t), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{2.7}$$

If the matrix function $X(t)$ has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$, then the orthogonal projector $P(t)$ defined in (2.4) is constant on $[\alpha,\infty)_{\mathbb{T}}$. Then we denote by $P$ the corresponding constant orthogonal projector in (2.4), i.e., we define

$$P := P(t) \quad \text{for } t \in [\alpha,\infty)_{\mathbb{T}}, \text{ where } \operatorname{Ker} X(t) \text{ is constant.} \tag{2.8}$$

In our aproach we take advantage of the so-called $S$-matrix associated with a conjoined basis $(X,U)$. Let $(X,U)$ be a conjoined basis of system ($\mathbb{S}$) with constant kernel on the interval $[\alpha,\infty)_{\mathbb{T}}$. Then we define

$$S(t) := \int_\alpha^t [X^\sigma(s)]^\dagger \, \mathcal{B}(s) \, [X^\dagger(s)]^T \, \Delta s, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{2.9}$$

Recall also that in this case the matrix

$$X(t) \, [X^\sigma(t)]^\dagger \, \mathcal{B}(t) \text{ is symmetric for all } t \in [\alpha,\infty)_{\mathbb{T}}. \tag{2.10}$$

when the kernel of $X(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$.

On intervals $[\beta,\infty)_{\mathbb{T}}$ where $\operatorname{Im} S(t)$ is constant we define the associated constant orthogonal projector

$$P_{S\infty} := \mathcal{P}_{\operatorname{Im} S(t)} = S(t)S^{\dagger}(t) = S^{\dagger}(t)S(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{2.11}$$

Then

$$\operatorname{Im} S(t) \subseteq \operatorname{Im} P_{S\infty} \subseteq \operatorname{Im} P, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{2.12}$$

which means that

$$P_{S\infty} S(t) = S(t) = S(t) P_{S\infty}, \quad t \in [\beta,\infty)_{\mathbb{T}}, \quad PP_{S\infty} = P_{S\infty} = P_{S\infty}P. \tag{2.13}$$

In some places we will use the time dependent orthogonal projector

$$P_S(t) := \mathcal{P}_{\operatorname{Im} S(t)}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \quad \lim_{t\to\infty} P_S(t) = P_{S\infty},$$

where the matrix on the right-hand side is given in (2.11).

While the limit of above defined $S$-matrix does not exists automatically as $t$ tends to infinity, it turns out that a limit of its pseudoinverse does. We use this property covered in the proposition below and later in our key definitions of principal and antiprincipal solutions of ($\mathbb{S}$) at infinity.

PROPOSITION 2.2. (Proposition 3.1, [14]) *Let $(X,U)$ be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$ and let the matrix $S(t)$ be given by (2.9). Then the limit of $S^{\dagger}(t)$ as $t\to\infty$ exists. Moreover, the matrix $T$ defined by*

$$T := \lim_{t\to\infty} S^{\dagger}(t) \tag{2.14}$$

*is symmetric and positive semidefinite, i.e., $T \geqslant 0$, and there exists $\beta \in [\alpha,\infty)_{\mathbb{T}}$ such that*

$$\operatorname{rank} T \leqslant \operatorname{rank} S(t) \leqslant \operatorname{rank} X(t) \quad \text{for all} \quad t \in [\beta,\infty)_{\mathbb{T}}.$$

Recall also that

$$P_{S\infty} T = T = T P_{S\infty}, \quad \text{i.e.,} \quad \operatorname{Im} T \subseteq \operatorname{Im} P_{S\infty}. \tag{2.15}$$

The next proposition is proven in [28, Theorem 3.2].

PROPOSITION 2.3. (Proposition 3.2, [14]) *Let $(X,U)$ be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and let the matrices $P$, $R(t)$, $T$ be defined in (2.8), (2.4), and (2.14). Then*

$$R^{\sigma}(t)\mathcal{B}(t) = \mathcal{B}(t), \quad \mathcal{B}(t)R(t) = \mathcal{B}(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{2.16}$$

*If in addition $(X,U)$ has no focal points in $(\alpha,\infty)$, then*

$$PT = T = TP, \quad PT^{\dagger} = T^{\dagger} = T^{\dagger}P. \tag{2.17}$$

### 2.3. Order of abnormality

Since we consider possibly abnormal symplectic system ($\mathbb{S}$), we use the order of abnormality. For any $\alpha \in [a,\infty)_{\mathbb{T}}$ we denote by $\Lambda[\alpha,\infty)_{\mathbb{T}}$ the linear space of $n$-vector functions $u : [\alpha,\infty)_{\mathbb{T}} \to \mathbb{R}^n$ such that $\mathcal{B}(t)\,u(t) = 0$ and $u^{\Delta} = \mathcal{D}(t)\,u(t)$ on $[\alpha,\infty)_{\mathbb{T}}$. The number $d[\alpha,\infty)_{\mathbb{T}} := \dim \Lambda[\alpha,\infty)_{\mathbb{T}}$ is called the *order of abnormality* of system ($\mathbb{S}$) on the interval $[\alpha,\infty)_{\mathbb{T}}$. The limit

$$d_{\infty} := \lim_{t\to\infty} d[t,\infty)_{\mathbb{T}} \quad \text{with } 0 \leqslant d[t,\infty)_{\mathbb{T}} \leqslant d_{\infty} \leqslant n \text{ for } t \in [a,\infty)_{\mathbb{T}},$$

is then called the *maximal order of abnormality* of system ($\mathbb{S}$).

### 2.4. Minimal conjoined bases

Another powerful tool we use are the properties of minimal conjoined bases of ($\mathbb{S}$). Its definition is based on Remark 2.4 below. A conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$ is called *minimal* on the interval $[\alpha,\infty)_{\mathbb{T}}$, if it has the smallest possible rank, i.e., if

$$\operatorname{rank} X(t) = n - d[\alpha,\infty)_{\mathbb{T}} = n - d_{\infty}, \quad t \in [\alpha,\infty)_{\mathbb{T}}.$$

On the other hand, if $\operatorname{rank} X(t) = n$ on $[\alpha,\infty)_{\mathbb{T}}$, then $(X,U)$ is called *maximal* on $[\alpha,\infty)_{\mathbb{T}}$. The existence of conjoined bases of ($\mathbb{S}$) with the range given in (2.19) is discussed in [28, Theorem 5.1], where it is shown that there exists a conjoined bases with any possible rank, thus also those with the minimal possible rank $n - d[\alpha,\infty)_{\mathbb{T}}$.

REMARK 2.4. (Remark 3.3, [14]) Let $(X,U)$ be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$. Let the matrices $P$, $R(t)$, $S(t)$, $P_{S\infty}$ be defined by (2.8), (2.4), (2.9), (2.11). Then from [28, Proposition 3.9] it follows that

$$\operatorname{rank} P_{S\infty} = n - d[\alpha,\infty)_{\mathbb{T}}, \tag{2.18}$$

$$n - d[\alpha,\infty)_{\mathbb{T}} \leqslant \operatorname{rank} X(t) \leqslant n, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{2.19}$$

The existence of conjoined bases of ($\mathbb{S}$) with the range given in (2.19) is discussed in [28, Theorem 5.1], where it is shown that there exists a conjoined bases with any possible rank, thus also those with the minimal possible rank $n - d[\alpha,\infty)_{\mathbb{T}}$.

The following result will be used in the proofs of Theorems 3.1, 4.1, and 5.1 below.

PROPOSITION 2.5. (Proposition 3.14, [14]) *Let $(X,U)$ be a minimal conjoined basis of system ($\mathbb{S}$) on the interval $[\alpha,\infty)_{\mathbb{T}}$, let $P_{S\infty}$ and $T$ defined by (2.11) and (2.14), and assume that $d[\alpha,\infty)_{\mathbb{T}} = d_{\infty}$. Then a solution $(\tilde{X},\tilde{U})$ is a minimal conjoined basis on $[\alpha,\infty)_{\mathbb{T}}$ if and only if there exist matrices $M,N \in \mathbb{R}^{n\times n}$ such that*

$$\tilde{X}(\alpha) = X(\alpha)\,M, \quad \tilde{U}(\alpha) = U(\alpha)\,M + X^{\dagger T}(\alpha)\,N, \tag{2.20}$$

$$M \text{ is nonsingular}, \quad M^T N = N^T M, \quad \operatorname{Im} N \subseteq \operatorname{Im} P_{S\infty}, \tag{2.21}$$

$$N M^{-1} + T \geqslant 0. \tag{2.22}$$

*In this case the matrix* $\tilde{T}$ *in* (2.14) *corresponding to* $(\tilde{X},\tilde{U})$ *satisfies*

$$\tilde{T} = M^T TM + M^T N, \quad \operatorname{rank}\tilde{T} = \operatorname{rank}(NM^{-1} + T). \tag{2.23}$$

A crucial and important relation follows in the proposition below.

PROPOSITION 2.6. (Proposition 3.13, [14]) *Let* $(X,U)$ *be a conjoined basis of* $(\mathbb{S})$ *on* $[\alpha,\infty)_{\mathbb{T}}$ *with constant kernel on* $[\alpha,\infty)_{\mathbb{T}}$ *and no focal points in* $(\alpha,\infty)$ *and assume that* $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. *Then* $(X,U)$ *is a minimal conjoined basis of* $(\mathbb{S})$ *on the interval* $[\alpha,\infty)_{\mathbb{T}}$ *if and only if the orthogonal projectors* $P$ *and* $P_{S\infty}$ *defined by* (2.8) *and* (2.11) *satisfy*

$$P = P_{S\infty}. \tag{2.24}$$

## 2.5. Special normalized conjoined bases

The following proposition provides the existence of special normalized conjoined bases, which possess useful additional properties.

PROPOSITION 2.7. (Proposition 3.9, [14]) *Let* $(X,U)$ *be a conjoined basis of* $(\mathbb{S})$ *with constant kernel on* $[\alpha,\infty)_{\mathbb{T}}$, *let the matrices* $P$ *and* $S(t)$ *defined by* (2.8) *and* (2.9). *Then there exists a conjoined basis* $(\bar{X},\bar{U})$ *of* $(\mathbb{S})$ *such that* $(X,U)$ *and* $(\bar{X},\bar{U})$ *satisfy*

*(i) the Wronskian* $\mathcal{W} := X^T(t)\bar{U}(t) - U^T(t)\bar{X}(t) \equiv I$ *on* $[a,\infty)_{\mathbb{T}}$, *and*

*(ii)* $X^\dagger(\alpha)\bar{X}(\alpha) = 0$.

*Moreover, such a conjoined basis* $(\bar{X},\bar{U})$ *then satisfies*

*(iii) the equality* $X^\dagger(t)\bar{X}(t)P = S(t)$ *for all* $t \in [\alpha,\infty)_{\mathbb{T}}$,

*(iv) the equalities* $\bar{X}(t)P = X(t)S(t)$ *for all* $t \in [\alpha,\infty)_{\mathbb{T}}$ *(in particular* $\bar{X}(\alpha)P = 0$*) and* $\bar{U}(t)P = U(t)S(t) + X^{\dagger T}(t) + U(t)(I-P)\bar{X}^T(t)X^{\dagger T}(t)$ *for all* $t \in [\alpha,\infty)_{\mathbb{T}}$,

*(v) the equality* $\operatorname{Ker}\bar{X}(t) = \operatorname{Im}[P - P_S(t)] = \operatorname{Im}P \cap \operatorname{Ker}S(t)$ *for all* $t \in [\alpha,\infty)_{\mathbb{T}}$,

*(vi) the equality* $\bar{P}(t) = I - P + P_S(t)$ *for all* $t \in [\alpha,\infty)_{\mathbb{T}}$, *where* $\bar{P}(t) := \bar{X}^\dagger(t)\bar{X}(t)$,

*(vii) the equalities* $S^\dagger(t) = \bar{X}^\dagger(t)X(t)P_S(t) = \bar{X}^\dagger(t)X(t)\bar{P}(t)$ *for all* $t \in [\alpha,\infty)_{\mathbb{T}}$.

## 2.6. Key definitions

In this subsection we provide the definitions of two main conpect we work with, those are antiprincipal and principal solution of $(\mathbb{S})$ at infinity. To avoid misunderstanding we also recall a definition of a principal solution of $(\mathbb{S})$ at point $\alpha$.

DEFINITION 2.8. (Definition 3.4, [14]) A conjoined basis $(X,U)$ of $(\mathbb{S})$ is said to be an *antiprincipal solution at infinity* with respect to the interval $[\alpha,\infty)_{\mathbb{T}} \subseteq [a,\infty)_{\mathbb{T}}$ if

(i) the order of abnormality of ($\mathbb{S}$) on the interval $[\alpha,\infty)_{\mathbb{T}}$ is maximal, i.e., $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$,

(ii) the conjoined basis $(X,U)$ has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$,

(iii) the matrix $T$ defined in (2.14) corresponding to $(X,U)$ satisfies $\operatorname{rank} T = n - d_\infty$.

DEFINITION 2.9. (Definition 3.5, [14]) A conjoined basis $(\hat{X},\hat{U})$ of ($\mathbb{S}$) is said to be a *principal solution at infinity* with respect to the interval $[\alpha,\infty)_{\mathbb{T}} \subseteq [a,\infty)_{\mathbb{T}}$ if

(i) the conjoined basis $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$,

(ii) the matrix $\hat{T}$ defined in (2.14) associated with $(\hat{X},\hat{U})$ satisfies $\operatorname{rank}\hat{T} = 0$, i.e., $\hat{T} = 0$.

The (anti)principal solution, which is at the same time minimal/maximal on $[\alpha,\infty)_{\mathbb{T}}$, is called *minimal/maximal (anti)principal solution at infinity*.

By the *principal solution of* ($\mathbb{S}$) *at the point* $\alpha \in [a,\infty)_{\mathbb{T}}$, denoted by $(\hat{X}^{[\alpha]},\hat{U}^{[\alpha]})$, we mean the conjoined basis of the system ($\mathbb{S}$) satisfying the initial conditions

$$\hat{X}^{[\alpha]}(\alpha) = 0 \text{ and } \hat{U}^{[\alpha]}(\alpha) = I. \tag{2.25}$$

The result of [15, Theorem 4.4] reveals that the existence of a limit of $S$-matrix associated with a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$ is actually a distinguishing mark for a solution to be an antiprincipal solution of ($\mathbb{S}$) at infinity. We will use it in the proof of Theorems 5.1 and 5.4.

PROPOSITION 2.10. (Proposition 3.8, [14]) *Let $(X,U)$ be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$, let the matrices $S(t)$ and $T$ be given by (2.9) and (2.14), and assume that $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. Then the following statements are equivalent.*

*(i) The conjoined basis $(X,U)$ is an antiprincipal solution of ($\mathbb{S}$) at $\infty$.*

*(ii) The limit of $S(t)$ for $t \to \infty$ exists.*

*(iii) The condition $\lim_{t\to\infty} S(t) = T^\dagger$ holds.*

Let $(\hat{X}_{\min},\hat{U}_{\min})$ be the minimal principal solution of ($\mathbb{S}$) at infinity, i.e., a principal solution of ($\mathbb{S}$) at infinity with $\operatorname{rank}\hat{X}_{\min}(t) = n - d_\infty$ for large $t$. We define the point

$$\hat{\alpha}_{\min} := \inf\left\{\alpha \in [a,\infty)_{\mathbb{T}},\ (\hat{X}_{\min},\hat{U}_{\min}) \text{ has constant kernel on } [\alpha,\infty)_{\mathbb{T}} \atop \text{and no focal points in } (\alpha,\infty)\right\}. \tag{2.26}$$

Note that then

$$d[\hat{\alpha}_{\min},\infty)_{\mathbb{T}} = d_\infty = d[\alpha,\infty)_{\mathbb{T}} \quad \text{for every } \alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}. \tag{2.27}$$

## 2.7. Containing and contained

We say that two solutions $(X_1,U_1)$ and $(X_2,U_2)$ of ($\mathbb{S}$) on some interval $[\alpha,\infty)_{\mathbb{T}}$ are equivalent, if $X_1(t)=X_2(t)$ on $[\alpha,\infty)_{\mathbb{T}}$. Let $(X,U)$ be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$ and let the matrices $P$ and $P_{S\infty}$ be defined by (2.8) and (2.11). Consider an orthogonal projector $P_*$ satisfying

$$\operatorname{Im} P_{S\infty} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P. \tag{2.28}$$

We say that a conjoined basis $(X_*,U_*)$ of ($\mathbb{S}$) is *contained* in $(X,U)$ on $[\alpha,\infty)_{\mathbb{T}}$ with respect to $P_*$, or that $(X,U)$ *contains* $(X_*,U_*)$ on $[\alpha,\infty)_{\mathbb{T}}$ with respect to $P_*$, if the solutions $(X_*,U_*)$ and $(XP_*,UP_*)$ are equivalent, that is, if $X_*(t)=X(t)P_*$ on $[\alpha,\infty)_{\mathbb{T}}$. See [28, Definition 4.1] for a roots of this definition.

## 2.8. Mutual representation

The mutual representation of special conjoined bases of ($\mathbb{S}$) is one of the most important tools we use in our approach. Here we recall the results from [14], see it for the additional details.

PROPOSITION 2.11. (Proposition 3.10, [14]) *Let $(X_i,U_i)$ for $i\in\{1,2\}$ be two conjoined bases of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$ and let $P_i$ be the constant orthogonal projector defined in (2.8) through the function $X_i$. Let the conjoined basis $(X_{3-i},U_{3-i})$ be expressed in terms of $(X_i,U_i)$ via the matrices $M_i$ and $N_i$, i.e.,*

$$\begin{pmatrix} X_{3-i}(t) \\ U_{3-i}(t) \end{pmatrix} = \begin{pmatrix} X_i(t) & \bar{X}_i(t) \\ U_i(t) & \bar{U}_i(t) \end{pmatrix} \begin{pmatrix} M_i \\ N_i \end{pmatrix}, \quad t\in[\alpha,\infty)_{\mathbb{T}}, \tag{2.29}$$

*where $(\bar{X}_i,\bar{U}_i)$ is the conjoined basis of ($\mathbb{S}$) satisfying the properties in Proposition 2.7 with respect to $(X_i,U_i)$. If the equality $\operatorname{Im} X_1(\alpha)=\operatorname{Im} X_2(\alpha)$ is satisfied, then for $i\in\{1,2\}$*

*(i) the matrix $M_i^T N_i$ is symmetric and $N_{3-i}=-N_i^T$,*

*(ii) the matrix $M_i$ is invertible and $M_{3-i}=M_i^{-1}$,*

*(iii) the inclusion $\operatorname{Im} N_i\subseteq \operatorname{Im} P_i$ holds.*

*Moreover, the matrices $M_i$ and $N_i$ do not depend on the choice of $(\bar{X}_i,\bar{U}_i)$ with*

$$N_i=\mathcal{W}[(X_i,U_i),(X_{3-i},U_{3-i})]. \tag{2.30}$$

The following properties complement the results in Proposition 2.11 with respect to the associated matrices $S_i(t)$. They are derived in [28, Remark 3.7].

REMARK 2.12. (Remark 3.11, Proposition 3.12, [14]) With the notation and the assumptions in Proposition 2.11, we set

$$L_1 := X_1^{\dagger}(\alpha)X_2(\alpha), \quad L_2 := X_2^{\dagger}(\alpha)X_1(\alpha),$$

and consider the associated matrix $S_i(t)$, which is defined for $t \in [\alpha,\infty)_{\mathbb{T}}$ in (2.9) through the matrix $X_i(t)$. Then the following properties hold for $i \in \{1,2\}$:

$$L_iL_{3-i} = P_i, \quad L_{3-i} = L_i^{\dagger}, \quad L_i = P_iM_i, \quad N_i = P_iN_i, \tag{2.31}$$

$$P_i = \mathcal{P}_{\operatorname{Im} L_i}, \quad L_i^T N_i = M_i^T P_iN_i = M_i^T N_i \text{ is symmetric}, \tag{2.32}$$

$$X_{3-i}(t) = X_i(t)[L_i + S_i(t)N_i], \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{2.33}$$

$$[L_i + S_i(t)N_i]^{\dagger} = L_{3-i} + S_{3-i}(t)N_{3-i}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{2.34}$$

$$\operatorname{Im}[L_i + S_i(t)N_i] = \operatorname{Im} P_i, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{2.35}$$

$$\operatorname{Im}[P_{3-i}M_{3-i}S_i(t)] = \operatorname{Im} S_{3-i}(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{2.36}$$

Note that some additional properties are derived in [28, Remark 3.7], which are not needed in this part of the paper.

### 2.9. Statement we directly use

Finally, we recall the most important statements from [14] connected with the genus of conjoined bases.

DEFINITION 2.13. We say that two conjoined bases $(X_1,U_1)$ and $(X_2,U_2)$ of ($\mathbb{S}$) belong to the same genus $\mathcal{G}$, or have the same genus $\mathcal{G}$, if the matrices $X_1(t)$ and $X_2(t)$ have eventually the same images, i.e., if there exists a point $\alpha \in [a,\infty)_{\mathbb{T}}$ such that

$$\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}.$$

THEOREM 2.14. (Theorem 7.3, [14]) *Assume that system* ($\mathbb{S}$) *is nonoscillatory, let* $\hat{\alpha}_{\min} \in [a,\infty)_{\mathbb{T}}$ *be defined in* (2.26). *Then if* $(\hat{X},\hat{U})$ *is a principal solution of system* ($\mathbb{S}$) *at infinity with respect to the interval* $[\alpha,\infty)_{\mathbb{T}}$ *for some* $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$, *then it is a principal solution of* ($\mathbb{S}$) *at infinity with respect to the interval* $[\beta,\infty)_{\mathbb{T}}$ *for all* $\beta \in (\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$.

THEOREM 2.15. (Theorem 5.3, [14]) *Assume that* ($\mathbb{S}$) *is nonoscillarory. Let* $(X_1,U_1)$ *and* $(X_2,U_2)$ *be conjoined bases of* ($\mathbb{S}$) *with constant kernel on* $[\alpha,\infty)_{\mathbb{T}}$. *Then the following statements are equivalent.*

*(i) The conjoined bases* $(X_1,U_1)$ *and* $(X_2,U_2)$ *belong to the same genus* $\mathcal{G}$.

*(ii) The equality* $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ *holds on some subinterval* $[\beta,\infty)_{\mathbb{T}}$ *of* $[\alpha,\infty)_{\mathbb{T}}$.

*(iii) The equality* $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ *holds on every subinterval* $[\beta,\infty)_{\mathbb{T}}$ *of* $[\alpha,\infty)_{\mathbb{T}}$.

THEOREM 2.16. (Theorem 4.1, [14]) *Assume that system* ($\mathbb{S}$) *is nonoscillatory and let* $(X,U)$ *be a conjoined basis of* ($\mathbb{S}$) *with constant kernel on an interval* $[\alpha,\infty)_{\mathbb{T}}$ *satisfying* $d[\alpha,\infty)_{\mathbb{T}}=d_\infty$ *and no focal points in* $(\alpha,\infty)$. *Then the associated conjoined basis* $(\bar{X},\bar{U})$ *from Proposition* 2.7 *is an antiprincipal solution of* ($\mathbb{S}$) *at infinity, and there exists* $\beta\in[\alpha,\infty)_{\mathbb{T}}$ *such that*

$$\operatorname{rank}\bar{X}(t)=2n-d_\infty-\operatorname{rank}X(t),\quad t\in[\beta,\infty)_{\mathbb{T}}. \tag{2.37}$$

REMARK 2.17. (Remark 5.5, [14]) From Proposition 2.5 and its proof displayed in [15, Theorem 5.1] it can be seen that any two minimal conjoined bases $(X^{(i)}_{\min},U^{(i)}_{\min})$ for $i\in\{1,2\}$ on $[\alpha,\infty)_{\mathbb{T}}$ can be mutually representable in the sense of Proposition 2.11. That is, there exist constant matrices $M^{(i)}_{\min}$ and $N^{(i)}_{\min}$ such that

$$\begin{pmatrix} X^{(3-i)}_{\min}(t)\\ U^{(3-i)}_{\min}(t)\end{pmatrix}=\begin{pmatrix} X^{(i)}_{\min}(t) & \bar{X}^{(i)}_{\max}(t)\\ U^{(i)}_{\min}(t) & \bar{U}^{(i)}_{\max}(t)\end{pmatrix}\begin{pmatrix} M^{(i)}_{\min}\\ N^{(i)}_{\min}\end{pmatrix},\quad t\in[\alpha,\infty)_{\mathbb{T}},$$

where $(\bar{X}^{(i)}_{\max},\bar{U}^{(i)}_{\max})$ is the conjoined basis of ($\mathbb{S}$) satisfying the properties in Proposition 2.7 with respect to $(X^{(i)}_{\min},U^{(i)}_{\min})$. Note that $(\bar{X}^{(i)}_{\max},\bar{U}^{(i)}_{\max})$ is indeed a maximal antiprincipal solutions of ($\mathbb{S}$) at infinity by Theorem 2.16, and

$$N^{(i)}_{\min}=\mathcal{W}[(X^{(i)}_{\min},U^{(i)}_{\min}),(X^{(3-i)}_{\min},U^{(3-i)}_{\min})]. \tag{2.38}$$

Notice that the matrices $M^{(i)}_{\min}$ and $N^{(i)}_{\min}$ satisfy the properties (i)–(iii) from Proposition 2.11 with the associated orthogonal projector $P^{(i)}_{\min}$ from (2.4). Moreover, if $d[\alpha,\infty)_{\mathbb{T}}=d_\infty$ and if we denote by $P_{S_{i\infty}}$ the orthogonal projector from (2.11) associated with $(X^{(i)}_{\min},U^{(i)}_{\min})$, then $P_{S_{i\infty}}=P^{(i)}_{\min}$ (see Proposition 2.6).

PROPOSITION 2.18. (Proposition 5.6, [14]) *Let* $(X_1,U_1)$ *and* $(X_2,U_2)$ *be conjoined bases of* ($\mathbb{S}$) *with constant kernel on* $[\alpha,\infty)_{\mathbb{T}}$ *and no focal points in* $(\alpha,\infty)$ *and let* $P_1$, $P_2$ *and* $P_{S_{1\infty}}$, $P_{S_{2\infty}}$ *be the corresponding orthogonal projectors from* (2.4) *and* (2.11) *associated with conjoined bases* $(X_1,U_1)$ *and* $(X_2,U_2)$, *respectively. Moreover, let* $(X^{(1)}_{\min},U^{(1)}_{\min})$ *be a minimal conjoined basis of* ($\mathbb{S}$), *which is contained in* $(X_1,U_1)$ *on* $[\alpha,\infty)_{\mathbb{T}}$ *with respect to* $P_{S_{1\infty}}$, *and* $(X^{(2)}_{\min},U^{(2)}_{\min})$ *be a minimal conjoined basis of* ($\mathbb{S}$), *which is contained in* $(X_1,U_1)$ *on* $[\alpha,\infty)_{\mathbb{T}}$ *with respect to* $P_{S_{2\infty}}$. *Suppose that* $(X_1,U_1)$ *and* $(X_2,U_2)$ *are mutually representable as in Proposition* 2.11 *on* $[\alpha,\infty)_{\mathbb{T}}$ *through the matrices* $M_1$, $N_1$, $M_2$, $N_2$, *i.e.,* (2.29) *holds. If* $M^{(1)}_{\min}$, $M^{(2)}_{\min}$, $N^{(1)}_{\min}$, $N^{(2)}_{\min}$ *are the corresponding matrices from Remark* 2.17, *then for* $i\in\{1,2\}$ *we have*

$$P_iM_iP_{S_{3-i\infty}}=P_{S_{i\infty}}M^{(i)}_{\min}, \tag{2.39}$$

$$N^{(i)}_{\min}(M^{(i)}_{\min})^{-1}=P_{S_{i\infty}}N_i(M_i)^{-1}P_{S_{i\infty}}. \tag{2.40}$$

The last statement of this section deals with the genera of conjoined bases.

LEMMA 2.19. (Lemma 5.1, [14]) *Let $(X_1,U_1)$ and $(X_2,U_2)$ be two conjoined bases of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ such that there exists $t_0 \in [\alpha,\infty)_{\mathbb{T}}$ such that*

$$\operatorname{Im} X_1(t_0) = \operatorname{Im} X_2(t_0). \tag{2.41}$$

*Then we have $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ for all $t \in [\alpha,\infty)_{\mathbb{T}}$.*

## 3. New results – characterization of principal solutions at infinity

In this section we characterize all principal solutions of $(\mathbb{S})$ at infinity belonging to a given genus $\mathcal{G}$ in terms of the initial conditions with a fixed principal solution at infinity from this genus. These initial conditions are similar to relations (2.20) and (2.21) in Proposition 2.5, which connect all minimal conjoined bases of system $(\mathbb{S})$ on $[\alpha,\infty)_{\mathbb{T}}$. Here we present a complete proof of this statement, thus providing a validation of the corresponding results in the continuous case in [24, Theorem 7.13] and the discrete case in [30, Theorem 5.6], see also [12, Theorem 6.139], with respect to the newly established (2.36). More precisely, by using Proposition 2.12 we correct an incomplete argument presented in the proofs of the above mentioned references. Thus the result below remains valid also in the discrete and continuous case.

THEOREM 3.1. *Assume that system $(\mathbb{S})$ is nonoscillatory and let $\hat{\alpha}_{\min} \in [a,\infty)_{\mathbb{T}}$ be defined in (2.26). Let $(\hat{X},\hat{U})$ be a principal solution of $(\mathbb{S})$ at infinity, which belongs to the genus $\mathcal{G}$, and let $\hat{P}$ and $\hat{R}(t)$ be the orthogonal projectors associated with $(\hat{X},\hat{U})$ defined in (2.4), and $P_{\hat{S}\infty}$ is an orthogonal projector associated with $(\hat{X},\hat{U})$ on $[\beta,\infty)_{\mathbb{T}}$ by (2.11) for a sufficiently large $\beta \in [\alpha,\infty)_{\mathbb{T}}$. Then the solution $(X,U)$ of $(\mathbb{S})$ is a principal solution at infinity belonging to the same genus $\mathcal{G}$ if and only if for some (and hence for any) point $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ there exist matrices $\hat{M},\hat{N} \in \mathbb{R}^{n\times n}$ such that*

$$X(\alpha) = \hat{X}(\alpha)\hat{M}, \quad U(\alpha) = \hat{U}(\alpha)\hat{M} + \hat{X}^{\dagger T}(\alpha)\hat{N}, \tag{3.1}$$

$$\hat{M} \text{ is nonsingular}, \quad \hat{M}^T\hat{N} = \hat{N}^T\hat{M}, \quad \operatorname{Im}\hat{N} \subseteq \operatorname{Im}\hat{P}, \tag{3.2}$$

$$P_{\hat{S}\infty}\hat{N}\hat{M}^{-1}P_{\hat{S}\infty} = 0. \tag{3.3}$$

*Proof.* Let $(\hat{X},\hat{U})$ be a principal solution at infinity, which belongs to the genus $\mathcal{G}$. Proving the first implication from the left to the right, let $(X,U)$ be a principal solution at infinity belonging to the genus $\mathcal{G}$, denote by $R(t)$ and $\hat{R}(t)$ the orthogonal projectors defined in (2.4) associated with conjoined bases $(X,U)$ and $(\hat{X},\hat{U})$. Then according to Theorems 2.14 and 2.15 we know that $(X,U)$ is a principal solution with respect to the interval $[\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$, and hence also $(X,U)$ has constant kernel on $[\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ and no focal points in $(\hat{\alpha}_{\min},\infty)$ and $\operatorname{Im} X(t) = \operatorname{Im}\hat{X}(t)$ on the whole interval $[\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$. Hence, we have $R(t) = \hat{R}(t)$ on $[\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$. Let now $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ be arbitrary but fixed. Then according to Proposition 2.11, where we put $(X_1,U_1) := (\hat{X},\hat{U})$ and $(X_2,U_2) := (X,U)$, the conjoined bases $(X,U)$ and $(\hat{X},\hat{U})$ are mutually representable, namely we can write

$$\begin{pmatrix} X(t) \\ U(t) \end{pmatrix} = \begin{pmatrix} \hat{X}(t) & \underline{\hat{X}}(t) \\ \hat{U}(t) & \underline{\hat{U}}(t) \end{pmatrix} \begin{pmatrix} \hat{M} \\ \hat{N} \end{pmatrix}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.4}$$

where $(\hat{\underline{X}}, \hat{\underline{U}})$ is the conjoined bases of ($\mathbb{S}$) satisfying the properties in Proposition 2.7 with respect to $(\hat{X}, \hat{U})$, and the matrices $\hat{M}$, $\hat{N}$ have the properties from Proposition 2.11 (here we have $M_2 := \hat{M}$ and $N_2 := \hat{N}$). Then the conditions (i)–(iii) in Proposition 2.11 imply that (3.2) holds. The first equation in (3.4) at $t = \alpha$, saying that $X(\alpha) = \hat{X}(\alpha)\hat{M} + \hat{\underline{X}}(\alpha)\hat{N}$, multiplied by $\hat{R}(\alpha)$ from the left, together with Proposition 2.7, $R(t) = \hat{R}(t)$ on $[\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$, and the relation $\hat{R}(\alpha)\hat{X}(\alpha) = \hat{X}(\alpha)$, provides that $X(\alpha) = \hat{X}(\alpha)\hat{M}$, which proves the first part of (3.1). Further, the second equation in (3.4) at the point $\alpha$ offers that

$$U(\alpha) = \hat{U}(\alpha)\hat{M} + \hat{\underline{U}}(\alpha)\hat{N}. \tag{3.5}$$

Since $(\hat{X}, \hat{U})$ and $(\hat{\underline{X}}, \hat{\underline{U}})$ are normalized, we get that $\hat{\underline{U}}(\alpha)\hat{X}^T(\alpha) - \hat{U}(\alpha)\hat{\underline{X}}^T(\alpha) = I$. Multiplying this equality by $\hat{X}^{\dagger T}(\alpha)\hat{N}$ from the right and using $\operatorname{Im}\hat{N} \subseteq \operatorname{Im}\hat{P}$, which implies $\hat{P}\hat{N} = \hat{N}$, we get $\hat{\underline{U}}(\alpha)\hat{N} = \hat{X}^{\dagger T}(\alpha)\hat{N}$, and hence (3.5) provides that (3.1) really holds. It remains to show (3.3). According to [28, Theorem 6.9], there exists a minimal principal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$, which is contained in $(\hat{X}, \hat{U})$ on $[\alpha, \infty)_{\mathbb{T}}$. We denote this minimal principal solution at infinity as $(\hat{X}_{\min}, \hat{U}_{\min})$. Similarly, there exists a minimal principal solution at infinity $(X_{\min}, U_{\min})$, which is contained in $(X, U)$ on $[\alpha, \infty)_{\mathbb{T}}$. But according to [28, Theorem 6.9] the minimal principal solution at infinity is unique up to a right invertible multiple, i.e., there exists a nonsingular constant matrix $\hat{M}_{\min}$ such that

$$X_{\min}(t) = \hat{X}_{\min}(t)\hat{M}_{\min}, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

It means that for the Wronskian of $(\hat{X}_{\min}, \hat{U}_{\min})$ and $(X_{\min}, U_{\min})$ we have

$$\hat{W}_{\min} := \mathcal{W}[(\hat{X}_{\min}, \hat{U}_{\min}), (X_{\min}, U_{\min})] = 0. \tag{3.6}$$

Note that since conjoined bases $(X_{\min}, U_{\min})$ and $(\hat{X}_{\min}, \hat{U}_{\min})$ belong to the same genus, (i.e., the minimal genus $\mathcal{G}_{\min}$), we have $R_{\min}(\alpha) = \hat{R}_{\min}(\alpha)$, where $R_{\min}(t)$ and $\hat{R}_{\min}(t)$ are the orthogonal projectors from (2.4) associated with conjoined bases $(X_{\min}, U_{\min})$ and $(\hat{X}_{\min}, \hat{U}_{\min})$ on $[\alpha, \infty)_{\mathbb{T}}$, respectively. Thus the conjoined bases $(X_{\min}, U_{\min})$ and $(\hat{X}_{\min}, \hat{U}_{\min})$ are mutually representable on $[\alpha, \infty)_{\mathbb{T}}$ in the sense of Proposition 2.11, which together with formula (2.30), in the very same way as we did above while deriving the second part of (3.1), we get $U_{\min}(\alpha) = \hat{U}_{\min}(\alpha)\hat{M}_{\min}$. Thus the minimal conjoined bases $(X_{\min}, U_{\min})$ and $(\hat{X}_{\min}, \hat{U}_{\min})$ are also mutually representable on $[\alpha, \infty)_{\mathbb{T}}$ in the sense of Remark 2.17 on $[\alpha, \infty)_{\mathbb{T}}$, where we put

$$(X^{(1)}_{\min}, U^{(1)}_{\min}) := (X_{\min}, U_{\min}) \quad \text{and} \quad (X^{(2)}_{\min}, U^{(2)}_{\min}) := (\hat{X}_{\min}, \hat{U}_{\min}).$$

Now from Proposition 2.18 with $(X_1, U_1) := (X, U)$ and $(X_2, U_2) := (\hat{X}, \hat{U})$, we get

$$\hat{N}_{\min}(\hat{M}_{\min})^{-1} = P_{\hat{S}_\infty}\hat{N}\hat{M}^{-1}P_{\hat{S}_\infty}. \tag{3.7}$$

But since $\hat{N}_{\min} = \hat{W}_{\min} = 0$ holds by (3.6), it follows from (3.7) that equality (3.3) also holds. The proof of the first implication is finished.

Proving the second implication from the right to the left, we fix $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$ and assume that $(X,U)$ is a solution of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$ such that (3.1)–(3.3) holds. Notice that the first equality in (3.1) and Lemma 2.19 implies that conjoined bases $(X,U)$ and $(\hat{X},\hat{U})$ belong to the same genus $\mathcal{G}$. Also, by the equality in (2.27) we have that $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$.

First we show that $(X,U)$ is a conjoined basis of ($\mathbb{S}$). For this it suffices to verify that $X^T(t)U(t)$ is symmetric and $\operatorname{rank}(X^T(t),U^T(t)) = n$ at the point $t = \alpha$. Using the third relation from (3.2) we get that

$$\hat{P}\hat{N} = \hat{N} \tag{3.8}$$

and (3.1) provides that

$$X^T(\alpha)U(\alpha) = [\hat{X}(\alpha)\hat{M}]^T[\hat{U}(\alpha)\hat{M} + \hat{X}^{\dagger T}(\alpha)\hat{N}] = \hat{M}^T\hat{X}^T(\alpha)\hat{U}(\alpha)\hat{M} + \hat{M}^T\hat{N}.$$

Thus, since $(\hat{X},\hat{U})$ is a conjoined basis of ($\mathbb{S}$) and (3.2) holds, we get that the matrix $X^T(\alpha)U(\alpha)$ is symmetric on $[\alpha,\infty)_{\mathbb{T}}$. Now, $\operatorname{rank}(X^T(\alpha),U^T(\alpha)) = n$ if and only if the conditions $X(\alpha)c = 0$ and $U(\alpha)c = 0$ imply that $c = 0$. Therefore, let $X(\alpha)c = 0$ and $U(\alpha)c = 0$. Then

$$\hat{X}(\alpha)\hat{M}c = 0 \quad \text{and} \quad \hat{U}(\alpha)\hat{M}c + \hat{X}^{\dagger T}(\alpha)\hat{N}c = 0. \tag{3.9}$$

Multiplying the second equality in (3.9) by $\hat{X}^T(\alpha)$ and using the first one from (3.9) and (3.8) we get that

$$\hat{U}(\alpha)\hat{M}c = 0. \tag{3.10}$$

But since $(\hat{X},\hat{U})$ is a conjoined basis of system ($\mathbb{S}$) on $[\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$, the first equality from (3.9) and (3.10) together imply that $\hat{M}c = 0$, and since $\hat{M}$ is nonsingular due to (3.2), we get $c = 0$. This finally proves that $(X,U)$ is a conjoined basis of system ($\mathbb{S}$). Now we show that the kernel of $X(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$. Let the matrix $\hat{S}(t)$ on $[\alpha,\infty)_{\mathbb{T}}$ be a $S$-matrix defined in (2.9) associated with $(\hat{X},\hat{U})$ and let $(\underline{\hat{X}},\underline{\hat{U}})$ be the conjoined basis from Proposition 2.7 associated with $(\hat{X},\hat{U})$, i.e., we put $(X,U) := (\hat{X},\hat{U})$ in Proposition 2.7. Then

$$\begin{pmatrix} X(t) \\ U(t) \end{pmatrix} = \hat{\Phi}(t)\begin{pmatrix} \underline{\hat{M}} \\ \underline{\hat{N}} \end{pmatrix}, \quad \hat{\Phi}(t) := \begin{pmatrix} \hat{X}(t) & \underline{\hat{X}}(t) \\ \hat{U}(t) & \underline{\hat{U}}(t) \end{pmatrix} \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.11}$$

for some matrices $\underline{\hat{M}}$ and $\underline{\hat{N}}$. Using (3.1), similarly as we did in the proof of Proposition 2.5, i.e., using the fact that the matrix $\hat{\Phi}(t)$ is symplectic on $[\alpha,\infty)_{\mathbb{T}}$, evaluating its inverse and using the identity (3.1), the equality $\hat{X}(\alpha)\underline{\hat{X}}(\alpha) = 0$ from Proposition 2.7, and (3.8), we get $\underline{\hat{M}} = \hat{M}$ and $\underline{\hat{N}} = \hat{N}$. Notice that it actually means that expression (3.4) holds, thus equations (3.11) and (3.4) coincide. Further, using Proposition 2.7(iv) together with (3.11), and considering $\hat{X}(t) = \hat{X}(t)\hat{P}$ on the interval $[\alpha,\infty)_{\mathbb{T}}$ from (2.6), we have

$$X(t) = \hat{X}(t)\hat{M} + \hat{X}(t)\hat{S}(t)\hat{N} = \hat{X}(t)[\hat{P}\hat{M} + \hat{S}(t)\hat{N}], \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.12}$$

We will show that

$$\operatorname{Ker} X(t) = \operatorname{Ker}(\hat{P}\hat{M}) \text{ on } [\alpha,\infty)_{\mathbb{T}}. \tag{3.13}$$

Since the symmetry of $\hat{M}^T\hat{N}$ and the invertibility of $\hat{M}$ implies the symmetry of $\hat{N}\hat{M}^{-1}$, and $\hat{N}\hat{M}^{-1} = \hat{N}\hat{M}^{-1}\hat{P}$, then (3.12) provides that

$$X(t) = \hat{X}(t)[I + \hat{S}(t)\hat{N}\hat{M}^{-1}]\hat{P}\hat{M}, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.14}$$

Equality (3.14) shows that $\operatorname{Ker} X(t) \supseteq \operatorname{Ker}(\hat{P}\hat{M})$ for all $t \in [\alpha,\infty)_{\mathbb{T}}$. To derive the opposite inclusion, we fix $t \in [\alpha,\infty)_{\mathbb{T}}$ and let $v \in \operatorname{Ker} X(t)$. Then by (2.12), (3.14), (2.7) we get

$$\hat{X}^{\dagger}(t)\hat{X}(t)[I + \hat{S}(t)\hat{N}\hat{M}^{-1}]\hat{P}\hat{M}v = \hat{P}\hat{M}v + \hat{S}(t)\hat{N}v = 0. \tag{3.15}$$

Put now $w := \hat{P}\hat{M}v$. Then (3.15) can be read as $w = -\hat{S}(t)\hat{N}\hat{M}^{-1}w$. Therefore due to (2.12) we get $w \in \operatorname{Im}\hat{S}(t) \subseteq \operatorname{Im} P_{\hat{S}\infty}$ and consequently

$$\hat{P}\hat{M}v = -\hat{S}(t)P_{\hat{S}\infty}\hat{N}\hat{M}^{-1}P_{\hat{S}\infty}w \overset{(3.3)}{=} 0, \quad t \in [\alpha,\infty)_{\mathbb{T}}.$$

We have just shown that $v \in \operatorname{Ker}(\hat{P}\hat{M})$, which means that $\operatorname{Ker} X(t) \subseteq \operatorname{Ker}(\hat{P}\hat{M})$ for all $t \in [\alpha,\infty)_{\mathbb{T}}$, therefore (3.13) really holds. This implies that $\operatorname{Ker} X(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$.

It remains to show that conjoined basis $(X,U)$ has no focal points in $(\alpha,\infty)$ and that it is a principal solution of ($\mathbb{S}$) at infinity. Once we know that $(X,U)$ has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$, it makes sense to denote by $P$ the orthogonal projector from (2.4) associated with $(X,U)$ on $[\alpha,\infty)_{\mathbb{T}}$. Since $(X,U)$ and $(\hat{X},\hat{U})$ belong to the same genus $\mathcal{G}$ on $[\alpha,\infty)_{\mathbb{T}}$, they are mutually representable in the spirit of Proposition 2.11 as above, namely we have that (3.4) and

$$\begin{pmatrix} \hat{X}(t) \\ \hat{U}(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.16}$$

for some matrices $M$, $N$, where $(\bar{X},\bar{U})$ is the conjoined basis of ($\mathbb{S}$) satisfying the properties in Proposition 2.7 with respect to $(X,U)$. Then Proposition 2.11 brings additional properties of the matrices $M$, $N$ and $\hat{M}$, $\hat{N}$, which are

$$\hat{N} = -N^T \text{ and } \hat{M} = M^{-1}. \tag{3.17}$$

This together with Remark 2.12, namely relations (2.34) and (2.35), provides that

$$\operatorname{Im}[\hat{P}\hat{M} + \hat{S}(t)\hat{N}] = \operatorname{Im}\hat{P}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.18}$$

$$[\hat{P}\hat{M} + \hat{S}(t)\hat{N}]^{\dagger} = PM + S(t)N = P\hat{M}^{-1} - S(t)\hat{N}^T, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.19}$$

Note also that (3.8) and the relation $\operatorname{Im} N \subseteq \operatorname{Im} P$ on $[\alpha,\infty)_{\mathbb{T}}$ brought from Proposition 2.11, and considering (3.16) and (3.17), implies that

$$\hat{N}P = -N^T P = -(PN)^T = -N^T = \hat{N}. \tag{3.20}$$

Equality (3.12) together with Remark 2.1(iv) gives that

$$X^{\dagger}(t) \overset{(2.3)}{=} \left[\mathcal{P}_{\mathrm{Im}\hat{X}^T(t)}\left(\hat{P}\hat{M}+\hat{S}(t)\hat{N}(t)\right)\right]^{\dagger}\left[\hat{X}(t)\mathcal{P}_{\mathrm{Im}[\hat{P}\hat{M}+\hat{S}(t)\hat{N}]}\right]^{\dagger}. \tag{3.21}$$

Moreover, we have that

$$\mathcal{P}_{\mathrm{Im}\hat{X}^T(t)} = \hat{P} = \hat{P}\hat{P} \quad \text{and} \quad \hat{P}\hat{S}(t) = \hat{S}(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.22}$$

Consider (3.18) with (2.6) and (2.5), then (3.21) and (3.22) implies that

$$X^{\dagger}(t) = [\hat{P}\hat{M}+\hat{S}(t)\hat{N}]^{\dagger}\hat{X}^{\dagger}(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.23}$$

Notice that the equality (3.18) means that

$$[\hat{P}\hat{M}+\hat{S}(t)\hat{N}][\hat{P}\hat{M}+\hat{S}(t)\hat{N}]^{\dagger} = \hat{P}, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.24}$$

We will use both equalities later in our subsequent computation. To derive one additional equality, notice that the defining property in (2.9) for the matrix $\hat{S}(t)$ on $[\alpha,\infty)_{\mathbb{T}}$ provides that

$$\hat{S}(t) = \hat{S}^{\sigma}(t) - \mu(t)[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)[\hat{X}^{\dagger}(t)]^T, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.25}$$

We prepared all the auxiliary relations to show that $(X,U)$ has no focal points in $(\alpha,\infty)$ directly by using the defining property (1.4). The idea of the computation is to show that

$$X(t)[X^{\sigma}(t)]^{\dagger}\mathcal{B}(t) = \hat{X}(t)[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.26}$$

In order to prove (3.26) we start the computation on the left-hand side and use the relations we derived above. For a fixed $t \in [\alpha,\infty)_{\mathbb{T}}$ we get

$$\begin{aligned}
&X(t)[X^{\sigma}(t)]^{\dagger}\mathcal{B}(t)\\
&\overset{(3.12)}{=} \hat{X}(t)[\hat{P}\hat{M}+\hat{S}(t)\hat{N}][\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)\\
&\overset{(3.23)}{=} \hat{X}(t)[\hat{P}\hat{M}+\hat{S}(t)\hat{N}][\hat{P}\hat{M}+\hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)\\
&\overset{(3.25)}{=} \hat{X}(t)\{\hat{P}\hat{M}+\hat{S}^{\sigma}(t)\hat{N}-\mu(t)[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)[\hat{X}^{\dagger}(t)]^T\hat{N}\}[\hat{P}\hat{M}+\hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)\\
&= \hat{X}(t)[\hat{P}\hat{M}+\hat{S}^{\sigma}(t)\hat{N}][\hat{P}\hat{M}+\hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)-\mu(t)Z(t)\\
&\overset{(3.24)}{=} \hat{X}(t)\hat{P}[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)-\mu(t)Z(t) \overset{(2.6)}{=} \hat{X}(t)[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)-\mu(t)Z(t),
\end{aligned}$$

where for $t \in [\alpha,\infty)_{\mathbb{T}}$ we denoted the matrix $Z(t)$ as

$$Z(t) := \hat{X}(t)[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t)[\hat{X}^{\dagger}(t)]^T\hat{N}[\hat{P}\hat{M}+\hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t).$$

We will show that $Z(t)=0$ by using the properties of the minimal principal solution of ($\mathbb{S}$) at infinity. Let $(\hat{X}_{\min},\hat{U}_{\min})$ be the minimal principal solution of ($\mathbb{S}$) at infinity which is contained in $(\hat{X},\hat{U})$ on $[\alpha,\infty)_{\mathbb{T}}$. Let $\hat{R}_{\min}(t)$ and $\hat{P}_{\min}$ be the orthogonal projectors from (2.4) associated with $(\hat{X}_{\min}, \hat{U}_{\min})$ on $[\alpha,\infty)_{\mathbb{T}}$. Notice that such a minimal principal solution at infinity exists according to [28, Theorem 6.9], since $(\hat{X}, \hat{U})$ itself is a principal solution of ($\mathbb{S}$) at infinity. Since $(\hat{X}_{\min}, \hat{U}_{\min})$ is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$, we have $\hat{P}_{\min} = P_{\hat{S}\infty}$, by Proposition 2.6, which together with [28, Proposition 4.2] and (2.6) yields that

$$P_{\hat{S}\infty}\hat{X}^{\dagger}_{\min}(t) = \hat{X}^{\dagger}_{\min}(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.27}$$

From the identity

$$\hat{P}_{\min} = \hat{P}\hat{P}_{\min} \text{ on } [\alpha,\infty)_{\mathbb{T}} \tag{3.28}$$

derived from the fact that $(\hat{X}, \hat{U})$ contains $(\hat{X}_{\min}, \hat{U}_{\min})$ on $[\alpha,\infty)_{\mathbb{T}}$, see (2.28), and from (2.6) we also get

$$\begin{aligned}\hat{X}^{\dagger}_{\min}(t) &= \hat{P}_{\min}\hat{X}^{\dagger}_{\min}(t) = \hat{P}\hat{P}_{\min}\hat{X}^{\dagger}_{\min}(t) = \hat{X}^{\dagger}(t)\hat{X}(t)\hat{P}_{\min}\hat{X}^{\dagger}_{\min}(t) \\ &= \hat{X}^{\dagger}(t)\hat{X}_{\min}(t)\hat{X}^{\dagger}_{\min}(t) = \hat{X}^{\dagger}(t)\hat{R}_{\min}(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}.\end{aligned} \tag{3.29}$$

Now equality (2.36) in Proposition 2.12 applied to our case gives that

$$\operatorname{Im}[\hat{P}\hat{M}S(t)] = \operatorname{Im}\hat{S}(t) \subseteq \operatorname{Im} P_{\hat{S}\infty}, \quad \text{i.e.,} \quad P_{\hat{S}\infty}\hat{P}\hat{M}S(t) = \hat{P}\hat{M}S(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.30}$$

Further, notice that Remark 2.12 (with $L_1 := L$ and $L_2 := \hat{L}$) together with the equalities in (3.28) and in (3.30) brings

$$\begin{aligned}S(t) = PS(t) = \hat{L}\hat{L}^{\dagger}S(t) = (\hat{P}\hat{M})^{\dagger}\hat{P}\hat{M}S(t) &\overset{(3.30)}{=} P\hat{M}^{-1}P_{\hat{S}\infty}\hat{P}\hat{M}S(t) \\ &= P\hat{M}^{-1}P_{\hat{S}\infty}\hat{M}S(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}.\end{aligned} \tag{3.31}$$

Now, putting the previous results together, using (2.16) providing

$$\mathcal{B}^T(t) = \mathcal{B}^T(t)\hat{R}^{\sigma}_{\min}(t) \quad \text{and} \quad \mathcal{B}(t) = \hat{R}^{\sigma}_{\min}(t)\mathcal{B}(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.32}$$

we get for $t \in [\alpha,\infty)_{\mathbb{T}}$

$$\begin{aligned}Z(t) &\overset{(2.10)}{=} \mathcal{B}^T(t)[\hat{X}^{\sigma}(t)]^{\dagger T}\hat{X}^T(t)[\hat{X}^{\dagger}(t)]^T\hat{N}[\hat{P}\hat{M} + \hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t) \\ &\overset{(2.4)}{=} \mathcal{B}^T(t)[\hat{X}^{\sigma}(t)]^{\dagger T}\hat{P}\hat{N}[\hat{P}\hat{M} + \hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t) \\ &\overset{(3.8)}{=} \mathcal{B}^T(t)[\hat{X}^{\sigma}(t)]^{\dagger T}\hat{N}[\hat{P}\hat{M} + \hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}(t)]^{\dagger}\mathcal{B}(t) \\ &\overset{(3.32)}{=} \mu(t)\mathcal{B}^T(t)\hat{R}^{\sigma}_{\min}(t)[\hat{X}^{\sigma}(t)]^{\dagger T}\hat{N}[\hat{P}\hat{M} + \hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}(t)]^{\dagger}\hat{R}^{\sigma}_{\min}(t)\mathcal{B}(t) \\ &\overset{(3.29)}{=} \mathcal{B}^T(t)[\hat{X}^{\sigma}_{\min}(t)]^{\dagger T}\hat{N}[\hat{P}\hat{M} + \hat{S}^{\sigma}(t)\hat{N}]^{\dagger}[\hat{X}^{\sigma}_{\min}(t)]^{\dagger}\mathcal{B}(t) \\ &\overset{(3.19)}{=} \mathcal{B}^T(t)[\hat{X}^{\sigma}_{\min}(t)]^{\dagger T}\hat{N}[P\hat{M}^{-1} - S^{\sigma}(t)\hat{N}^T][\hat{X}^{\sigma}_{\min}(t)]^{\dagger}\mathcal{B}(t) \\ &\overset{(3.31)}{=} \mathcal{B}^T(t)[\hat{X}^{\sigma}_{\min}(t)]^{\dagger T}\hat{N}[P\hat{M}^{-1} - P\hat{M}^{-1}P_{\hat{S}\infty}\hat{M}S^{\sigma}(t)\hat{N}^T][\hat{X}^{\sigma}_{\min}(t)]^{\dagger}\mathcal{B}(t) \\ &\overset{(3.20)}{=} \mathcal{B}^T(t)[\hat{X}^{\sigma}_{\min}(t)]^{\dagger T}\hat{N}\hat{M}^{-1}[I - P_{\hat{S}\infty}\hat{M}S^{\sigma}(t)\hat{N}^T][\hat{X}^{\sigma}_{\min}(t)]^{\dagger}\mathcal{B}(t)\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.27)}{=} \mathcal{B}^T(t)[\hat{X}^{\sigma}_{\min}(t)]^{\dagger T} P_{\hat{S}_\infty} \hat{N}\hat{M}^{-1}[I - P_{\hat{S}_\infty}\hat{M}S^{\sigma}(t)\hat{N}^T] P_{\hat{S}_\infty}[\hat{X}^{\sigma}_{\min}(t)]^{\dagger}\mathcal{B}(t) \\
&= \mathcal{B}^T(t)[\hat{X}^{\sigma}_{\min}(t)]^{\dagger T} P_{\hat{S}_\infty} \hat{N}\hat{M}^{-1} P_{\hat{S}_\infty}[I - \hat{M}S^{\sigma}(t)\hat{N}^T P_{\hat{S}_\infty}][\hat{X}^{\sigma}_{\min}(t)]^{\dagger}\mathcal{B}(t) \\
&\overset{(3.3)}{=} 0.
\end{aligned}
$$

This proves that (3.26) is valid. Finally, since $(\hat{X},\hat{U})$ has no focal points in $(\alpha,\infty)$, the conjoined basis $(X,U)$ possess the same property on $(\alpha,\infty)$.

To finish the proof we show that $(X,U)$ is a principal solution of ($\mathbb{S}$) at infinity on $[\alpha,\infty)_{\mathbb{T}}$. Notice that there exists a minimal conjoined basis $(X_*,U_*)$ which is contained in $(X,U)$ on $[\alpha,\infty)_{\mathbb{T}}$. Since all minimal conjoined bases belong to the genus $\mathcal{G}_{\min}$, we can represent them mutually using Proposition 2.11, namely using $(\hat{X}_{\min},\hat{U}_{\min})$ as it was defined above, we can write

$$
\begin{pmatrix} X_*(t) \\ U_*(t) \end{pmatrix} = \begin{pmatrix} \hat{X}_{\min}(t) & \underline{\hat{X}}_{\min}(t) \\ \hat{U}_{\min}(t) & \underline{\hat{U}}_{\min}(t) \end{pmatrix} \begin{pmatrix} \tilde{M} \\ \tilde{N} \end{pmatrix}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.33}
$$

where $(\underline{\hat{X}}_{\min},\underline{\hat{U}}_{\min})$ is the conjoined bases of ($\mathbb{S}$) satisfying the properties in Proposition 2.7 with respect to $(\hat{X}_{\min},\hat{U}_{\min})$, and the matrices $\tilde{M}$, $\tilde{N}$ have the properties from Proposition 2.11. Specifically we need that the matrix $\tilde{M}$ is invertible and $\tilde{N} = \mathcal{W}[(\hat{X}_{\min},\hat{U}_{\min}),(X_*,U_*)]$. Using the fact that both $(X_*,U_*)$ and $(\hat{X}_{\min},\hat{U}_{\min})$ are minimal conjoined bases of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$, Proposition 2.18 reveals that

$$
\tilde{N}\tilde{M}^{-1} = P_{\hat{S}_\infty}\hat{N}\hat{M}^{-1}P_{\hat{S}_\infty} \overset{(3.3)}{=} 0. \tag{3.34}
$$

Equality (3.34) gives that $\tilde{N} = 0$ and hence, it follows from (3.33) that

$$
(X_*,U_*) = (\hat{X}_{\min}\tilde{M},\hat{U}_{\min}\tilde{M}) \quad \text{on } [\alpha,\infty)_{\mathbb{T}},
$$

where the matrix $\tilde{M}$ is invertible. But since $(\hat{X}_{\min},\hat{U}_{\min})$ is a principal solution of ($\mathbb{S}$) at infinity with respect to $[\alpha,\infty)_{\mathbb{T}}$, which is unique up to the right invertible multiple by [28, Theorem 6.9], we get that also $(X_*,U_*)$ is a principal solution of ($\mathbb{S}$) at infinity with respect to $[\alpha,\infty)_{\mathbb{T}}$. Finally, considering [28, Theorem 6.9], it shows of that $(X,U)$ is really a principal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha,\infty)_{\mathbb{T}}$. The proof is complete. $\square$

REMARK 3.2. If we deal with $\mathbb{T} = \mathbb{R}$, then the proof of Theorem 3.1 turns out to be simpler. Note that in this case $\mu(t)Z(t) = 0$ on $[\alpha,\infty)_{\mathbb{T}}$ holds automatically, thus all the additional computations remain necessary only in the other cases of arbitrary time scale $\mathbb{T}$.

If we consider the situation $\mathcal{G} = \mathcal{G}_{\max}$ in Theorem 3.1, then we get the following corollary, which provides a characterization of all maximal principal solutions at infinity in the maximal genus $\mathcal{G}_{\max}$. Since all maximal principal solutions at infinity belong to the same genus $\mathcal{G}_{\max}$, we get that in this case $\hat{P} = I = \hat{R}(t)$ on $[\alpha,\infty)_{\mathbb{T}}$, where $\hat{P}$ and $\hat{R}(t)$ are the orthogonal projectors associated with $(\hat{X},\hat{U})$ defined in (2.4) on $[\alpha,\infty)_{\mathbb{T}}$. The relation $\operatorname{Im}\hat{N} \subseteq \operatorname{Im}\hat{P} = \mathbb{R}^n$ from (3.2) then holds automatically, thus it can be dropped.

COROLLARY 3.3. *Assume that system* ($\mathbb{S}$) *is nonoscillatory and let* $\hat{\alpha}_{\min} \in [a,\infty)_{\mathbb{T}}$ *be defined in* (2.26). *Let* $(\hat{X},\hat{U})$ *be a maximal principal solution of* ($\mathbb{S}$) *at infinity. Let* $P_{\hat{S}\infty}$ *be an orthogonal projector associated with* $(\hat{X},\hat{U})$ *on* $[\alpha,\infty)_{\mathbb{T}}$ *by* (2.11). *Then the solution* $(X,U)$ *of* ($\mathbb{S}$) *is a maximal principal solution at infinity if and only if for some (and hence for any)* $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ *there exist matrices* $\hat{M},\hat{N} \in \mathbb{R}^{n\times n}$ *such that*

$$X(\alpha) = \hat{X}(\alpha)\hat{M}, \quad U(\alpha) = \hat{U}(\alpha)\hat{M} + \hat{X}^{T-1}(\alpha)\hat{N},$$
$$\hat{M} \text{ is nonsingular}, \quad \hat{M}^T\hat{N} = \hat{N}^T\hat{M},$$
$$P_{\hat{S}\infty}\hat{N}\hat{M}^{-1}P_{\hat{S}\infty} = 0.$$

REMARK 3.4. In the same way, as we did above, we can consider the situation when $\mathcal{G} = \mathcal{G}_{\min}$ in Theorem 3.1. Once again, since all minimal principal solutions at infinity belong to the same genus $\mathcal{G}_{\min}$, we get the classification of all minimal principal solutions of ($\mathbb{S}$) at infinity by the given minimal principal solution of ($\mathbb{S}$) at infinity. Then in the notation of Theorem 3.1, considering Proposition 2.6, we have that $\hat{P} = \hat{P}_{\hat{S}\infty}$, where $\hat{P}_{\hat{S}\infty}$ is an orthogonal projector defined in (2.11) associated with $(\hat{X},\hat{U})$ on $[\alpha,\infty)_{\mathbb{T}}$. The third equality in (3.3) for $\mathcal{G}_{\min}$ actually provides that $\hat{N} = \mathcal{W}[(X,U),(\hat{X},\hat{U})] = 0$, showing in (3.1) the solution $(X,U)$ is a right nonsingular multiple of $(\hat{X},\hat{U})$ . This complies with the conclusion of [28, Theorem 6.9].

## 4. Characterization of antiprincipal solutions at infinity

In this section we characterize all antiprincipal solutions of ($\mathbb{S}$) at infinity belonging to a given genus $\mathcal{G}$ in terms of the rank of the Wronskian with a fixed principal solution at infinity from this genus. The following main result is a unification and extension of the continuous case in [25, Theorem 5.13] and the discrete case in [30, Theorem 5.8], see also [12, Theorem 6.141].

THEOREM 4.1. *Assume that system* ($\mathbb{S}$) *is nonoscillatory. Let* $(\hat{X},\hat{U})$ *be a principal solution of* ($\mathbb{S}$) *at infinity belonging to a genus* $\mathcal{G}$ *and let* $(X,U)$ *be a conjoined basis of* ($\mathbb{S}$) *from the same genus* $\mathcal{G}$. *Denote by* $P_{\hat{S}\infty}$ *and* $P_{S\infty}$ *an orthogonal projectors from* (2.11) *associated with* $(\hat{X},\hat{U})$ *and* $(X,U)$ *on* $[\alpha,\infty)_{\mathbb{T}}$, *respectively. Then the conjoined basis* $(X,U)$ *is an antiprincipal solution of* ($\mathbb{S}$) *at infinity if and only if*

$$\operatorname{rank}\left[P_{\hat{S}\infty}\mathcal{W}[(\hat{X},\hat{U}),(X,U)]P_{S\infty}\right] = n - d_\infty. \tag{4.1}$$

*Proof.* Let $(\hat{X},\hat{U})$ be a principal solution of ($\mathbb{S}$) at infinity belonging to a genus $\mathcal{G}$ and let $(X,U)$ be a conjoined basis of ($\mathbb{S}$) from the same genus $\mathcal{G}$. Note that there exists a point $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ such that both $(X,U)$ and $(\hat{X},\hat{U})$ have constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$. We can assume that $(X,U)$ and $(\hat{X},\hat{U})$ belong to $\mathcal{G}$ on $[\alpha,\infty)_{\mathbb{T}}$ due to Theorem 2.15, and $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$ due to [28, Proposition 6.4] and Theorem 2.14. Denote by $P$ and $\hat{P}$ the orthogonal projectors from (2.4) defined on $[\alpha,\infty)_{\mathbb{T}}$ associated with $(X,U)$ and $(\hat{X},\hat{U})$, respectively. Then according to Proposition 2.11 together with Theorem 2.15, these conjoined bases are mutually representable and moreover, some additional properties hold. Specifically, put

$$\begin{pmatrix} X(t) \\ U(t) \end{pmatrix} = \begin{pmatrix} \hat{X}(t) & \underline{\hat{X}}(t) \\ \hat{U}(t) & \underline{\hat{U}}(t) \end{pmatrix} \begin{pmatrix} \hat{M} \\ \hat{N} \end{pmatrix}, \quad \begin{pmatrix} \hat{X}(t) \\ \hat{U}(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{4.2}$$

where $(\bar{X},\bar{U})$ and $(\underline{\hat{X}},\underline{\hat{U}})$ are the conjoined bases of ($\mathbb{S}$) satisfying the properties in Proposition 2.7 with respect to $(X,U)$ and $(\hat{X},\hat{U})$, respectively. Then Proposition 2.11 provides that the matrices $\hat{M}^T\hat{N}$ and $M^TN$ are symmetric and $N = -\hat{N}^T$, the matrices $M$ and $\hat{M}$ are invertible and $M = \hat{M}^{-1}$, and the inclusions

$$\operatorname{Im}\hat{N} \subseteq \operatorname{Im}\hat{P} \quad \text{and} \quad \operatorname{Im}N \subseteq \operatorname{Im}P \tag{4.3}$$

hold. The latter relations show that

$$\hat{P}\hat{N} = \hat{N} = -N^T = -(PN)^T = -N^TP = \hat{N}P. \tag{4.4}$$

Moreover, formula (2.30) says that

$$\hat{N} = \mathcal{W}[(\hat{X},\hat{U}),(X,U)]. \tag{4.5}$$

Let $(X_{\min},U_{\min})$ and $(\hat{X}_{\min},\hat{U}_{\min})$ be the minimal conjoined bases contained in $(X,U)$ and $(\hat{X},\hat{U})$ on $[\alpha,\infty)_{\mathbb{T}}$, respectively, and denote by $\hat{N}_{\min} := \mathcal{W}[(\hat{X}_{\min},\hat{U}_{\min}),(X_{\min},U_{\min})]$ their Wronskian. We now apply Remark 2.17 and Proposition 2.18, where we put

$$(X_1,U_1) := (X,U), \qquad (X_2,U_2) := (\hat{X},\hat{U}),$$
$$(X^{(1)}_{\min},U^{(1)}_{\min}) := (X_{\min},U_{\min}), \qquad (X^{(2)}_{\min},U^{(2)}_{\min}) := (\hat{X}_{\min},\hat{U}_{\min}),$$

and we use relations (4.2) and the relations

$$\begin{pmatrix} X_{\min}(t) \\ U_{\min}(t) \end{pmatrix} = \begin{pmatrix} \hat{X}_{\min}(t) & \underline{\hat{X}}_{\min}(t) \\ \hat{U}_{\min}(t) & \underline{\hat{U}}_{\min}(t) \end{pmatrix} \begin{pmatrix} \hat{M}_{\min} \\ \hat{N}_{\min} \end{pmatrix}, \quad t \in [\alpha,\infty)_{\mathbb{T}},$$
$$\begin{pmatrix} \hat{X}_{\min}(t) \\ \hat{U}_{\min}(t) \end{pmatrix} = \begin{pmatrix} X_{\min}(t) & \bar{X}_{\min}(t) \\ U_{\min}(t) & \bar{U}_{\min}(t) \end{pmatrix} \begin{pmatrix} M_{\min} \\ N_{\min} \end{pmatrix}, \quad t \in [\alpha,\infty)_{\mathbb{T}},$$

where $(\bar{X}_{\min},\bar{U}_{\min})$ and $(\underline{\hat{X}}_{\min},\underline{\hat{U}}_{\min})$ are the conjoined bases of ($\mathbb{S}$) satisfying the properties in Proposition 2.7 with respect to $(X_{\min},U_{\min})$ and $(\hat{X}_{\min},\hat{U}_{\min})$, respectively. Note that Proposition 2.11 provides that the matrices $M_{\min}$ and $\hat{M}_{\min}$ are invertible and $M_{\min} = \hat{M}_{\min}^{-1}$. Then Proposition 2.18 brings that

$$\hat{N}_{\min}(\hat{M}_{\min})^{-1} \overset{(2.40)}{=} P_{\hat{S}\infty}\hat{N}\hat{M}^{-1}P_{\hat{S}\infty}, \tag{4.6}$$

$$P\hat{M}^{-1}P_{\hat{S}\infty} = PMP_{\hat{S}\infty} \overset{(2.39)}{=} P_{S\infty}M_{\min} = P_{S\infty}\hat{M}_{\min}^{-1}. \tag{4.7}$$

Using (4.6) we obtain that

$$\hat{N}_{\min}\hat{M}_{\min}^{-1} \overset{(4.4)}{=} P_{\hat{S}\infty}\hat{N}P\hat{M}^{-1}P_{\hat{S}\infty} \overset{(4.7)}{=} P_{\hat{S}\infty}\hat{N}P_{S\infty}\hat{M}_{\min}^{-1}. \tag{4.8}$$

Moreover, due to [15, Theorems 4.6] we know that $(\hat{X}_{\min}, \hat{U}_{\min})$ is a minimal principal solution of ($\mathbb{S}$) at infinity, i.e., the associated matrix $\hat{T}_{\min}$ in (2.14) satisfies

$$\hat{T}_{\min} = 0. \tag{4.9}$$

Denote by $T$ and $T_{\min}$ the $T$-matrix associated with $(X,U)$ and $(X_{\min}, U_{\min})$ from (2.14), respectively. Notice that Proposition 2.5 and the relation (2.23) reveals the connection between the ranks of $T$-matrices of the above two minimal conjoined bases $(X_{\min}, U_{\min})$ and $(\hat{X}_{\min}, \hat{U}_{\min})$ and that is

$$\operatorname{rank} T_{\min} = \operatorname{rank}(\hat{N}_{\min} \hat{M}_{\min}^{-1} + \hat{T}_{\min}) \overset{(4.9)}{=} \operatorname{rank}(\hat{N}_{\min} \hat{M}_{\min}^{-1}). \tag{4.10}$$

Now we prove the theorem by putting all the previous preparatory considerations together.

Proving the first implication from the left to the right, let $(X,U)$ be an antiprincipal solution of ($\mathbb{S}$) at infinity. Then according to [15, Theorems 4.6] we have that $(X_{\min}, U_{\min})$ is also an antiprincipal solution of ($\mathbb{S}$) at infinity. Notice that (4.10) is valid. On the other hand Definition 2.8 declares that

$$\operatorname{rank} T_{\min} = n - d_\infty, \tag{4.11}$$

since $(X_{\min}, U_{\min})$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. Connecting equalities (4.10) and (4.11), together with the aid of (4.8), we get

$$\begin{aligned}&\operatorname{rank}[P_{\hat{S}\infty} \mathcal{W}[(\hat{X},\hat{U}),(X,U)] P_{S\infty}] \\ &\quad \overset{(4.5)}{=} \operatorname{rank}(P_{\hat{S}\infty} \hat{N} P_{S\infty} \hat{M}_{\min}^{-1}) \overset{(4.8)}{=} \operatorname{rank}(\hat{N}_{\min} \hat{M}_{\min}^{-1}) \overset{(4.10)}{=} \operatorname{rank} T_{\min} \overset{(4.11)}{=} n - d_\infty,\end{aligned}$$

i.e., formula (4.1) holds.

Proving the second implication from the right to the left, assume that (4.1) holds and consider (4.5). But then, since (4.8) and (4.10) hold, we have

$$\operatorname{rank} T_{\min} \overset{(4.10)}{=} \operatorname{rank}(\hat{N}_{\min} \hat{M}_{\min}^{-1}) \overset{(4.8)}{=} \operatorname{rank}(P_{\hat{S}\infty} \hat{N} P_{S\infty} \hat{M}_{\min}^{-1}) \overset{(4.1)}{=} n - d_\infty.$$

This equality shows that $(X_{\min}, U_{\min})$ is a minimal antiprincipal solution of ($\mathbb{S}$) at infinity. Consequently, this proves due to [15, Theorem 6.3] that also $(X,U)$ possesses the same property. The proof is complete. □

If we deal with the minimal genus $\mathcal{G}_{\min}$ in Theorem 4.1, we obtain the following corollary.

COROLLARY 4.2. *Assume that system ($\mathbb{S}$) is nonoscillatory. Let $(\hat{X},\hat{U})$ be a minimal principal solution of ($\mathbb{S}$) belonging to a genus $\mathcal{G}_{\min}$ and let $(X,U)$ be a minimal conjoined basis of ($\mathbb{S}$) from the same genus $\mathcal{G}_{\min}$. Then $(X,U)$ is a minimal antiprincipal solution of ($\mathbb{S}$) at infinity if and only if*

$$\operatorname{rank} \mathcal{W}[(\hat{X},\hat{U}),(X,U)] = n - d_\infty. \tag{4.12}$$

*Proof.* Denote by $\hat{P}$, $P$, $P_{\hat{S}_\infty}, P_{S_\infty}$ the orthogonal projectors from (2.4) and (2.11) associated with $(\hat{X}, \hat{U})$ and $(X, U)$ on $[\alpha, \infty)_{\mathbb{T}}$, respectively. Then, since the conjoined bases $(\hat{X}, \hat{U})$ and $(X, U)$ belong to the minimal genus $\mathcal{G}_{\min}$, we have according to Proposition 2.6 that $P_{\hat{S}_\infty} = \hat{P}$ and $P_{S_\infty} = P$. We use the notation from the proof of Theorem 4.1. Equations (4.3) and (4.5), which are naturally valid also in this case, then imply

$$P_{\hat{S}_\infty} \mathcal{W}[(\hat{X}, \hat{U}), (X, U)] P_{S_\infty} = \mathcal{W}[(\hat{X}, \hat{U}), (X, U)].$$

This together with Theorem 4.1 completes the proof. □

Two corollaries of Theorem 4.1 now follow. We focus on the situation when the maximal order of abnormality $d_\infty = 0$. This is the situation of an eventually controllable system ($\mathbb{S}$) considered in [10].

COROLLARY 4.3. *Assume that* ($\mathbb{S}$) *is nonoscillatory and* $d_\infty = 0$. *Let* $(\hat{X}, \hat{U})$ *be a principal solution of* ($\mathbb{S}$) *at infinity. Then a conjoined basis* $(X, U)$ *of* ($\mathbb{S}$) *is an antiprincipal solution at infinity if and only if the Wronskian* $\mathcal{W}[(\hat{X}, \hat{U}), (X, U)]$ *is invertible.*

*Proof.* If $d_\infty = 0$, then according to Remark 2.4, namely equation (2.19), we have that $X(t)$ is eventually invertible. Moreover, then also $\mathcal{G}_{\min} = \mathcal{G}_{\max}$, i.e., there is only one genus of conjoined bases of ($\mathbb{S}$). The statement of this corollary now follows directly from Corollary 4.2. □

The following corollary is in some sense an extension of [15, Theorem 6.4]. We move the point $\alpha$ to the left beyond the point $\hat{\alpha}_{\min}$ and we add the condition which guarantees the same statement as [15, Theorem 6.4].

COROLLARY 4.4. *Assume that system* ($\mathbb{S}$) *is nonoscillatory and* $d_\infty = 0$. *Let* $(\hat{X}, \hat{U})$ *be a principal solution of* ($\mathbb{S}$) *at infinity. Then for every point* $\alpha \in [a, \infty)_{\mathbb{T}}$ *the principal solution* $(X^{[\alpha]}, U^{[\alpha]})$ *at the point* $\alpha$ *is an antiprincipal solution of* ($\mathbb{S}$) *at infinity if and only if the matrix* $\hat{X}(\alpha)$ *is invertible.*

*Proof.* As well as in the proof of Corollary 4.3 we deduce from the condition on the maximal order of abnormality $d_\infty = 0$ that $\mathcal{G}_{\min} = \mathcal{G}_{\max}$ holds. Let $(X^{[\alpha]}, U^{[\alpha]})$ be the principal solution at the point $\alpha$ and let $(\hat{X}, \hat{U})$ be a principal solution of ($\mathbb{S}$) at infinity. Let $\alpha \in [a, \infty)_{\mathbb{T}}$. Then

$$\mathcal{W}[(\hat{X}, \hat{U}), (X^{[\alpha]}, U^{[\alpha]})] = \hat{X}^T(\alpha) \hat{U}^{[\alpha]}(\alpha) - \hat{U}^T(\alpha) \hat{X}^{[\alpha]}(\alpha) \overset{(2.25)}{=} \hat{X}^T(\alpha). \tag{4.13}$$

Let $\hat{\alpha}_{\min} \in [a, \infty)_{\mathbb{T}}$ be defined in (2.26). We divide the proof into two cases.

If $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$, then $(X^{[\alpha]}, U^{[\alpha]})$ is a minimal (and hence also maximal) antiprincipal solution of ($\mathbb{S}$) at infinity according to [15, Theorem 6.4]. Then due to Corollary 4.2 (in our case with $(X, U) := (X^{[\alpha]}, U^{[\alpha]})$) we have that

$$\operatorname{rank} \hat{X}^T(\alpha) \overset{(4.13)}{=} \operatorname{rank} \mathcal{W}[(\hat{X}, \hat{U}), (X^{[\alpha]}, U^{[\alpha]})] \overset{(4.12)}{=} n - d_\infty = n,$$

and hence $\hat{X}(\alpha)$ is invertible. The statement of the corollary is true.

Let $\alpha \in [a, \hat{\alpha}_{\min})_{\mathbb{T}}$. If we put $(X,U) := (X^{[\alpha]}, U^{[\alpha]})$ in Corollary 4.3, then we directly receive that $(X^{[\alpha]}, U^{[\alpha]})$ is a (minimal) antiprincipal solution of ($\mathbb{S}$) at infinity if and only if the Wronskian $\mathcal{W}[(\hat{X},\hat{U}),(X^{[\alpha]},U^{[\alpha]})]$ is invertible. But this happens, in view of (4.13), if and only if the matrix $\hat{X}(\alpha)$ is invertible. The previous two steps together complete the proof. □

Note that the principal solution $(\hat{X},\hat{U})$ of ($\mathbb{S}$) at infinity used in Corollary 4.4 is in fact the unique minimal (and at the same time maximal) principal solution at infinity from [28, Theorem 6.9]. This follows from the assumption $d_\infty = 0$, which yields that $\mathcal{G}_{\min} = \mathcal{G}_{\max}$.

## 5. Limit properties of principal and antiprincipal solutions at infinity

In this section we deal with the limit properties of principal and antiprincipal solutions of ($\mathbb{S}$) at infinity. These are related to the minimality property of the principal solutions of ($\mathbb{S}$) at infinity, which is known in the eventually controllable case in [10]. The following theorem provides a limit characterization of a principal solution in the genus $\mathcal{G}$. It a unification and extension of the continuous case in [25, Theorem 6.1] and the discrete case in [30, Theorem 6.1], see also [12, Theorem 6.146].

THEOREM 5.1. *Assume that system* ($\mathbb{S}$) *is nonoscillatory. Let* $\hat{\alpha}_{\min} \in [a,\infty)_{\mathbb{T}}$ *be defined in* (2.26) *and let* $(X,U)$ *and* $(\hat{X},\hat{U})$ *be two conjoined bases of* ($\mathbb{S}$), *both from the same genus* $\mathcal{G}$. *Further, let* $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ *be such that* $(X,U)$ *and* $(\hat{X},\hat{U})$ *have constant kernel on* $[\alpha,\infty)_{\mathbb{T}}$ *and no focal points in* $(\alpha,\infty)$. *Denote by* $\hat{P}$, $P_{\hat{S}\infty}$, *and* $P_{S\infty}$ *their associated projectors from* (2.4) *and* (2.11). *Then* $(\hat{X},\hat{U})$ *is a principal solution of* ($\mathbb{S}$) *at infinity and*

$$\operatorname{rank}[P_{\hat{S}\infty}\mathcal{W}[(\hat{X},\hat{U}),(X,U)]P_{S\infty}] = n - d_\infty \tag{5.1}$$

*if and only if*

$$\lim_{t\to\infty} X^\dagger(t)\hat{X}(t) = V, \quad \textit{where} \quad \operatorname{Im} V^T = \operatorname{Im}(\hat{P} - P_{\hat{S}\infty}). \tag{5.2}$$

*Moreover, in this case* $(X,U)$ *is an antiprincipal solution of* ($\mathbb{S}$) *at infinity.*

*Proof.* We divide the proof into two main parts. The first part is a preparatory one, where we derive some additional results, which we will use later. The second part is devoted to the proofs of the implications, which we need to show.

*Preparatory part.* Let $(X,U)$ and $(\hat{X},\hat{U})$ be conjoined bases of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$ belonging to the genus $\mathcal{G}$. Denote by $P$ and $\hat{P}$ the orthogonal projectors associated with $(X,U)$ and $(\hat{X},\hat{U})$ defined in (2.4). Then according to Proposition 2.11 together with Theorem 2.15, these conjoined bases are mutually representable, i.e., for $t \in [\alpha,\infty)_{\mathbb{T}}$ we have

$$\begin{pmatrix} X(t) \\ U(t) \end{pmatrix} = \begin{pmatrix} \hat{X}(t) & \underline{\hat{X}}(t) \\ \hat{U}(t) & \underline{\hat{X}}(t) \end{pmatrix} \begin{pmatrix} \hat{M} \\ \hat{N} \end{pmatrix}, \quad \begin{pmatrix} \hat{X}(t) \\ \hat{U}(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}, \tag{5.3}$$

where $(\bar{X},\bar{U})$ and $(\underline{\hat{X}},\underline{\hat{U}})$ are the conjoined bases of ($\mathbb{S}$) satisfying the properties in Proposition 2.7 with respect to $(X,U)$ and $(\hat{X},\hat{U})$, respectively. Then Proposition 2.11 provides that the matrices $\hat{M}^T\hat{N}$ and $M^TN$ are symmetric and $N=-\hat{N}^T$, the matrices $M$ and $\hat{M}$ are invertible and $M=\hat{M}^{-1}$, and the inclusions

$$\operatorname{Im}\hat{N}\subseteq\operatorname{Im}\hat{P} \quad \text{and} \quad \operatorname{Im}N\subseteq\operatorname{Im}P \tag{5.4}$$

hold. Moreover, formula (2.30) yields that

$$\hat{N}=\mathcal{W}[(\hat{X},\hat{U}),(X,U)]. \tag{5.5}$$

Inclusion (5.4) together with Remark 2.12 (here with conjoined bases $(X_1,U_1):=(X,U)$ and $(X_2,U_2):=(\hat{X},\hat{U})$) imply that

$$N\hat{P}=-\hat{N}^T\hat{P}=-(\hat{P}\hat{N})^T=-\hat{N}^T=N. \tag{5.6}$$

Similarly,

$$\hat{P}\hat{N}=\hat{N}=-N^T=-(PN)^T=-N^TP=\hat{N}P. \tag{5.7}$$

Next, note that the relation $P=\mathcal{P}_{\operatorname{Im}(PM)}$ received from (2.31) and (2.32) in Remark 2.12 can be read as

$$P=PM(PM)^{\dagger}. \tag{5.8}$$

Then it follows that

$$PM\hat{P}=PM\hat{P}\hat{M}\hat{M}^{-1}\overset{(2.31)}{=}PM(PM)^{\dagger}\hat{M}^{-1}\overset{(5.8)}{=}P\hat{M}^{-1}. \tag{5.9}$$

Denote by $S(t)$ and $\hat{S}(t)$ the $S$-matrices from (2.9) corresponding to $(X,U)$ and $(\hat{X},\hat{U})$, respectively. Then equations (2.31) and (2.33) give that

$$X(t)=\hat{X}(t)[\hat{P}\hat{M}+\hat{S}(t)\hat{N}], \quad t\in[\alpha,\infty)_{\mathbb{T}}. \tag{5.10}$$

Notice that Remark 2.12, namely relation (2.35), directly provides that

$$\operatorname{Im}[\hat{P}\hat{M}+\hat{S}(t)\hat{N}]=\operatorname{Im}\hat{P} \quad \text{on } [\alpha,\infty)_{\mathbb{T}}.$$

Moreover, $\hat{P}\hat{P}=\hat{P}$, $\hat{X}(t)\hat{P}=\hat{X}(t)$, and $\hat{P}\hat{S}(t)=\hat{S}(t)$ on $[\alpha,\infty)_{\mathbb{T}}$. Then from equation (5.10) we obtain, by using Remark 2.1(iii) and considering the equalities above, that

$$X^{\dagger}(t)=[\hat{P}\hat{M}+\hat{S}(t)\hat{N}]^{\dagger}\hat{X}^{\dagger}(t), \quad t\in[\alpha,\infty)_{\mathbb{T}}. \tag{5.11}$$

Finally, by combining equations (5.11), (5.6), (5.9), and Remark 2.12 we get

$$\begin{aligned}X^{\dagger}(t)\hat{X}(t)&=[\hat{P}\hat{M}+\hat{S}(t)\hat{N}]^{\dagger}\hat{X}^{\dagger}(t)\hat{X}(t)\overset{(2.34)}{=}[PM+S(t)N]\hat{P}\\&=PM\hat{P}+S(t)N\hat{P}\overset{(5.6)}{=}P\hat{M}^{-1}\hat{P}+S(t)N\\&=P\hat{M}^{-1}-S(t)\hat{N}^T, \quad t\in[\alpha,\infty)_{\mathbb{T}}.\end{aligned} \tag{5.12}$$

Let $(X_{\min},U_{\min})$ and $(\hat{X}_{\min},\hat{U}_{\min})$ be minimal conjoined bases contained in $(X,U)$ and $(\hat{X},\hat{U})$ on $[\alpha,\infty)_{\mathbb{T}}$, respectively. Then $(X_{\min},U_{\min})$ and $(\hat{X}_{\min},\hat{U}_{\min})$ have constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$, since $(X,U)$ and $(\hat{X},\hat{U})$ satisfy the same by the assumptions of the theorem. Denote by $\hat{N}_{\min} := \mathcal{W}[(\hat{X}_{\min},\hat{U}_{\min}),(X_{\min},U_{\min})]$. Now we apply Remark 2.17 and Proposition 2.18 (here with $(X_1,U_1) := (X,U)$ and $(X_2,U_2) := (\hat{X},\hat{U})$ for the first two conjoined bases, and for the associated minimal conjoined bases $(X^{(1)}_{\min},U^{(1)}_{\min}) := (X_{\min},U_{\min})$, and $(X^{(2)}_{\min},U^{(2)}_{\min}) := (\hat{X}_{\min},\hat{U}_{\min})$). We will use the relations in (5.3) and the relations

$$\begin{pmatrix} X_{\min}(t) \\ U_{\min}(t) \end{pmatrix} = \begin{pmatrix} \hat{X}_{\min}(t) & \underline{\hat{X}}_{\min}(t) \\ \hat{U}_{\min}(t) & \underline{\hat{X}}_{\min}(t) \end{pmatrix} \begin{pmatrix} \hat{M}_{\min} \\ \hat{N}_{\min} \end{pmatrix},$$
$$\begin{pmatrix} \hat{X}_{\min}(t) \\ \hat{U}_{\min}(t) \end{pmatrix} = \begin{pmatrix} X_{\min}(t) & \bar{X}_{\min}(t) \\ U_{\min}(t) & \bar{U}_{\min}(t) \end{pmatrix} \begin{pmatrix} M_{\min} \\ N_{\min} \end{pmatrix},$$

for $t \in [\alpha,\infty)_{\mathbb{T}}$, where $(\bar{X}_{\min},\bar{U}_{\min})$ and $(\underline{\hat{X}}_{\min},\underline{\hat{U}}_{\min})$ are the conjoined bases of $(\mathbb{S})$ satisfying the properties in Proposition 2.7 with respect to $(X_{\min},U_{\min})$ and $(\hat{X}_{\min},\hat{U}_{\min})$, respectively. Note that Proposition 2.11 provides that the matrices $M_{\min}$ and $\hat{M}_{\min}$ are invertible and $M_{\min} = \hat{M}^{-1}_{\min}$ and $\hat{M}^T_{\min}\hat{N}_{\min}$ is symmetric. Then Proposition 2.18 brings that

$$PMP_{\hat{S}_\infty} \overset{(2.39)}{=} P_{S_\infty}M_{\min} \quad \text{and} \quad \hat{P}\hat{M}P_{S_\infty} \overset{(2.39)}{=} P_{\hat{S}_\infty}\hat{M}_{\min}, \tag{5.13}$$
$$\hat{N}_{\min}(\hat{M}_{\min})^{-1} \overset{(2.40)}{=} P_{\hat{S}_\infty}\hat{N}\hat{M}^{-1}P_{\hat{S}_\infty}, \tag{5.14}$$
$$P\hat{M}^{-1}P_{\hat{S}_\infty} = PMP_{\hat{S}_\infty} \overset{(5.13)}{=} P_{S_\infty}M_{\min} = P_{S_\infty}\hat{M}^{-1}_{\min}. \tag{5.15}$$

Using (5.14) we obtain that

$$\hat{N}_{\min}\hat{M}^{-1}_{\min} \overset{(5.7)}{=} P_{\hat{S}_\infty}\hat{N}P\hat{M}^{-1}P_{\hat{S}_\infty} \overset{(5.15)}{=} P_{\hat{S}_\infty}\hat{N}P_{S_\infty}\hat{M}^{-1}_{\min}. \tag{5.16}$$

Denote by $T_{\min}$ and by $\hat{T}_{\min}$ the $T$-matrices associated with the considered conjoined bases $(X_{\min},U_{\min})$ and $(\hat{X}_{\min},\hat{U}_{\min})$ from (2.14), respectively. Notice that Proposition 2.5 and the relations in (2.23) reveal the connection between the $T$-matrices of the above two minimal conjoined bases $(X_{\min},U_{\min})$ and $(\hat{X}_{\min},\hat{U}_{\min})$, that is

$$T_{\min} = \hat{M}^T_{\min}\hat{T}_{\min}\hat{M}_{\min} + \hat{M}^T_{\min}\hat{N}_{\min}. \tag{5.17}$$

We now stress that if we denote by $T$ and $\hat{T}$ as the $T$-matrix from (2.14) associated with $(X,U)$ and $(\hat{X},\hat{U})$, then

$$T = T_{\min} \quad \text{and} \quad \hat{T} = \hat{T}_{\min}. \tag{5.18}$$

This follows from [28, Proposition 4.2]. We will often use this important fact in the following computation without an additional warning. Notice now that everything from the beginning of the proof to this place is valid for any two conjoined bases $(X,U)$ and $(\hat{X},\hat{U})$ of $(\mathbb{S})$ with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$, both belonging to the genus $\mathcal{G}$.

Proving the first implication "$\Rightarrow$", let $(\hat{X},\hat{U})$ be a principal solution of ($\mathbb{S}$) at infinity and let (5.1) hold. Then $\hat{T}=0$ from Definition 2.9, and hence also $\hat{T}_{\min}=0$. First we show that $(X,U)$ is an antiprincipal solution of ($\mathbb{S}$) at infinity using the first equality from (5.18). The result in Proposition 2.5 together with (5.5) guarantees that

$$\begin{aligned}\operatorname{rank} T_{\min} &\overset{(2.23)}{=} \operatorname{rank}(\hat{N}_{\min}\hat{M}_{\min}^{-1}+\hat{T}_{\min}) = \operatorname{rank}(\hat{N}_{\min}\hat{M}_{\min}^{-1}) \\ &\overset{(5.16)}{=} \operatorname{rank}(P_{\hat{S}\infty}\hat{N}P_{S\infty}\hat{M}_{\min}^{-1}) = \operatorname{rank}(P_{\hat{S}\infty}\hat{N}P_{S\infty}) \overset{(5.1)}{=} n-d_\infty.\end{aligned}$$

This proves that $(X_{\min},U_{\min})$ is an antiprincipal solution of ($\mathbb{S}$) at infinity and hence, due to (5.18) and the relation being contained in [15, Theorem 6.3], $(X,U)$ is also an antiprincipal solution of ($\mathbb{S}$) at infinity. It remains to show that (5.2) holds. Using (5.12) and Proposition 2.10(iii) about the existence of the limit of the matrix $S(t)$ we get

$$\lim_{t\to\infty} X^\dagger(t)\hat{X}(t) = \lim_{t\to\infty}[P\hat{M}^{-1}-S(t)\hat{N}^T] = P\hat{M}^{-1}-T^\dagger\hat{N}^T =: V. \tag{5.19}$$

In the above computation we have used the fact that we already know that $(X,U)$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. Now we show that

$$\operatorname{Im} V^T \subseteq \operatorname{Im}(\hat{P}-P_{\hat{S}\infty}). \tag{5.20}$$

We will do so by showing the equivalent inclusion, which is

$$\operatorname{Im} V^T \subseteq \operatorname{Im}\hat{P}\cap\operatorname{Ker}P_{\hat{S}\infty}. \tag{5.21}$$

Let $v\in\operatorname{Im}V^T$. Then there exists $w\in\mathbb{R}^n$ such that $V^Tw=v$. But since the equality $\hat{N}=\hat{P}\hat{N}$ holds from (5.4), considering the symmetry of $\hat{P}$, $P$ and $T$, and (5.19), and the defining property of $V$ in (5.19), then we get

$$V^T = (P\hat{M}^{-1})^T-(T^\dagger\hat{N}^T)^T \overset{(5.9)}{=} \hat{P}(PM)^T-\hat{P}(T^\dagger\hat{N}^T)^T = \hat{P}V^T, \tag{5.22}$$

i.e., $\hat{P}V^Tw=V^Tw=v$. It shows that $v\in\operatorname{Im}\hat{P}$. At the same time, from (5.17) with $\hat{T}_{\min}=0$ we deduce that

$$T=T_{\min}=\hat{M}_{\min}^T\hat{N}_{\min}. \tag{5.23}$$

Then considering the symmetry of $\hat{M}_{\min}^T\hat{N}_{\min}$ we get

$$P_{S\infty}=T^\dagger T \overset{(5.23)}{=} T^\dagger\hat{M}_{\min}^T\hat{N}_{\min} = T^\dagger\hat{N}_{\min}^T\hat{M}_{\min},$$

which due to the invertibility of $\hat{M}_{\min}$ leads to the identity

$$\hat{N}_{\min}T^\dagger = \hat{M}_{\min}^{T-1}P_{S\infty}. \tag{5.24}$$

Using (2.17) in Proposition 2.3 we then obtain

$$\begin{aligned}P_{\hat{S}\infty} &= P_{\hat{S}\infty}V^Tw = [P_{\hat{S}\infty}(P\hat{M}^{-1})^T-P_{\hat{S}\infty}(T^\dagger\hat{N}^T)^T]w \overset{(2.15)}{=} (P_{\hat{S}\infty}\hat{M}^{T-1}P-P_{\hat{S}\infty}\hat{N}P_{S\infty}T^\dagger)w \\ &\overset{(5.16)}{=} (P_{\hat{S}\infty}M^TP-\hat{N}_{\min}T^\dagger)w \overset{(5.13)}{=} (\hat{M}_{\min}^{T-1}P_{S\infty}-\hat{N}_{\min}T^\dagger)w \overset{(5.24)}{=} 0.\end{aligned}$$

The latter identity implies that $v \in \operatorname{Ker} P_{\hat{S}\infty}$. But since also $v \in \operatorname{Im} \hat{P}$, we proved that (5.21), and mainly (5.20), holds. Now we show the opposite inclusion

$$\operatorname{Im} V^T \supseteq \operatorname{Im}(\hat{P} - P_{\hat{S}\infty}). \tag{5.25}$$

Similarly, as above, we will do so by showing the equivalent inclusion, see formula (2.1),

$$\operatorname{Ker} V \subseteq \operatorname{Ker} \hat{P} \oplus \operatorname{Im} P_{\hat{S}\infty}. \tag{5.26}$$

Let $v \in \operatorname{Ker} V$. Then it is possible to write $v$ uniquely as

$$v = v_1 + v_2, \quad \text{where} \quad v_1 \in \operatorname{Ker} \hat{P} \quad \text{and} \quad v_2 \in \operatorname{Im} \hat{P}. \tag{5.27}$$

We investigate $v_2 \in \operatorname{Im} \hat{P}$. Then there exists $w_2 \in \mathbb{R}^n$ such that $\hat{P} w_2 = v_2$. But then form the decomposition in (5.27) and from the identity $V\hat{P} = V$ obtained from (5.22) we get

$$Vv = V\hat{P}v_1 + V\hat{P}w_2.$$

Since $v \in \operatorname{Ker} V$ and $v_1 \in \operatorname{Ker} \hat{P}$, the latter equality shows that

$$V\hat{P}w_2 = Vv_2 = (P\hat{M}^{-1} - T^{\dagger}\hat{N}^T)v_2 = 0. \tag{5.28}$$

Denote by $w := P\hat{M}^{-1}v_2$. Then (5.28) implies that $w = T^{\dagger}\hat{N}^T v_2$, and hence we obtain that $w \in \operatorname{Im} T^{\dagger} = \operatorname{Im} P_{S\infty}$. Therefore, there exists the vector $\tilde{w} \in \mathbb{R}^n$ such that $P_{S\infty}\tilde{w} = w$. Finally, for the vector $v_2$ we also get

$$\begin{aligned} v_2 = \hat{P}v_2 &\overset{(2.32)}{=} \hat{P}\hat{M}(\hat{P}\hat{M})^{\dagger}v_2 \overset{(2.31)}{=} \hat{P}\hat{M}PMv_2 = \hat{P}\hat{M}P\hat{M}^{-1}v_2 \\ &= \hat{P}\hat{M}w = \hat{P}\hat{M}P_{S\infty}\tilde{w} \overset{(5.13)}{=} P_{\hat{S}\infty}\hat{M}_{\min}\tilde{w}. \end{aligned}$$

This means that $v_2 \in \operatorname{Im} P_{\hat{S}\infty}$, which shows that (5.26), and hence mainly (5.25), holds. Altogether, the second equality from (5.2) is now proven, which completes the proof of the implication from the left to the right.

Conversely, we prepare the opposite implication "$\Leftarrow$". Let $(X,U)$ and $(\hat{X},\hat{U})$ be as in the assumptions of the theorem and let (5.2) be valid. First we show that $(\hat{X},\hat{U})$ is a principal solution of ($\mathbb{S}$) at infinity. Since it is given that $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$, it remains to show that $\operatorname{rank}\hat{T} = 0$, where $\hat{T}$ is the $T$-matrix defined in (2.14) corresponding to $(\hat{X},\hat{U})$. In proving this it appears to be useful to know that $(X,U)$ is an antiprincipal solution of ($\mathbb{S}$) at infinity.

We show this as the first step. To do so we will use a minimal conjoined basis $(X_{\min},U_{\min})$, which is contained in $(X,U)$ on $[\alpha,\infty)_{\mathbb{T}}$. Let

$$V_0 := P\hat{M}^{-1} - V, \tag{5.29}$$

where $V$ is given by (5.2). Then

$$\lim_{t\to\infty} S(t)\hat{N}^T \overset{(5.12)}{=} \lim_{t\to\infty}[P\hat{M}^{-1} - X^{\dagger}(t)\hat{X}(t)] \overset{(5.2)}{=} P\hat{M}^{-1} - V \overset{(5.29)}{=} V_0. \tag{5.30}$$

The set of equalities

$$S(t) = S(t)P_{S\infty} = P_{S\infty}S(t), \quad t \in [\beta, \infty)_{\mathbb{T}}, \tag{5.31}$$

which holds automatically for some sufficiently large point $\beta \in [\alpha, \infty)_{\mathbb{T}}$, such that $\operatorname{Im} S(t)$ is constant on $[\beta, \infty)_{\mathbb{T}}$, see (2.13), yields that

$$\operatorname{Ker}(P_{S\infty}\hat{N}^T) \subseteq \operatorname{Ker} V_0. \tag{5.32}$$

Indeed, if $P_{S\infty}\hat{N}^T u = 0$ for some $u \in \mathbb{R}^n$, then also

$$V_0 u \overset{(5.30)}{=} \lim_{t\to\infty} S(t)\hat{N}^T u \overset{(5.31)}{=} \lim_{t\to\infty} S(t)P_{S\infty}\hat{N}^T u = 0.$$

Similarly, due to the computation

$$V_0 \overset{(5.30)}{=} \lim_{t\to\infty} S(t)\hat{N}^T \overset{(5.31)}{=} \lim_{t\to\infty} P_{S\infty}S(t)\hat{N}^T = P_{S\infty}V_0,$$

we obtain

$$\operatorname{Im} V_0 \subseteq \operatorname{Im} P_{S\infty}, \quad \text{and hence} \quad \operatorname{rank} V_0 \leqslant \operatorname{rank} P_{S\infty}. \tag{5.33}$$

Further, the inclusion $\operatorname{Im} V^T \subseteq \operatorname{Im}(\hat{P} - P_{\hat{S}\infty})$, which is derived from the equality in (5.2), can be read as $V^T = (\hat{P} - P_{\hat{S}\infty})V^T$, and hence $VP_{S\infty} = V(\hat{P} - I)$. This directly provides that

$$VP_{S\infty} \overset{(2.13)}{=} VP_{S\infty}\hat{P} = V(\hat{P} - I)\hat{P} \overset{(2.7)}{=} V(\hat{P} - \hat{P}) = 0.$$

The latter equality leads to

$$V_0 P_{\hat{S}\infty} = P\hat{M}^{-1} + VP_{\hat{S}\infty} \overset{(5.15)}{=} P_{S\infty}M_{\min}. \tag{5.34}$$

Moreover, if $u \in \operatorname{Im} P_{S\infty}$, then there exists $v \in \mathbb{R}^n$ such that

$$P_{S\infty}v \overset{(5.34)}{=} V_0 P_{\hat{S}\infty} M_{\min}^{-1} v = u,$$

hence $u \in \operatorname{Im} V_0$. Thus, we showed that

$$\operatorname{Im} P_{S\infty} \subseteq \operatorname{Im} V_0, \quad \text{and hence} \quad \operatorname{rank} P_{S\infty} \leqslant \operatorname{rank} V_0. \tag{5.35}$$

The relations in (5.33) and (5.35) together reveal that $\operatorname{Im} V_0 = \operatorname{Im} P_{S\infty}$ and

$$\operatorname{rank} V_0 = \operatorname{rank} P_{S\infty} \overset{(2.18)}{=} n - d_\infty. \tag{5.36}$$

Altogether, we have

$$n - d_\infty \overset{(5.36)}{=} \operatorname{rank} V_0 \overset{(5.32)}{\leqslant} \operatorname{rank}(P_{S\infty}\hat{N}^T) \leqslant \operatorname{rank} P_{S\infty} = n - d_\infty, \tag{5.37}$$

where the last inequality holds automatically. Thus in (5.37) we showed that

$$\operatorname{rank}(P_{S\infty}\hat{N}^T) = n - d_\infty. \tag{5.38}$$

Finally, if we focus on the rank of the matrix $T$, we get

$$\operatorname{rank} T \geqslant \operatorname{rank}(T\hat{N}^T) = \operatorname{rank} \lim_{t\to\infty} S^\dagger(t)S(t)\hat{N}^T = \operatorname{rank}(P_{S\infty}\hat{N}^T) \overset{(5.38)}{=} n - d_\infty.$$

But according to [15, Theorem 5.2], since $T$ is the $T$-matrix of the conjoined basis $(X,U)$, we must have $\operatorname{rank} T = n - d_\infty$. This proves that $(X,U)$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. Then we know by Proposition 2.10 that the limit of $S(t)$ exists as $t$ tends to infinity. Then the relation (5.30) implies

$$TV_0 = \lim_{t\to\infty} S^\dagger(t)S(t)\hat{N}^T = P_{S\infty}\hat{N}^T. \tag{5.39}$$

Next, considering the symmetry of $\hat{N}_{\min}\hat{M}_{\min}^{-1}$ and using $V_0P_{\hat{S}\infty} = P_{S\infty}\hat{M}_{\min}^{-1}$ received directly from (5.34), we get

$$\begin{aligned}
\hat{N}_{\min}\hat{M}_{\min}^{-1} &\overset{(5.14)}{=} P_{\hat{S}\infty}\hat{N}\hat{M}^{-1}P_{\hat{S}\infty} \overset{(5.7)}{=} P_{\hat{S}\infty}\hat{N}P\hat{M}^{-1}P_{\hat{S}\infty} \overset{(5.15)}{=} P_{\hat{S}\infty}\hat{N}P_{S\infty}\hat{M}_{\min}^{-1} \\
&\overset{(\text{symmetry})}{=} \hat{M}_{\min}^{T-1}P_{S\infty}\hat{N}^T P_{\hat{S}\infty} \overset{(5.39)}{=} \hat{M}_{\min}^{T-1}\,T\,V_0\,P_{\hat{S}\infty} \\
&\overset{(5.34)}{=} \hat{M}_{\min}^{T-1}\,T\,P_{S\infty}\hat{M}_{\min}^{-1} \overset{(2.15)}{=} \hat{M}_{\min}^{T-1}\,T\hat{M}_{\min}^{-1}.
\end{aligned}$$

It follows that $T = \hat{N}_{\min}^T\hat{M}_{\min}$. Inserting this into (5.17) we get

$$T_{\min} = \hat{M}_{\min}^T\hat{T}_{\min}\hat{M}_{\min} + T. \tag{5.40}$$

But since the equality $T = T_{\min}$ holds and the matrix $\hat{M}_{\min}$ is invertible, then we obtain from (5.40) that $\hat{T}_{\min} = 0$, and then $\hat{T} = \hat{T}_{\min} = 0$ as well. This proves that $(\hat{X},\hat{U})$ is a principal solution of ($\mathbb{S}$) at infinity. At the very end, notice that Theorem 4.1 guarantees that condition (5.1) holds. The proof is complete. $\square$

The relations from Theorem 5.1 become simpler if we deal with the minimal genus $\mathcal{G}_{\min}$. The statement covering this situation follows. In this case the matrix $V$ in (5.19) is the zero matrix (i.e., $V = 0$). Such a limit property of the principal solutions of ($\mathbb{S}$) at infinity is known in [10] for an eventually controllable system ($\mathbb{S}$).

COROLLARY 5.2. *Assume that system ($\mathbb{S}$) is nonoscillatory. Let $(X,U)$ and $(\hat{X},\hat{U})$ be two conjoined bases from the minimal genus $\mathcal{G}_{\min}$. Then $(\hat{X},\hat{U})$ is a minimal principal solution of ($\mathbb{S}$) at infinity and*

$$\operatorname{rank} \mathcal{W}[(\hat{X},\hat{U}),(X,U)] = n - d_\infty \tag{5.41}$$

*if and only if*

$$\lim_{t\to\infty} X^\dagger(t)\hat{X}(t) = 0. \tag{5.42}$$

*In this case, $(X,U)$ is a minimal antiprincipal solution of ($\mathbb{S}$) at infinity.*

*Proof.* The statement of this corollary follows directly from Theorem 5.1. Specifically, let both $(X,U)$ and $(\hat{X},\hat{U})$ belong to the genus $\mathcal{G}_{\min}$. Then, by Proposition 2.6

we know that $P_{\hat{S}\infty} = \hat{P}$ and $P_{S\infty} = P$. Using the notation from the proof of Theorem 5.1 we get

$$P_{\hat{S}\infty}\mathcal{W}[(\hat{X},\hat{U}),(X,U)]P_{S\infty} \overset{(2.24)}{=} \hat{P}\mathcal{W}[(\hat{X},\hat{U}),(X,U)]P \overset{(5.5)}{=} \hat{P}\hat{N}P \overset{(5.7)}{=} \hat{N}P$$
$$\overset{(5.7)}{=} \hat{N} \overset{(5.5)}{=} \mathcal{W}[(\hat{X},\hat{U}),(X,U)].$$

This shows that condition (5.1) reduces to (5.41) when $\mathcal{G} = \mathcal{G}_{\min}$. Further, we have

$$\hat{P} - P_{\hat{S}\infty} \overset{(2.24)}{=} \hat{P} - \hat{P} = 0,$$

so that condition (5.2) reduces in this case to condition (5.42). □

Similar simplifications as in the previous corollary occur if we deal with the maximal genus $\mathcal{G}_{\max}$. As well as the previous one, the following corollary is a special case of Theorem 5.1.

COROLLARY 5.3. *Assume that system* ($\mathbb{S}$) *is nonoscillatory. Let* $\hat{\alpha}_{\min} \in [a,\infty)_{\mathbb{T}}$ *be defined in* (2.26) *and let* $(X,U)$ *and* $(\hat{X},\hat{U})$ *be two conjoined bases of* ($\mathbb{S}$) *from the genus* $\mathcal{G}_{\max}$. *Further, let* $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ *be such that both* $(X,U)$ *and* $(\hat{X},\hat{U})$ *have invertible* $X(t)$ *and* $\hat{X}(t)$ *for all* $t \in [\alpha,\infty)_{\mathbb{T}}$ *and no focal points in* $(\alpha,\infty)$. *Denote by* $P_{\hat{S}\infty}$ *and* $P_{S\infty}$ *their associated orthogonal projectors from* (2.11). *Then* $(\hat{X},\hat{U})$ *is a maximal principal solution of* ($\mathbb{S}$) *at infinity and*

$$\operatorname{rank}\left[P_{\hat{S}\infty}\mathcal{W}[(\hat{X},\hat{U}),(X,U)]P_{S\infty}\right] = n - d_\infty$$

*if and only if*

$$\lim_{t\to\infty} X^\dagger(t)\hat{X}(t) = V, \quad \textit{where} \quad \operatorname{Im} V^T = \operatorname{Ker} P_{\hat{S}\infty}. \tag{5.43}$$

*In this case* $(X,U)$ *is a maximal antiprincipal solution of* ($\mathbb{S}$) *at infinity.*

*Proof.* Using the notation from the proof Theorem 5.1, we get that

$$\hat{P} = \hat{X}^\dagger(t)\hat{X}(t) = \hat{X}^{-1}(t)\hat{X}(t) = I, \quad t \in [\alpha,\infty)_{\mathbb{T}}.$$

Then $\operatorname{Im} V^T = \operatorname{Im}(I - P_{\hat{S}\infty}) = \operatorname{Ker} P_{\hat{S}\infty}$, hence conditions (5.43) and (5.2) coincide for $\mathcal{G} = \mathcal{G}_{\max}$. The statement now follows from Theorem 5.1. □

The following theorem deals with the existence of the limit in (5.2). This theorem is an extension and unification of [25, Theorem 6.3] in the continuous case and of [30, Theorem 6.4] in discrete case, see also [12, Theorem 6.149].

THEOREM 5.4. *Assume that system* ($\mathbb{S}$) *is nonoscillatory. Let* $\hat{\alpha}_{\min} \in [a,\infty)_{\mathbb{T}}$ *be defined in* (2.26) *and let* $(X,U)$ *and* $(\tilde{X},\tilde{U})$ *be two conjoined bases of* ($\mathbb{S}$) *from the same genus* $\mathcal{G}$. *Let* $(X,U)$ *be an antiprincipal solution of* ($\mathbb{S}$) *at infinity. Further, let* $\alpha \in [\hat{\alpha}_{\min},\infty)_{\mathbb{T}}$ *be such that* $(X,U)$ *and* $(\tilde{X},\tilde{U})$ *have constant kernel on* $[\alpha,\infty)_{\mathbb{T}}$ *and no focal points in* $(\alpha,\infty)$. *Denote by* $\tilde{P}$, $P_{\tilde{S}\infty}$, $\tilde{T}$ *the matrices from* (2.4), (2.11), *and* (2.14) *which are associated with* $(\tilde{X},\tilde{U})$. *Then the limit of* $X^\dagger(t)\tilde{X}(t)$ *for* $t$ *tending to infinity exists and satisfies*

$$\lim_{t\to\infty} X^\dagger(t)\tilde{X}(t) = V, \quad \textit{where} \quad \operatorname{Im} V^T = \operatorname{Im}\tilde{T} \oplus \operatorname{Im}(\tilde{P} - P_{\tilde{S}\infty}). \tag{5.44}$$

*Proof.* Let $(X,U)$ and $(\tilde{X},\tilde{U})$ be two conjoined bases from the genus $\mathcal{G}$ such that they have constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in $(\alpha,\infty)$. Let $(X_{\min},U_{\min})$ and $(\tilde{X}_{\min},\tilde{U}_{\min})$ be minimal conjoined bases of ($\mathbb{S}$), which are contained in $(X,U)$ and $(\tilde{X},\tilde{U})$ on $[\alpha,\infty)_{\mathbb{T}}$, respectively. We will use the preparatory part of the proof of Theorem 5.1, where we replace the symbol *hat* by the symbol *tilde* on every single place. We also use the same natural notation for the orthogonal projectors and $T$-matrices and $S$-matrices defined in (2.4), (2.11), (2.14), and (2.9). That is, we use the notation $P$, $P_{S\infty}$, $T$, and $S(t)$ for those matrices associated with $(X,U)$ on $[\alpha,\infty)_{\mathbb{T}}$, the notation $\tilde{P}$, $P_{\tilde{S}\infty}$, $\tilde{T}$, and $\tilde{S}(t)$ for those matrices associated with $(\tilde{X},\tilde{U})$ on $[\alpha,\infty)_{\mathbb{T}}$, the notation $P_{\min}$, $P_{S_{\min}\infty}$, $T_{\min}$, and $S_{\min}(t)$ for those associated with $(X_{\min},U_{\min})$ on $[\alpha,\infty)_{\mathbb{T}}$, and finally the notation $\tilde{P}_{\min}$, $P_{\tilde{S}_{\min}\infty}$, $\tilde{T}_{\min}$, and $\tilde{S}_{\min}(t)$ for those matrices associated with $(\tilde{X}_{\min},\tilde{U}_{\min})$ on $[\alpha,\infty)_{\mathbb{T}}$. We stress that all the relations from (5.3) to (5.18) remain valid also in this modified context. In addition, notice also that (5.6)–(5.9) remain valid if we replace the matrices $N$, $P$, $M$, $\hat{N}$, $\hat{P}$, $\hat{M}$ by the matrices $N_{\min}$, $P_{\min}$, $M_{\min}$, $\tilde{N}_{\min}$, $\tilde{P}_{\min}$, $\tilde{M}_{\min}$, respectively.

Suppose now that $(X,U)$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. Then, in addition, also other relations from the proof of Theorem 5.1 hold with the tilde instead of the hat, namely equations (5.19), (5.22), (5.25), and hence also equation (5.26). According to Proposition 2.10(iii), the limit of the matrix $S(t)$ as $t$ tends to infinity exists and equals $T^{\dagger}$. Then it is clear that the limit

$$\lim_{t\to\infty} X^{\dagger}(t)\tilde{X}(t) \overset{(5.12)}{=} \lim_{t\to\infty}[P\tilde{M}^{-1} - S(t)\hat{N}^{T}] = P\tilde{M}^{-1} - T^{\dagger}\tilde{N}^{T} =: V \tag{5.45}$$

exists. The first part of (5.44) is proven. Now we prove the second part, i.e.,

$$\operatorname{Im} V^{T} = \operatorname{Im}\tilde{T} \oplus \operatorname{Im}(\tilde{P} - P_{\tilde{S}\infty}). \tag{5.46}$$

Notice that since we know that $(X,U)$ is an antiprincipal solution of ($\mathbb{S}$) at infinity, then relation (5.22) is valid and hence,

$$V^{T} = \tilde{P}V^{T}. \tag{5.47}$$

Moreover, as was mentioned above, we can be also sure that

$$\operatorname{Im} V^{T} \supseteq \operatorname{Im}(\tilde{P} - P_{\tilde{S}\infty}), \quad \text{or equivalently} \quad \operatorname{Ker} V \subseteq \operatorname{Ker}\tilde{P} \oplus \operatorname{Im} P_{\tilde{S}\infty}, \tag{5.48}$$

by (2.1). In addition, the inclusion $\operatorname{Im}\tilde{T} \subseteq \operatorname{Im}\tilde{P}$ holds automatically by (2.15) and (2.12), and

$$\operatorname{Im}\tilde{T} \cap \operatorname{Ker} P_{\tilde{S}\infty} \overset{(2.15)}{\subseteq} \operatorname{Im} P_{\tilde{S}\infty} \cap \operatorname{Ker} P_{\tilde{S}\infty} = \{0\}. \tag{5.49}$$

But then we can transform equality (5.46) as

$$\begin{aligned}(\operatorname{Ker} V)^{\perp} = \operatorname{Im} V^{T} &\overset{(5.46)}{=} \operatorname{Im}\tilde{T} \oplus (\operatorname{Im}\tilde{P} \cap \operatorname{Ker} P_{\tilde{S}\infty}) \\ &= (\operatorname{Im}\tilde{P} \cap \operatorname{Im}\tilde{T}) \oplus (\operatorname{Im}\tilde{P} \cap \operatorname{Ker} P_{\tilde{S}\infty}) \overset{(5.49)}{=} \operatorname{Im}\tilde{P} \cap (\operatorname{Im}\tilde{T} \oplus \operatorname{Ker} P_{\tilde{S}\infty}).\end{aligned}$$

Now we see that the second equality (5.46) is equivalent with

$$\operatorname{Ker} V = \operatorname{Ker} \tilde{P} \oplus (\operatorname{Ker} \tilde{T} \cap \operatorname{Im} P_{\tilde{S}\infty}), \tag{5.50}$$

so that it is enough to show that (5.50) holds. Let us show the first inclusion

$$\operatorname{Ker} V \subseteq \operatorname{Ker} \tilde{P} \oplus (\operatorname{Ker} \tilde{T} \cap \operatorname{Im} P_{\tilde{S}\infty}). \tag{5.51}$$

Let $v \in \operatorname{Ker} V$. Then (5.48) guarantees that $v$ can be uniquely decomposed as

$$v = v_1 + v_2, \quad \text{where} \quad v_1 \in \operatorname{Ker} \tilde{P} \text{ and } v_2 \in \operatorname{Im} P_{\tilde{S}\infty}. \tag{5.52}$$

We focus on the vector $v_2 \in \operatorname{Im} P_{\tilde{S}\infty}$. From (5.52) we get

$$0 = Vv = Vv_1 + Vv_2 \overset{(5.47)}{=} V\tilde{P}v_1 + Vv_2 = Vv_2 \overset{(5.45)}{=} (P\tilde{M}^{-1} - T^{\dagger}\tilde{N}^T)v_2,$$

which yields that the vector

$$w := P\tilde{M}^{-1}v_2 \quad \text{satisfies} \quad w = T^{\dagger}\tilde{N}^T v_2. \tag{5.53}$$

In addition, since $v_2 \in \operatorname{Im} P_{\tilde{S}\infty}$, and since $P_{\tilde{S}\infty}$ is idempotent, and $\tilde{M}^{-1} = M$, we have

$$w \overset{(5.53)}{=} P\tilde{M}^{-1}v_2 = PMP_{\tilde{S}\infty}v_2 \overset{(5.13)}{=} P_{S\infty}M_{\min}v_2.$$

Moreover, because $\tilde{S}(t) = \tilde{S}_{\min}(t)$ on $[\alpha,\infty)_{\mathbb{T}}$ and $P_{\tilde{S}_{\min}\infty} = \tilde{P}_{\min}$, see Proposition 2.6, relations (2.32) and (2.31) provide that

$$\tilde{P}_{\min} = \tilde{P}_{\min}\tilde{M}_{\min}(\tilde{P}_{\min}\tilde{M}_{\min})^{\dagger} = \tilde{P}_{\min}\tilde{M}_{\min}P_{\min}M_{\min}.$$

Now $v_2 \in \operatorname{Im} P_{\tilde{S}\infty}$ brings that

$$\begin{aligned} v_2 &= P_{\tilde{S}\infty}v_2 = P_{\tilde{S}_{\min}\infty}v_2 = P_{\min}v_2 = \tilde{P}_{\min}\tilde{M}_{\min}P_{\min}M_{\min}v_2 \\ &= \tilde{P}_{\min}\tilde{M}_{\min}P_{S\infty}\tilde{M}_{\min}^{-1}v_2 \overset{(5.15)}{=} \tilde{P}_{\min}\tilde{M}_{\min}P\tilde{M}^{-1}P_{\tilde{S}\infty}v_2 = \tilde{P}_{\min}\tilde{M}_{\min}P\tilde{M}^{-1}v_2. \end{aligned}$$

Hence, it follows that

$$v_2 = P_{\tilde{S}\infty}\tilde{M}_{\min}w. \tag{5.54}$$

The latter equalities allow us to derive

$$w \overset{(5.53)}{=} T^{\dagger}\tilde{N}^T v_2 \overset{(5.54)}{=} T^{\dagger}\tilde{N}^T P_{\tilde{S}\infty}\tilde{M}_{\min}w \overset{(2.15)}{=} T^{\dagger}P_{S\infty}\tilde{N}^T P_{\tilde{S}\infty}\tilde{M}_{\min}w \overset{(5.16)}{=} T^{\dagger}\tilde{N}_{\min}^T\tilde{M}_{\min}w.$$

The above expression for $w$ together with the symmetry of $\tilde{N}_{\min}^T\tilde{M}_{\min}$ yields that

$$\begin{aligned} Tw &= TT^{\dagger}\tilde{N}_{\min}^T\tilde{M}_{\min}w = P_{S\infty}\tilde{N}_{\min}^T\tilde{M}_{\min}w = P_{S_{\min}\infty}\tilde{N}_{\min}^T\tilde{M}_{\min}w \\ &= P_{\min}\tilde{N}_{\min}^T\tilde{M}_{\min}w \overset{(5.7)}{=} \tilde{N}_{\min}^T\tilde{M}_{\min}w = \tilde{M}_{\min}^T\tilde{N}_{\min}w. \end{aligned}$$

On the other hand, relation (5.17) implies that

$$Tw = T_{\min} w \overset{(5.17)}{=} \tilde{M}_{\min}^T \tilde{T}_{\min} \tilde{M}_{\min} w + \tilde{M}_{\min}^T \tilde{N}_{\min} w.$$

Combining the latter two equalities together, while considering the existence of the matrix $\tilde{M}_{\min}^{-1}$, yields that

$$\tilde{T}_{\min} \tilde{M}_{\min} w = 0. \tag{5.55}$$

Finally, this leads to

$$\tilde{T} v_2 \overset{(5.54)}{=} \tilde{T} P_{\tilde{S}\infty} \tilde{M}_{\min} w = \tilde{T}_{\min} \tilde{M}_{\min} w \overset{(5.55)}{=} 0,$$

and hence, $v_2 \in \operatorname{Ker} \tilde{T}$. But at the same time $v_2 \in \operatorname{Im} P_{\tilde{S}\infty}$ by (5.52). This shows that the inclusion in (5.51) holds. Now we show the opposite inclusion

$$\operatorname{Ker} V \supseteq \operatorname{Ker} \tilde{P} \oplus (\operatorname{Ker} \tilde{T} \cap \operatorname{Im} P_{\tilde{S}\infty}). \tag{5.56}$$

Let $u \in \operatorname{Ker} \tilde{P} \oplus (\operatorname{Ker} \tilde{T} \cap \operatorname{Im} P_{\tilde{S}\infty})$. Then there exists a unique decomposition $u = u_1 + u_2$, where $u_1 \in \operatorname{Ker} \tilde{T} \cap \operatorname{Im} P_{\tilde{S}\infty}$ and $u_2 \in \operatorname{Ker} \tilde{P}$. But then

$$Vu = Vu_1 + Vu_2 \overset{(5.47)}{=} Vu_1 + V\tilde{P}u_2 = Vu_1.$$

Now since $u_1 \in \operatorname{Im} P_{\tilde{S}\infty}$, we have $u_1 = P_{\tilde{S}\infty} u_1$ and we can continue the computation as

$$\begin{aligned}
Vu &\overset{(5.45)}{=} (P\tilde{M}^{-1} - T^{\dagger} \tilde{N}^T) u_1 = (PMP_{\tilde{S}\infty} - T^{\dagger} \tilde{N}^T P_{\tilde{S}\infty}) u_1 \\
&\overset{(5.13)}{=} (P_{S\infty} M_{\min} - T^{\dagger} P_{S\infty} \tilde{N}^T P_{\tilde{S}\infty}) u_1 \overset{(5.16)}{=} (P_{S\infty} M_{\min} - T^{\dagger} \tilde{N}_{\min}^T) u_1 \\
&= (T^{\dagger} T M_{\min} - T^{\dagger} \tilde{N}_{\min}^T) u_1 = T^{\dagger} (T_{\min} M_{\min} - \tilde{N}_{\min}^T) u_1 \\
&= T^{\dagger} (M_{\min}^{T-1} M_{\min}^T T_{\min} M_{\min} - M_{\min}^{T-1} M_{\min}^T \tilde{N}_{\min}^T) u_1 \\
&= T^{\dagger} M_{\min}^{T-1} (M_{\min}^T T_{\min} M_{\min} - M_{\min}^T \tilde{N}_{\min}^T) u_1 \\
&= T^{\dagger} M_{\min}^{T-1} (M_{\min}^T T_{\min} M_{\min} + M_{\min}^T N_{\min}) u_1 \\
&\overset{(2.23)}{=} T^{\dagger} M_{\min}^{T-1} \tilde{T}_{\min} u_1 = T^{\dagger} M_{\min}^{T-1} \tilde{T} u_1.
\end{aligned}$$

Since $u_1 \in \operatorname{Ker} \tilde{T}$, the latter equality yields that $Vu = 0$ and hence, $u \in \operatorname{Ker} V$. The inclusion (5.56) is therefore proven. If we put the previous two steps together, then we get that equality (5.50) holds, and thus the second condition in (5.44) is valid. The proof is complete. □

The result in Theorem 5.4 shows that a comparison of two conjoined bases $(X, U)$ and $(\tilde{X}, \tilde{U})$ of ($\mathbb{S}$) in the sense of the limit in (5.2) or (5.44) is possible for any conjoined basis $(\tilde{X}, \tilde{U})$, when the conjoined basis $(X, U)$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. This complies with the results in Theorem 5.1 and Corollary 5.2.

## 6. Conclusions

In this paper we focused on certain topics from the theory of genera of conjoined bases of dynamic symplectic systems. We derived new properties of principal and antiprincipal solutions at infinity which belong to a given genus G. More precisely, in Theorems 3.1 and 4.1 we provided classifications of all principal and antiprincipal solutions of ($\mathbb{S}$) at infinity in the genus $\mathcal{G}$ in terms of some known principal solution of ($\mathbb{S}$) at infinity belonging to the same genus $\mathcal{G}$. The main tools to prove these theorems are Propositions 2.11 and 2.18, namely it is the mutual representation of some special conjoined bases and the relation to be contained and its properties related to the inheritance of the property to be a principal or antiprincipal solution at infinity.

In our investigations we did not use any controllability (normality) assumption, which leads in natural way to using the Moore–Penrose pseudoinverse in the situations, where the considered matrices are not invertible. The article opens a door for future research.

It seems to be possible to use the above mentioned tools for deriving the classifications of all principal and antiprincipal solutions at infinity in the genus $\mathcal{G}$ in terms of some known antiprincipal solution at infinity belonging to the same genus $\mathcal{G}$. We leave this topic, letting this kind of classification as an open problem. Note that it is an open problem even in the continuous case and also in the discrete case.

A next natural step could be the investigation of an ordering in the set of equivalence classes given by the relation to belong to the same genus. Once we know that there exists some minimal genus $\mathcal{G}_{\min}$ and the maximal genus $\mathcal{G}_{\max}$, it seems to be a good idea to investigate what happens in between. In the continuous case the ordering on the set of all genera of conjoined bases is described in [27, Theorem 4.8]. Such the result would be new even in the discrete case.

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