Differential
Equations
& Applications
Volume 15, Number 4 (2023), 361–380
doi:10.7153/dea-2023-15-18

QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH DAMAGE FOR THERMO–ELECTRO–ELASTIC–VISCOPLASTIC BODIES

AHMED HAMIDAT* AND ADEL AISSAOUI

(*Communicated by I. Velčić*)

Abstract. The aim of present paper is to study the process of a quasistatic frictional contact between a thermo-electro-elastic-viscoplastic body with damage, and an obstacle, the so-called foundation. We assume that the normal stress is prescribed on the contact surface and we use the quasistatic version of Coulomb's law of dry friction. We establish a variational formulation of the model, which is set as a system involving the displacement field, the stress field, the electric potential field, the temperature field and the damage field. Existence and uniqueness of a weak solution of the problem is proved. The proof is based on arguments of evolutionary variational inequalities, parabolic inequalities, differential equations and fixed point.

1. Introduction

Situations of frictional contact abound in the industry and everyday life (contacts of the braking pads with the wheel or the tire with the road are usual examples). As a result, a considerable effort has been done in its modelling and numerical simulations. see for instance [10, 16, 18] and the references therein.

The piezoelectric effect is characterized by the coupling between the mechanical and electrical properties of the materials. In simplest terms, when a piezoelectric material is squeezed, an electric charge collects on its surface, conversely, when a piezoelectric material is subjected to a voltage drop, it mechanically deforms. Piezoelectric materials are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and measuring equipments. There is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, see for instance [2, 12, 17] and the references therein.

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Mathematics subject classification (2020): 74C10, 49J40, 74M10, 74M15, 47H10.
Keywords and phrases: Piezoelectric, elastic-viscoplastic, quasistatic, friction contact, temperature, damage, differential equations, fixed point.

* Corresponding author.

in radioelectronics, electroacoustics and measuring equipments. There is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, see for instance [2, 12, 17] and the references therein. Different models have been proposed to describe the interaction between the thermal and mechanical field, see for instance [3, 14, 11] and the references therein. A thermo-elastic-viscoplastic body is considered in [5, 14]. Initial and boundary value problems for thermo mechanical models were studied by many authors. Therefore, existence and uniqueness result concerning the uncoupled thermo viscoelastic was obtained in [13] using a monotony method.

Damage is a very important phenomenon in engineering because it directly affects the structure of machines. There exists a very large engineering literature on it. Early models for mechanical damage derived from the thermodyamical considerations appeared in [6, 7], where numerical simulations were included. The mathematical analysis of one-dimensional problems can be found in [8]. In all these results, the damage of the material is described with a damage function α, restricted to have values between zero and one. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$, the material is completely damaged, when $0 < \alpha < 1$ there is partial damage and the system has a reduced load carrying capacity. Quasistatic contact problems with damage have been investigated in [9, 10, 15].

Quasi-static processes for electro-viscoelastic with long-term memory and damage have been studied in [11], such that electrical conditions are introduced in cases where the foundation conductive. In this paper, we consider a general model for the a quasistatic process of frictional contact between a deformable body and an obstacle. The material obeys a general electro elastic-viscoplastic constitutive law with damage and thermal effects. On the contact surface the body can arrive in frictional contact with an obstacle, the so-called foundation which is electrically nonconducting and the contact is given by

$$-\sigma_\nu = F, \qquad \begin{cases} \|\boldsymbol{\sigma}_\tau\| \leqslant \mu\,|\sigma_\nu|, \\ \boldsymbol{\sigma}_\tau = -\mu\,|\sigma_\nu|\,\dfrac{\dot{\boldsymbol{u}}_\tau}{\|\dot{\boldsymbol{u}}_\tau\|} \quad \text{if} \quad \dot{\boldsymbol{u}}_\tau \neq 0, \end{cases}$$

where F is a given positive function. The above relations assert that the tangential stress is bounded by the normal stress multiplied by the value of the friction coefficient μ.

The rest of the article is structured as follows. In Section 2 we present contact model and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. We prove in Section 4 the existence and uniqueness of the solution.

2. Problem statement

The physical setting is the following. A body occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with outer Lipschitz surface which is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$ and the displacement vanishes there. Surface tractions of density f_2 act on $\Gamma_2 \times (0,T)$ and a volume force of density f_0 is applied in $\Omega \times (0,T)$.

We also assume that the electrical potential vanishes on $\Gamma_a \times (0,T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0,T)$. On Γ_3 the potential contact surface, the body is in contact with an insulator obstacle, the so-called foundation.

The classical formulation of the mechanical problem of electro elastic-viscoplastic with damage and thermal effects, be stated as follows.

Problem P

Find a displacement field $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times (0,T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$, an electric displacement field $\boldsymbol{D} : \Omega \times (0,T) \to \mathbb{R}^d$, and a damage field $\alpha : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathscr{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathscr{B}(\varepsilon(\boldsymbol{u}(t)),\alpha(t)) - \mathscr{E}^* E(\varphi)(t) \\ &\quad + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\varepsilon(\dot{\boldsymbol{u}}(s)) + \mathscr{E}^* E(\varphi)(s), \varepsilon(\boldsymbol{u}(s)))\, ds - C_e\theta \qquad \text{in } \Omega \times (0,T),\end{aligned} \tag{1}$$

$$\boldsymbol{D} = \mathscr{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}E(\varphi) \qquad \text{in } \Omega \times (0,T), \tag{2}$$

$$\dot{\theta} - \operatorname{div} K(\nabla\theta) = r(\dot{\boldsymbol{u}},\alpha) + \mathbf{q}, \qquad \text{in } \Omega \times (0,T), \tag{3}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_{\mathscr{Y}}(\alpha) \ni S(\varepsilon(\boldsymbol{u}),\alpha), \qquad \text{in } \Omega \times (0,T), \tag{4}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{5}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{6}$$

$$\boldsymbol{u} = \mathbf{0} \qquad \text{on } \Gamma_1 \times (0,T), \tag{7}$$

$$\sigma\nu = f_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{8}$$

$$-\sigma_\nu = F \qquad \text{on } \Gamma_3 \times (0,T) \tag{9}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \leqslant \mu\,|\sigma_\nu| \\ \boldsymbol{\sigma}_\tau = -\mu\,|\sigma_\nu|\,\dfrac{\dot{\boldsymbol{u}}_\tau}{\|\dot{\boldsymbol{u}}_\tau\|} \qquad \text{if} \quad \dot{\boldsymbol{u}}_\tau \neq 0 \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{10}$$

$$-k_{ij}\frac{\partial\theta}{\partial x_i}\nu_j = k_e(\theta - \theta_R) + h_\tau(|\dot{u}_\tau|) \qquad \text{on } \Gamma_3 \times (0,T), \tag{11}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \qquad \text{on } \Gamma \times (0,T), \tag{12}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), \tag{13}$$

$$\boldsymbol{D}.\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), \tag{14}$$

$$\theta = 0 \qquad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{15}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0, \qquad \text{in } \Omega. \tag{16}$$

First, equations (1)–(4) represent the electro-elastic-viscoplastic constitutive law with damage and thermal effects, were $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{G}$ are, respectively, nonlinear operators describing the purely viscous, the elastic and the viscoplastic properties of the material, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathscr{E} = (e_{ijk})$ represent the third order piesoelectric tensor, $\mathscr{E}^*$ is its transposition and $\boldsymbol{B}$ denotes the electric permittivity tensor, $C_e = (c_{ij)}$ represents the thermal expansion tensor, K represent the thermal conductivity tensor, $\operatorname{div}(K\nabla\theta) = (k_{ij}\theta_{,i})_{,i}$, $\mathbf{q}$ represent the density of volume heat source and r is non linear function of velocity and damage.

α, θ represent the damage, and the temperature. $\varphi_{\mathscr{Y}}(\alpha)$ denotes the subdifferential of the indicator function of the set $\mathscr{Y}$ of admissible damage functions defined by

$$\mathscr{Y} = \left\{\alpha \in H^1(\Omega) \mid 0 \leqslant \alpha \leqslant 1 \ \text{ a.e. in } \ \Omega\right\},$$

and S is the mechanical source of the damage.

Equations (5) and (6) represent the equilibrium equations for the stress and electric displacement fields. Equations (7)–(8) are the displacement-traction conditions.

Frictional contact conditions of the form (9) and (10) describe the contact on the surface Γ_3, (11), (12) represent, respectively on Γ, a Fourier boundary condition for the temperature and an homogeneous Neumann boundary condition for the damage field on Γ. (13) and (14) represent the electric boundary conditions. Equation (15) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0,T)$. Finally, The functions $\boldsymbol{u}_0$, θ_0 and α_0 in (16) are the initial data.

3. Variational formulation and preliminaries

For a weak formulation of the problem, first we introduce some notation. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. We define the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively, by

$$\|\boldsymbol{u}\| = (\boldsymbol{u}\cdot\boldsymbol{u})^{\frac{1}{2}}, \quad \forall \boldsymbol{u} \in \mathbb{R}^d \quad \text{ and } \quad \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d.$$

Here and below, the indices i and j run from 1 to d and the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We define the function spaces

$$H = L^2(\Omega)^d = \left\{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\right\}, \quad H_1 = \{\boldsymbol{u} = (u_i) \mid \varepsilon(\boldsymbol{u}) \in \mathscr{H}\},$$
$$\mathscr{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \quad \mathscr{H}_1 = \{\boldsymbol{\sigma} \in \mathscr{H} \mid \operatorname{Div}\boldsymbol{\sigma} \in H\}.$$

Here $\varepsilon : H_1 \to \mathscr{H}$ and $\operatorname{Div} : \mathscr{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}), \quad \operatorname{Div}(\boldsymbol{\sigma}) = \sigma_{ij,j}.$$

The sets H, H_1, $\mathscr{H}$ and $\mathscr{H}_1$ are real Hilbert spaces endowed with the canonical inner products

$$(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega u_i v_i dx \quad \forall u,v \in H, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathscr{H},$$
$$(u,\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathscr{H}}, \quad \forall \boldsymbol{u},\boldsymbol{v} \in H_1,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathscr{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathscr{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_H, \quad \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathscr{H}_1.$$

The associated norms are denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathscr{H}}$ and $\|.\|_{\mathscr{H}_1}$. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\boldsymbol{u} \in H_1$, we also write $\boldsymbol{u}$ for the trace $\gamma\boldsymbol{u}$ of $\boldsymbol{u}$ on Γ and we denote by u_ν and $\boldsymbol{u}_\tau$ the normal and tangential components of $\boldsymbol{u}$ on Γ given by

$$u_\nu = \boldsymbol{u}.\boldsymbol{\nu}, \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}. \tag{17}$$

We recall that when $\boldsymbol{\sigma}$ is a regular function then the normal component and the tangential part of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by

$$\sigma_\nu = \sigma\nu \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{18}$$

and for all $\boldsymbol{\sigma} \in \mathscr{H}_1$ the following Green's formula holds

$$(\boldsymbol{\sigma},\varepsilon(\boldsymbol{v}))_{\mathscr{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu.\boldsymbol{v} da, \quad \forall \boldsymbol{v} \in H_1. \tag{19}$$

Now, let $\mathscr{V}$ denote the closed subspace of $H^1(\Omega)$ given by

$$\mathscr{V} = \left\{\gamma \in H^1(\Omega) \mid \gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2\right\},$$

and we denote by $\mathscr{V}'$ the dual space of $\mathscr{V}$.

We use the notation $(.,.)_{\mathscr{V}\times\mathscr{V}'}$ to represent the duality pairing between $\mathscr{V}$ and $\mathscr{V}'$.

Let V denote the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{\boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1\right\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_0 > 0$, that depends only on Ω and Γ_1 such that

$$\|\varepsilon(\boldsymbol{v})\|_{\mathscr{H}} \geqslant C_0 \|\boldsymbol{v}\|_{H^1(\Omega)^d}, \quad \forall \boldsymbol{v} \in V.$$

On V, we consider the inner product and the associated norm given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathscr{H}}, \ \|\boldsymbol{v}\|_V = \|\varepsilon(\boldsymbol{v})\|_{\mathscr{H}}, \ \boldsymbol{u},\boldsymbol{v} \in V. \tag{20}$$

It follows that $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent norms on V and therefore $(V,(.,.)_V)$ is a real Hilbert space.

For the electric displacement field we use two Hilbert spaces

$$\mathscr{W} = \left\{D \in H \mid \mathrm{div}\, D \in L^2(\Omega)\right\},$$

endowed with the inner products

$$(D,E)_{\mathscr{W}} = (D,E)_H + (\operatorname{div} D, \operatorname{div} E)_{L^2(\Omega)},$$

and the associated norm $\|.\|_{\mathscr{W}}$. The electric potential field is to be found in

$$W = \left\{ \xi \in H^1(\Omega), \xi = 0 \text{ on } \Gamma_a \right\}.$$

Since $meas(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality holds:

$$\|\nabla\zeta\|_H \geqslant c_F \|\zeta\|_{H^1(\Omega)}, \quad \forall \zeta \in W, \tag{21}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W we use the inner product

$$(\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H, \tag{22}$$

and $\|.\|_W$ the associated norm. It follows from (21) that $\|.\|_{H^1(\Omega)}$ and $\|.\|_W$ are equivalent norms on W and therefore $(W, \|.\|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace theorem, there exist two positive constants c_0 and $\tilde{c}_0$ such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leqslant c_0 \|\boldsymbol{v}\|_V, \quad \forall \boldsymbol{v} \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leqslant \tilde{c}_0 \|\psi\|_W, \quad \forall \psi \in W. \tag{23}$$

Moreover, when $\boldsymbol{D} \in \mathscr{W}$ is a regular function, the following Green's type formula holds

$$(\boldsymbol{D}, \nabla\zeta)_H + (\operatorname{div} \boldsymbol{D}, \zeta)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D} \cdot \boldsymbol{\nu} \zeta \, da, \quad \forall \zeta \in H^1(\Omega). \tag{24}$$

For any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leqslant p \leqslant \infty$ and $k \geqslant 1$. For $T > 0$ we denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\boldsymbol{f}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\boldsymbol{f}(t)\|_X,$$
$$\|\boldsymbol{f}\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|\boldsymbol{f}(t)\|_X + \max_{t\in[0,T]} \|\dot{\boldsymbol{f}}(t)\|_X.$$

In the study of the problem P, we consider the following assumptions

The viscosity operator $\mathscr{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \text{ There exists } L_{\mathscr{A}} > 0 \quad \text{such that} \\ \qquad \|\mathscr{A}(\boldsymbol{x}, \boldsymbol{\omega}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\omega}_2)\| \leqslant L_{\mathscr{A}} \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|, \\ \qquad \text{for all } \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathscr{A}} > 0 \quad \text{such that} \\ \qquad (\mathscr{A}(\boldsymbol{x}, \boldsymbol{\omega}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\omega}_2)).(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \geqslant m_{\mathscr{A}} \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|^2, \\ \qquad \text{for all } \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, \boldsymbol{\omega}) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for any } \boldsymbol{\omega} \in \mathbb{S}^d. \\ (d) \text{ The mapping } \boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, 0) \in \mathscr{H}. \end{cases} \tag{25}$$

The elasticity operator $\mathscr{B} : \Omega \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \text{ There exists } L_{\mathscr{B}} > 0 \quad \text{such that} \\ \qquad \|\mathscr{B}(\boldsymbol{x}, \boldsymbol{\omega}_1, \alpha_1) - \mathscr{B}(\boldsymbol{x}, \boldsymbol{\omega}_2, \alpha_2)\| \leqslant L_{\mathscr{B}}(\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\| + \|\alpha_1 - \alpha_2\|), \\ \qquad \text{for all } \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ The mapping } \boldsymbol{x} \mapsto \mathscr{B}(\boldsymbol{x}, \boldsymbol{\omega}, \alpha) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\omega} \in \mathbb{S}^d, \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \mathscr{B}(\boldsymbol{x}, 0, 0) \in \mathscr{H}. \end{cases} \tag{26}$$

The visco-plasticity operator $\mathscr{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathscr{G}} > 0 \quad \text{such that} \\ \qquad \|\mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\omega}_1) - \mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\omega}_2)\| \leqslant L_{\mathscr{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|), \\ \qquad \text{for all } t \in (0,T), \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ The mapping } \boldsymbol{x} \mapsto \mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\omega}) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\omega}, \in \mathbb{S}^d, t \in (0,T), \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \mathscr{G}(\boldsymbol{x}, 0, 0) \in \mathscr{H}. \end{cases} \tag{27}$$

The function $S : \Omega \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_S > 0 \quad \text{such that} \\ \qquad \|S(\boldsymbol{x}, \boldsymbol{\omega}_1, \alpha_1) - S(\boldsymbol{x}, \boldsymbol{\omega}_2, \alpha_2)\| \leqslant L_S(\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\| + \|\alpha_1 - \alpha_2\|), \\ \qquad \text{for all } \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{ for all } \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ The mapping } \boldsymbol{x} \mapsto S(\boldsymbol{x}, \boldsymbol{\omega}, \alpha) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\omega} \in \mathbb{S}^d, \text{ for all } \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto S(\boldsymbol{x}, 0, 0) \in L^2(\Omega). \end{cases} \tag{28}$$

The thermal expansion operator $C_e : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) & \text{There exists } L_{C_e} > 0 \text{ such that} \\ & \|C_e(\boldsymbol{x},\theta_1) - C_e(\boldsymbol{x},\theta_2)\| \leqslant L_{C_e} \|\theta_1 - \theta_2\| \text{ for all } \theta_1,\theta_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) & C_e = (c_{ij}), c_{ij} = c_{ji} \in L^\infty(\Omega). \\ (c) & \text{The mapping } \boldsymbol{x} \mapsto C_e(\boldsymbol{x},\theta) \text{ is Lebesgue measurable on } \Omega, \\ & \text{for any } \theta \in \mathbb{R}. \\ (d) & \text{The mapping } \boldsymbol{x} \mapsto C_e(\boldsymbol{x},0) \in \mathscr{H}. \end{cases} \tag{29}$$

The thermal conductivity operator $K = (k_{ij}) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) & \text{There exists } L_K > 0 \text{ such that} \\ & \|K(\boldsymbol{x},r_1) - K(\boldsymbol{x},r_2)\| \leqslant L_K \|r_1 - r_2\|, \text{ for all } r_1,r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) & k_{ij} = k_{ji} \in L^\infty(\Omega), k_{ij}\alpha_i\alpha_j \leqslant c_k\alpha_i\alpha_j \text{ for some } c_k > 0, \\ & \text{for all } (\alpha_i) \in \mathbb{R}. \\ (c) & \text{The mapping } \boldsymbol{x} \mapsto k(\boldsymbol{x},0) \text{ belongs to } L^2(\Omega). \end{cases} \tag{30}$$

Electric permittivity operator $\boldsymbol{B} = (b_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) & \boldsymbol{B}(x,E) = (b_{ij}(x)E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. } x \in \Omega. \\ (b) & b_{ij} = b_{ji} \in L^\infty(\Omega), 1 \leqslant i,j \leqslant d. \\ (c) & \text{There exists a constant } m_{\boldsymbol{B}} > 0 \text{ such that} \\ & \boldsymbol{B}E.E \geqslant m_{\boldsymbol{B}}\|E\|^2, \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. in } \Omega. \end{cases} \tag{31}$$

The piezoelectric operator $\mathscr{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) & \mathscr{E} = (e_{ijk}), e_{ijk} \in L^\infty(\Omega), 1 \leqslant i,j,k \leqslant d. \\ (b) & \mathscr{E}(\mathbf{x})\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathscr{E}^*\boldsymbol{\tau}, \text{ for all } \boldsymbol{\sigma} \in \mathbb{S}^d, \text{ and all } \boldsymbol{\tau} \in \mathbb{R}^d. \end{cases} \tag{32}$$

The tangential function $h_\tau : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{cases} (a) & \text{There exists } L_\tau > 0 \text{ such that} \\ & \|h_\tau(\boldsymbol{x},u_1) - h_\tau(\boldsymbol{x},u_2)\| \leqslant L_\tau \|u_1 - u_2\|, \\ & \text{for all } u_1,u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ (b) & \text{For any } u \in \mathbb{R}, \boldsymbol{x} \mapsto h_\tau(\boldsymbol{x},u) \text{ is Lebesgue measurable on } \Gamma_3. \\ (c) & \text{The mapping } \boldsymbol{x} \mapsto h_\tau(\boldsymbol{x},0) \text{ belongs to } L^2(\Gamma_3). \end{cases} \tag{33}$$

We assume that the friction coefficient μ, the normal stress F, the boundary and initial data θ_R, k_e, α_0, $\boldsymbol{u}_0$ and θ_0 the volume of forces f_0 and f_2 and the charges densities q_0, q_2 the heat source density $\mathbf{q}$ the microcrack diffusion coefficient k_0 satisfy

$$\begin{aligned} &\mu \in L^\infty(\Gamma_3),\ \mu \geqslant 0 \text{ a.e. on } \Gamma_3, \\ &F \in L^2(\Gamma_3),\ F \geqslant 0 \text{ a.e. on } \in \Gamma_3, \end{aligned} \tag{34}$$

$$\theta_R \in C(0,T;L^2(\Gamma_3)),\ k_e \in L^\infty(\Omega,\mathbb{R}_+), \tag{35}$$

$$\boldsymbol{u}_0 \in V, \quad \alpha_0 \in \mathscr{Y}, \quad \theta_0 \in \mathscr{V}, \tag{36}$$

$$f_0 \in C\left(0,T;L^2(\Omega)^d\right),\ f_2 \in C\left(0,T;L^2(\Gamma_2)^d\right), \tag{37}$$

$$q_0 \in C\left(0,T;L^2(\Omega)\right),\ q_2 \in C\left(0,T;L^2(\Gamma_b)\right), \tag{38}$$

$$k_0 > 0, \quad \mathbf{q} \in C\left(0,T;L^2(\Omega)\right). \tag{39}$$

The function $r : V \times \mathbb{R} \to L^2(\Omega)$ satisfies that there exists a constant $L_r > 0$ such that

$$\begin{gathered} \|r(\boldsymbol{u}_1,\xi_1) - r(\boldsymbol{u}_2,\xi_2)\|_{L^2(\Omega)} \leqslant L_r\left(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\xi_1 - \xi_2)\|\right) \\ \forall \boldsymbol{u}_1,\boldsymbol{u}_2 \in V, \quad \xi_1,\xi_2 \in \mathbb{R}. \end{gathered} \tag{40}$$

We introduce the following bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, by

$$a(\zeta,\xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi\, dx. \tag{41}$$

Now we consider the mappings $j : V \to \mathbb{R}$, $f : [0,T] \to V$, $q : [0,T] \to W$, $Q : [0,T] \to \mathscr{V}'$, $K : \mathscr{V} \to \mathscr{V}'$, and $R : V \to \mathscr{V}'$ respectively, by

$$j(\boldsymbol{w}) = \int_{\Gamma_3} \mu F \|\boldsymbol{w}_\tau\|\, da, \quad \forall \boldsymbol{w} \in V, \tag{42}$$

$$(\mathbf{f}(t),\boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{w} dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{w} da, \tag{43}$$

$$(q(t),\upsilon)_W = \int_\Omega q_0(t)\upsilon dx - \int_{\Gamma_b} q_2(t)\upsilon da, \tag{44}$$

$$(Q(t),\phi)_{\mathscr{V}'\times\mathscr{V}} = \int_{\Gamma_3} k_e\theta_R(t)\phi da + \int_\Omega \mathbf{q}(t)\phi dx. \tag{45}$$

$$(K\rho,\phi)_{\mathscr{V}'\times\mathscr{V}} = \sum_{i,j=1}^d \int_\Omega k_{ij}\frac{\partial\rho}{\partial x_j}\frac{\partial\phi}{\partial x_i}dx + \int_{\Gamma_3} k_e\rho\phi da. \tag{46}$$

$$(R\boldsymbol{w},\phi)_{\mathscr{V}'\times\mathscr{V}} = \int_\Omega r(\boldsymbol{w})\phi dx + \int_{\Gamma_3} h_\tau(|\boldsymbol{w}_\tau|)\,\phi da. \tag{47}$$

for all $\boldsymbol{w} \in V$, $\upsilon \in W$, $\phi,\rho \in \mathscr{V}$ and $t \in [0,T]$. Note that

$$\mathbf{f} \in C(0,T;V), \quad q \in C(0,T;W). \tag{48}$$

Using standard arguments based on Green's formula, we obtain the following variational formulation (1)–(16).

Problem *PV*

Find a displacement field $\boldsymbol{u}:[0,T]\to V$, a stress field $\boldsymbol{\sigma}:[0,T]\to\mathscr{H}$, an electric potential $\varphi:[0,T]\to W$, a damage field $\alpha:[0,T]\to H^1(\Omega)$, and a temperature $\theta:[0,T]\to\mathscr{V}$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t)=&\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}(t))+\mathscr{B}(\varepsilon(\boldsymbol{u}(t)),\alpha(t))+\mathscr{E}^*\nabla\varphi(t)\\&+\int_0^t\mathscr{G}(\boldsymbol{\sigma}(s)-\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}(s))-\mathscr{E}^*\nabla\varphi(s),\varepsilon(\boldsymbol{u}(s)))\,ds-C_e\theta(t),\end{aligned}\tag{49}$$

$$\boldsymbol{D}=\mathscr{E}\varepsilon(\boldsymbol{u})-\boldsymbol{B}\nabla(\varphi),\tag{50}$$

$$(\boldsymbol{\sigma}(t),\varepsilon(\boldsymbol{v})-\varepsilon(\dot{\boldsymbol{u}}(t))_{\mathscr{H}}+j(\boldsymbol{v})-j(\dot{\boldsymbol{u}}(t))\geqslant(\mathbf{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_V,\tag{51}$$

$$(\boldsymbol{B}\nabla\varphi(t),\nabla\phi)_H-(\mathscr{E}\varepsilon(\boldsymbol{u}(t)),\nabla\phi)_H=(q(t),\phi)_W,\quad\forall\phi\in W,\ t\in(0,T)\tag{52}$$

$$\dot{\theta}(t)+K\theta(t)=R\dot{\boldsymbol{u}}(t)+Q(t),\quad\text{in }\mathscr{V}',\tag{53}$$

$$\begin{aligned}&\alpha(t)\in\mathscr{Y},(\dot{\alpha}(t),\xi-\alpha(t))_{L^2(\Omega)}+a(\alpha(t),\xi-\alpha(t))\\&\qquad\geqslant(S(\varepsilon(\boldsymbol{u}(t)),\alpha(t)),\xi-\alpha(t))_{L^2(\Omega)},\forall\xi\in\mathscr{Y},\ t\in(0,T),\end{aligned}\tag{54}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad\theta(0)=\theta_0,\quad\alpha(0)=\alpha_0.\tag{55}$$

Our main existence and uniqueness result for Problem PV is in the following section.

4. Existence and uniqueness

THEOREM 1. *Assume that (25)–(40) hold, Then there exists a unique solution $(\boldsymbol{u},\boldsymbol{\sigma},\varphi,\theta,\alpha,\boldsymbol{D})$ to problem PV. Moreover, the solution has the regularity*

$$\boldsymbol{u}\in C^1(0,T;V),\tag{56}$$

$$\varphi\in C(0,T;W),\tag{57}$$

$$\boldsymbol{\sigma}\in C(0,T;\mathscr{H}),\tag{58}$$

$$\theta\in C\left(0,T;L^2(\Omega)\right)\cap L^2(0,T;\mathscr{V})\cap W^{1,2}\left(0,T;\mathscr{V}'\right),\tag{59}$$

$$\alpha\in W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right),\tag{60}$$

$$\boldsymbol{D}\in C(0,T;\mathscr{W}).\tag{61}$$

A set of functions $(\boldsymbol{u},\ \boldsymbol{\sigma},\ \varphi,\ \theta,\ \alpha,\boldsymbol{D})$, satisfying (1)–(16), (49)–(55) is called a weak solution of the mechanical problem of electro elastic-viscoplastic with damage and thermal effects P. We conclude that, under the conditions specified in Theorem 1, the mechanical problem P has a unique weak solution satisfying (56)–(61).

The proof of Theorem 1 will be carried out in several steps, From now on, in this section, we always suppose that the assumptions of Theorem 1 hold, and we always assume that C is a generic positive constant may change from place to place. Let $\boldsymbol{\eta}\in C(0,T;\mathscr{H})$ and $\lambda\in C(0,T;L^2(\Omega))$ we consider the following variational problem.

Problem $\mathscr{P}_\eta^1$

Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ such that for all $t \in [0,T]$

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}_\eta(t)), \varepsilon(\boldsymbol{w}) - \varepsilon(\dot{\boldsymbol{u}}_\eta(t)))_{\mathscr{H}} + (\mathscr{B}(\varepsilon(\boldsymbol{u}_\eta(t)), \alpha_\lambda(t)), \varepsilon(\boldsymbol{w}) - \varepsilon(\dot{\boldsymbol{u}}_\eta(t)))_{\mathscr{H}}\\ &+ (\boldsymbol{\eta}(t), \varepsilon(\boldsymbol{w}) - \varepsilon(\dot{\boldsymbol{u}}_\eta(t)))_{\mathscr{H}} + j(\boldsymbol{w}) - j(\dot{\boldsymbol{u}}_\eta(t)) \geqslant (\mathbf{f}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}_\eta(t))_V,\\ &\forall \boldsymbol{w} \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{62}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{63}$$

We have the following result for $\mathscr{P}_\eta^1$

LEMMA 1. 1) *There exists a unique solution $\boldsymbol{u}_\eta \in C^1(0,T;V)$ to the problem* (62) *and* (63).

2) *If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of* (62) *and* (63) *corresponding to the data $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0,T];\mathscr{H})$, then there exists $C > 0$ such that*

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leqslant C\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathscr{H}}\, ds, \quad t \in [0,T]. \tag{64}$$

Proof. We define the operators $A : V \to V$ and $B : V \times H^1(\Omega) \to V$ by

$$(A\boldsymbol{u}, \boldsymbol{w})_V = (\mathscr{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{w}))_{\mathscr{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{w} \in V, \tag{65}$$

$$(B(\boldsymbol{u}, \alpha), \boldsymbol{w})_V = (\mathscr{B}(\varepsilon(\boldsymbol{u}), \alpha)\,\varepsilon(\boldsymbol{w}))_{\mathscr{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{w} \in V, \quad \alpha \in H^1(\Omega). \tag{66}$$

Therefore, (62) can be rewritten as follows

$$\begin{aligned}(A\dot{\boldsymbol{u}}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_V + (B(\boldsymbol{u}(t), \alpha(t)), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_V + j(\boldsymbol{w})&\\ - j(\boldsymbol{u}(t)) \geqslant (\mathbf{f}_\eta(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_V,&\end{aligned} \tag{67}$$

where

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \boldsymbol{\eta}(t), \quad \text{a.e. } t \in [0,T].$$

We use assumption (25) to show that A is a strongly monotone Lipschitz continuous operator. Also, it follows from (26) that B is a Lipschitz continuous operator and we use (23) to see that the functional j defined in (42) satisfies

$$j(\boldsymbol{w}) \leqslant c_0\|\mu\|_{L^\infty(\Gamma_3)}\|F\|_{L^2(\Gamma_3)}\|\boldsymbol{w}\|_V, \quad \forall \boldsymbol{w} \in V.$$

So the seminorm j is continuous and, therefore, it is a convex lower semicontinuous function on V. Finally, note that $\mathbf{f}_\eta \in C([0,T];V)$ and $\boldsymbol{u}_0 \in V$ and we use classical arguments of functional analysis concerning evolutionary variational inequalities [4, 19] we can easily prove the existence and uniqueness of $\boldsymbol{u}_\eta$ satisfying (56). Using inequality (62) for $\boldsymbol{\eta} = \boldsymbol{\eta}_1$, $\boldsymbol{u}_{\eta_1} = \boldsymbol{u}_1$, $\dot{\boldsymbol{u}}_{\eta_1} = \dot{\boldsymbol{u}}_1$, we find

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}_1(t)), \varepsilon(\boldsymbol{w}) - \varepsilon(\dot{\boldsymbol{u}}_1(t)))_{\mathscr{H}} + (\mathscr{B}(\varepsilon(\boldsymbol{u}_1(t)), \alpha_\lambda(t)), \varepsilon(\boldsymbol{w}) - \varepsilon(\dot{\boldsymbol{u}}_1(t)))_{\mathscr{H}}\\ &+ (\boldsymbol{\eta}_1(t), \varepsilon(\boldsymbol{w}) - \varepsilon(\dot{\boldsymbol{u}}_1(t)))_{\mathscr{H}} + j(\boldsymbol{w}) - j(\dot{\boldsymbol{u}}_1(t)) \geqslant (\mathbf{f}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}_1(t))_V,\\ &\forall \boldsymbol{w} \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{68}$$

for $\boldsymbol{\eta} = \boldsymbol{\eta}_2$, $\boldsymbol{u}_{\boldsymbol{\eta}_2} = \boldsymbol{u}_2$, $\dot{\boldsymbol{u}}_{\eta_2} = \dot{\boldsymbol{u}}_2$, we obtain

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}_2(t)),\varepsilon(\boldsymbol{w})-\varepsilon(\dot{\boldsymbol{u}}_2(t)))_{\mathscr{H}}+(\mathscr{B}(\varepsilon(\boldsymbol{u}_2(t)),\alpha_\lambda(t)),\varepsilon(\boldsymbol{w})-\varepsilon(\dot{\boldsymbol{u}}_2(t)))_{\mathscr{H}}\\&+(\boldsymbol{\eta}_2(t),\varepsilon(\boldsymbol{w})-\varepsilon(\dot{\boldsymbol{u}}_2(t)))_{\mathscr{H}}+j(\boldsymbol{w})-j(\dot{\boldsymbol{u}}_2(t))\geqslant(\mathbf{f}(t),\boldsymbol{w}-\dot{\boldsymbol{u}}_2(t))_V,\\&\forall\boldsymbol{w}\in V,\ \text{a.e. } t\in(0,T),\end{aligned}\tag{69}$$

we take $\boldsymbol{w} = \dot{\boldsymbol{u}}_2(t)$ in (68) and $\boldsymbol{w} = \dot{\boldsymbol{u}}_1(t)$ in (69), add the two inequalities to obtain

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}_1(t))-\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}_2(t)),\varepsilon(\dot{\boldsymbol{u}}_1(t))-\varepsilon(\dot{\boldsymbol{u}}_2(t)))_{\mathscr{H}}\\&\qquad\leqslant(\mathscr{B}(\varepsilon(\boldsymbol{u}_1(t)),\alpha_\lambda(t))-\mathscr{B}(\varepsilon(\boldsymbol{u}_2(t)),\alpha_\lambda(t)),\varepsilon(\dot{\boldsymbol{u}}_2(t))-\varepsilon(\dot{\boldsymbol{u}}_1(t)))_{\mathscr{H}}\\&\qquad\quad+(\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\varepsilon(\dot{\boldsymbol{u}}_1(t))-\varepsilon(\dot{\boldsymbol{u}}_2(t))),\end{aligned}$$

then we use assumptions (25) and (26) to find

$$\|\dot{\boldsymbol{u}}_1-\dot{\boldsymbol{u}}_2\|_V\leqslant C(\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V+\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_{\mathscr{H}}).\tag{70}$$

Since $\boldsymbol{u}_i(t) = \int_0^t \dot{\boldsymbol{u}}_i(s)ds + \boldsymbol{u}_0, \forall t \in [0,T]$, we have

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V\leqslant\int_0^t\|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V\,ds.\tag{71}$$

Using (70), (71) and the Gronwall's inequality, we find

$$\int_0^t\|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V\,ds\leqslant\int_0^t\|\boldsymbol{\eta}(s)-\boldsymbol{\eta}(s)\|_{\mathscr{H}}ds,\tag{72}$$

which concludes the proof of Lemma 1. □

In the second step we use the solution $\boldsymbol{u}_\eta$, obtained in Lemma 1, and consider the following variational problem for the electrical potential.

Problem $\mathscr{P}_\eta^2$

Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$(\boldsymbol{B}\nabla\varphi_\eta(t),\nabla\zeta)_H-(\mathscr{E}\varepsilon(\boldsymbol{u}_\eta(t)),\nabla\zeta)_H=(q(t),\zeta)_W,\ \text{for all } \zeta\in W,\ t\in(0,T).\tag{73}$$

LEMMA 2. *Problem (73) has unique solution φ_η which satisfies the regularity (57). Moreover, if $\varphi_{\boldsymbol{\eta}_1}$ and $\varphi_{\boldsymbol{\eta}_2}$ are the solutions of (73) corresponding to $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2 \in C([0,T];\mathscr{H})$, then there exists $C > 0$ such that*

$$\left\|\varphi_{\boldsymbol{\eta}_1}(t)-\varphi_{\boldsymbol{\eta}_2}(t)\right\|_W\leqslant C\left\|\boldsymbol{u}_{\boldsymbol{\eta}_1}(t)-\boldsymbol{u}_{\boldsymbol{\eta}_2}(t)\right\|_V,\quad\forall t\in[0,T].\tag{74}$$

Proof. We consider the form $L : W \times W \to \mathbb{R}$

$$L(\varphi,\phi)=(\boldsymbol{B}\nabla\varphi,\nabla\phi)_H,\quad\forall\varphi,\phi\in W,\tag{75}$$

we use (21), (22), (31) and (75) to show that the form L is bilinear continuous, symmetric and coercive on W, moreover using (44) and the Riesz representation theorem we may define an element $\xi_\eta : [0,T] \to W$ such that

$$(\xi_\eta(t), \phi)_W = (q(t), \phi)_W + (\mathscr{E}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\phi)_H, \quad \forall \phi \in W,\ t \in (0,T),$$

we apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$L(\varphi_\eta(t), \phi) = (\xi_\eta(t), \phi)_W, \quad \forall \phi \in W. \tag{76}$$

It follows from (76) that φ_η is a solution of the equation (73). Let $\varphi_{\eta_i} = \varphi_i$, and $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$ for $i = 1,2$. We use (73) to obtain

$$\|\varphi_1(t) - \varphi_2(t)\|_W \leqslant C\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V, \quad \forall t \in [0,T].$$

Now since $\boldsymbol{u}_\eta \in C^1(0,T;V)$, so implies that $\varphi_\eta \in C(0,T;W)$. This completes the proof. □

In the third step, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 1 to consider the following variational problem.

Problem $\mathscr{P}_\eta^3$

Find the temperature field $\theta_\eta : (0,T) \to L^2(\Omega)$

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{\boldsymbol{u}}_\eta(t) + Q(t), \quad \text{in } \mathscr{V}', \quad \text{a.e. } t \in [0,T], \tag{77}$$

$$\theta_\eta(0) = \theta_0. \tag{78}$$

LEMMA 3. *There exists a unique solution θ_η to the auxiliary problem $\mathscr{P}_\eta^3$ satisfying (59). Moreover $\exists C > 0$ such that $\forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;\mathscr{H})$.*

$$\left\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\right\|^2_{L^2(\Omega)} \leqslant C\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|^2_{\mathscr{H}} ds, \quad \forall t \in [0,T]. \tag{79}$$

Proof. The result follows from classical first order evolution equation given in Refs. [1, 18]. Here the Gelfand triple is given by

$$\mathscr{V} \subset L^2(\Omega) = \left(L^2(\Omega)\right)' \subset \mathscr{V}'.$$

The operator K is linear and coercive. By Korn's inequality, we have

$$(K\tau, \tau)_{\mathscr{V}'\times\mathscr{V}} \geqslant C\|\tau\|^2_{\mathscr{V}}.$$

Let $\theta_{\eta_i} = \theta_i$, and $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$ for $i = 1,2$. Let $t \in \mathbb{R}^+$ be fixed. Then, we have

$$\left(\dot{\theta}_1(t) - \dot{\theta}_2(t), \theta_1(t) - \theta_2(t)\right)_{\mathscr{V}'\times\mathscr{V}} + (K\theta_1(t) - K\theta_2(t), \theta_1(t) - \theta_2(t))_{\mathscr{V}'\times\mathscr{V}}$$
$$= (R\dot{u}_1(t) - R\dot{u}_2(t), \theta_1(t) - \theta_2(t))_{\mathscr{V}'\times\mathscr{V}}.$$

We integrate the above equality over $(0,t)$ and we use the strong monotonicity of K and the Lipschitz continuity of $R : V \to \mathscr{V}'$ to deduce that

$$\|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} ds \leqslant C \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|^2_V ds,$$

It follows now from (72), that

$$\|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} \leqslant C \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|^2_{\mathscr{H}} ds, \quad \forall t \in [0,T]. \quad \square$$

In the fourth step we let $\lambda \in C(0,T;L^2(\Omega))$

Problem $\mathscr{P}_\lambda$

Find the damage field $\alpha_\lambda : (0,T) \to L^2(\Omega)$ such that $\alpha_\lambda(t) \in \mathscr{Y}$ and

$$\begin{aligned} &(\dot{\alpha}_\lambda(t), \xi - \alpha_\lambda)_{L^2(\Omega)} + a(\alpha_\lambda(t), \xi - \alpha_\lambda(t)) \\ &\qquad \geqslant (\lambda(t), \xi - \alpha_\lambda(t))_{L^2(\Omega)} \quad \forall \xi \in \mathscr{Y}, \text{ a.e. } t \in (0,T), \end{aligned} \tag{80}$$

$$\alpha_\lambda(0) = \alpha_0. \tag{81}$$

For the study of problem $\mathscr{P}_\lambda$, we have the following result.

LEMMA 4. *There exists a unique solution α_λ to the auxiliary problem $\mathscr{P}_\lambda$ satisfying* (60).

Proof. The inclusion mapping of $\left(H^1(\Omega), \|\cdot\|_{H^1(\Omega)}\right)$ into $\left(L^2(\Omega), \|\cdot\|_{L^2(\Omega)}\right)$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset \left(H^1(\Omega)\right)'.$$

We use the notation $(.,.)_{(H^1(\Omega))' \times H^1(\Omega)}$ to represent the duality pairing between $(H^1(\Omega))'$ and $(H^1(\Omega))$. We have

$$(\alpha, \rho)_{(H^1(\Omega))' \times H^1(\Omega)} = (\alpha, \rho)_{L^2(\Omega)}, \quad \forall \alpha \in L^2(\Omega), \rho \in H^1(\Omega),$$

and we note that K is a closed convex set in $(H^1(\Omega))$, using the definition (53) of the bilinear form a, for all $\upsilon, \rho \in H^1(\Omega)$, we have $a(\upsilon, \rho) = a(\rho, \upsilon)$ and

$$|a(\upsilon, \rho)| \leqslant k\|\nabla \upsilon\|_H \|\nabla \rho\|_H \leqslant c\|\upsilon\|_{H^1(\Omega)} \|\rho\|_{H^1(\Omega)},$$

Therefore, a is continuous and symmetric. Thus, for all $\upsilon \in H^1(\Omega)$, we have

$$a(\upsilon, \upsilon) = k\|\nabla \upsilon\|^2_H,$$

so

$$a(\upsilon,\upsilon)+(k+1)\|\upsilon\|^2_{L^2(\Omega)} \geqslant k\left(\|\nabla\upsilon\|^2_H+\|\upsilon\|^2_{L^2(\Omega)}\right),$$

which implies

$$a(\upsilon,\upsilon)+c_0l\|\upsilon\|^2_{L^2(\Omega)} \geqslant c_1\|\upsilon\|^2_{H^1(\Omega)} \text{ with } c_0=k+1 \text{ and } c_1=k.$$

Finally, we use (40), (46) to see that $\lambda \in L^2(0,T;L^2(\Omega))$ and $\alpha_0 \in K$, and we use a standard result for parabolic variational inequalities (see [1], p. 124), we find that there exists a unique function $\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right)$, such that $\alpha(0)=\alpha_0$, $\alpha(t)\in K$ for all $t\in[0,T]$ and for almost all $t\in(0,T)$

$$(\dot{\alpha}_\lambda(t),\rho-\alpha_\lambda)_{(H^1(\Omega))'\times H^1(\Omega)}+a(\alpha_\lambda(t),\rho-\alpha_\lambda(t)) \geqslant (\lambda(t),\rho-\alpha_\alpha(t))_{L^2(\Omega)},$$

$\forall\rho\in K.$ $\square$

In the fifth step, we use $\boldsymbol{u}_\eta$, φ_η, θ_η and α_λ obtained in Lemmas 1, 2, 3 and 4, respectively to construct the following Cauchy problem for the stress field.

Problem $\mathscr{P}_{\eta,\lambda}$

Find the stress field $\boldsymbol{\sigma}_{\eta,\lambda}:[0,T]\to\mathscr{H}$ which is a solution of the problem

$$\boldsymbol{\sigma}_{\eta,\lambda}(t)=\mathscr{B}(\varepsilon(u_\eta(t),\alpha_\lambda(t)))+\int_0^t\mathscr{G}\left(\boldsymbol{\sigma}_{\eta,\lambda}(s),\varepsilon(u_\eta(s))\right)ds-C_e\theta_\eta(t), \quad \text{a.e. } t\in(0,T). \tag{82}$$

LEMMA 5. *$\mathscr{P}_{\eta,\lambda}$ has a unique solutions $\boldsymbol{\sigma}_{\eta,\lambda}\in C(0,T;\mathscr{H})$. Moreover, if $\boldsymbol{\sigma}_{\eta_i,\lambda_i}$, $\boldsymbol{u}_{\eta_i}$, θ_{η_i} and α_{λ_i} represent the solutions of Problems $\mathscr{P}_{\eta,\lambda}$, $\mathscr{P}^1_\eta$, $\mathscr{P}^3_\eta$ and, $\mathscr{P}_\lambda$ respectively, for $(\boldsymbol{\eta}_i,\lambda_i)\in C(0,T;\mathscr{H}\times L^2(\Omega))$, $i=1,2$, then there exists $C>0$ such that*

$$\begin{aligned}\left\|\boldsymbol{\sigma}_{\eta_1,\lambda_1}(t)-\boldsymbol{\sigma}_{\eta_2,\lambda_2}(t)\right\|^2_{\mathscr{H}} \leqslant C\Big(&\left\|\boldsymbol{u}_{\eta_1}(t)-\boldsymbol{u}_{\eta_2}(t)\right\|^2_V+\left\|\alpha_{\lambda_1}(t)-\alpha_{\lambda_2}(t)\right\|^2_{L^2(\Omega)}\\&+\left\|\theta_{\eta_1}(t)-\theta_{\eta_2}(t)\right\|^2_{L^2(\Omega)}+\int_0^t\left\|\boldsymbol{u}_{\eta_1}(s)-\boldsymbol{u}_{\eta_2}(s)\right\|^2_V\Big).\end{aligned} \tag{83}$$

Proof. Let $\boldsymbol{\Sigma}_{\eta,\lambda}:C(0,T;\mathscr{H})\to C(0,T;\mathscr{H})$ be the operator given by

$$\boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}(t)=\mathscr{B}(\varepsilon(u_\eta(t),\alpha_\lambda(t)))+\int_0^t\mathscr{G}\left(\boldsymbol{\sigma}_{\eta,\lambda}(s),\varepsilon(u_\eta(s))\right)ds-C_e\theta_\eta(t), \tag{84}$$

Let $\boldsymbol{\sigma}_i\in W^{1,\infty}(0,T;\mathscr{H})$, $i=1,2$ and $t_1\in(0,T)$. Using hypothesis (27) and Holder's inequality, we find

$$\left\|\boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_1)-\Sigma_{\eta,\lambda}\boldsymbol{\sigma}_2(t_1)\right\|^2_{\mathscr{H}} \leqslant L^2_{\mathscr{G}}T\int_0^{t_1}\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathscr{H}}ds.$$

Integration on the time interval $(0,t_2) \subset (0,T)$, it follows that

$$\int_0^{t_2} \left\|\boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_1) - \boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_1)\right\|_{\mathscr{H}}^2 dt_1 \leqslant L_{\mathscr{G}}^2 T \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathscr{H}}^2 ds dt_1.$$

Therefore

$$\left\|\boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_2) - \boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_2)\right\|_{\mathscr{H}}^2 \leqslant L_{\mathscr{G}}^4 T^2 \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathscr{H}}^2 ds dt_1.$$

For $t_1, t_2, \ldots, t_n \in (0,T)$, we generalize the procedure above by recurrence on n. We obtain the inequality

$$\begin{aligned}&\left\|\boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_n) - \boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_n)\right\|_{\mathscr{H}}^2 \\ &\qquad \leqslant L_{\mathscr{G}}^{2n} T^n \int_0^{tn} \cdots \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathscr{H}}^2 ds dt_1 \ldots dt_{n-1}.\end{aligned}$$

Which implies

$$\left\|\boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_n) - \boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_n)\right\|_{\mathscr{H}}^2 \leqslant \frac{L_{\mathscr{G}}^{2n} T^{n+1}}{n!} \int_0^T \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathscr{H}}^2 ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\left\|\boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_1 - \boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma}_2\right\|_{C(0,T;\mathscr{H})}^2 \leqslant \frac{L_{\mathscr{G}}^{2n} T^{n+2}}{n!} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{C(0,T;\mathscr{H})}^2.$$

It follows from this inequality that for large n enough, the operator $\boldsymbol{\Sigma}_{\eta,\lambda}^n$ is a contraction on the Banach space $C(0,T;\mathscr{H})$, and therefore there exists a unique element $\boldsymbol{\sigma} \in C(0,T;\mathscr{H})$ such that $\boldsymbol{\Sigma}_{\eta,\lambda}\boldsymbol{\sigma} = \boldsymbol{\sigma}$. Moreover, $\boldsymbol{\sigma}$ is the unique solution of Problem $\mathscr{P}_{\eta,\lambda}$. Consider now $(\boldsymbol{\eta}_1,\lambda_1),(\boldsymbol{\eta}_2,\lambda_2) \in C(0,T;\mathscr{H} \times L^2(\Omega))$ and for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\theta_{\eta_i} = \theta_i$, $\alpha_{\lambda_i} = \alpha_i$ and $\boldsymbol{\sigma}_{\eta_i,\lambda_i} = \boldsymbol{\sigma}_i$. We have

$$\begin{aligned}\boldsymbol{\sigma}_i(t) = \mathscr{B}(\varepsilon(\boldsymbol{u}_i(t),\alpha_i(t))) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}_i(s),\varepsilon(\boldsymbol{u}_i(s)))ds - C_e\theta_i(t),\\ \text{a.e. } t \in (0,T).\end{aligned} \tag{85}$$

and using the properties (26), (27) and (29) of $\mathscr{B}$, $\mathscr{G}$ and C_e we find

$$\begin{aligned}&\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathscr{H}}^2 \\ &\leqslant C\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V^2 + \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \\ &\quad + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V^2 ds + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathscr{H}}^2 ds\Big), \quad \forall t \in [0,T].\end{aligned} \tag{86}$$

We use Gronwall argument in the previous inequality to deduce (83), which concludes the proof of Lemma 5. □

Finally, as a consequence of these results and using the properties of the operators $\mathscr{G}$, $\mathscr{E}$, C_e and the function S, for $t \in (0,T)$, we consider the element

$$\Lambda(\boldsymbol{\eta},\lambda)(t) = \left(\Lambda^1(\boldsymbol{\eta},\lambda)(t), \Lambda^2(\boldsymbol{\eta},\lambda)(t)\right) \in \mathscr{H} \times L^2(\Omega), \tag{87}$$

defined by

$$\begin{aligned}\left(\Lambda^1(\boldsymbol{\eta},\lambda)(t), \boldsymbol{v}\right)_{\mathscr{H}\times V} &= (\mathscr{E}^*\nabla\varphi_\eta(t), \varepsilon(\boldsymbol{v}))_{\mathscr{H}} + (C_e\theta_\eta(t), \varepsilon(\boldsymbol{v}))_{\mathscr{H}} \\ &\quad + \left(\int_0^t \mathscr{G}\left(\boldsymbol{\sigma}_{\eta,\lambda}, \boldsymbol{\varepsilon}\left(\boldsymbol{u}_\eta(s)\right)\right) ds, \varepsilon(\boldsymbol{v})\right)_{\mathscr{H}}, \forall \boldsymbol{v} \in V,\end{aligned} \tag{88}$$

$$\Lambda^2(\boldsymbol{\eta},\lambda)(t) = S\left(\varepsilon\left(\boldsymbol{u}_\eta(t)\right), \alpha_\lambda(t)\right). \tag{89}$$

Here, for every $(\boldsymbol{\eta},\lambda) \in C(0,T;\mathscr{H} \times L^2(\Omega))$. $\boldsymbol{u}_\eta$, φ_η, θ_η, α_λ and $\boldsymbol{\sigma}_{\eta,\lambda}$ represent the displacement field, the electric potential field, the temperature field, the damage field and the stress field, obtained in Lemmas 1, 2, 3, 4 and 5 respectively. We have the following result.

LEMMA 6. *The mapping* Λ *has a fixed point* $(\boldsymbol{\eta}^*,\lambda^*) \in C(0,T;\mathscr{H} \times L^2(\Omega))$, *such that* $\Lambda(\boldsymbol{\eta}^*,\lambda^*) = (\boldsymbol{\eta}^*,\lambda^*)$.

Proof. Let $t \in (0,T)$ and $(\boldsymbol{\eta}_1,\lambda_1), (\boldsymbol{\eta}_2,\lambda_2) \in C\left(0,T;\mathscr{H} \times L^2(\Omega)\right)$. We use the notation that $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \dot{\boldsymbol{u}}_i$, $\theta_{\eta_i} = \theta_i$, $\varphi_{\eta_i} = \varphi_i$, $\alpha_{\lambda_i} = \alpha_i$ and $\boldsymbol{\sigma}_{\eta_i,\lambda_i} = \boldsymbol{\sigma}_i$ for $i = 1,2$.

Let us start by using (23), (27), (29) and (32), we have

$$\begin{aligned}\|\Lambda^1(\boldsymbol{\eta}_1,\lambda_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\lambda_2)(t)\|_{\mathscr{H}}^2 &\leqslant \|\mathscr{E}^*\nabla\varphi_1(t) - \mathscr{E}^*\nabla\varphi_2(t)\|_{\mathscr{H}}^2 \\ &\quad + \|C_e\theta_1(t)) - C_e\theta_2(t)\|_{\mathscr{H}}^2 \\ &\quad + \int_0^t \|\mathscr{G}\left(\boldsymbol{\sigma}_1(s), \varepsilon(\boldsymbol{u}_1(s))\right) - \mathscr{G}\left(\boldsymbol{\sigma}_2(s), \varepsilon(\boldsymbol{u}_2(s))\right)\|_{\mathscr{H}}^2 ds \\ &\leqslant C\bigg(\|\varphi_1(t) - \varphi_2(t)\|_W^2 + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \\ &\quad + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathscr{H}}^2 ds + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V^2 ds\bigg).\end{aligned} \tag{90}$$

We use estimates (74), (83) to obtain

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1,\lambda_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\lambda_2)(t)\|_{\mathscr{H}}^2 \\ &\quad \leqslant C\bigg(\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \\ &\qquad + \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V^2 + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V^2 ds\bigg).\end{aligned} \tag{91}$$

By similar arguments, from (89) and (28) we obtain

$$\begin{aligned}&\left\|\Lambda^2(\boldsymbol{\eta}_1,\lambda_1)(t) - \Lambda^2(\boldsymbol{\eta}_2,\lambda_2)(t)\right\|_{\mathscr{H}}^2 \\ &\quad \leqslant C\left(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V^2 + \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2\right), \quad \text{a.e. } t \in (0,T).\end{aligned} \tag{92}$$

It follows now from (92), (91) and (87) that

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\lambda_1)(t)-\Lambda(\boldsymbol{\eta}_2,\lambda_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)}\\&\quad\leqslant C\Big(\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}\\&\qquad+\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V\,ds\Big).\end{aligned}\tag{93}$$

Form (80), deduced that

$$\begin{aligned}&(\dot{\alpha}_1-\dot{\alpha}_2,\alpha_1-\alpha_2)_{L^2(\Omega)}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\\&\qquad\leqslant(\lambda_1-\lambda_2,\alpha_1-\alpha_2)_{L^2(\Omega)},\ \text{a.e. } t\in(0,T).\end{aligned}$$

integrate inequality with respect to time, using the initial conditions $\alpha_1(0)=\alpha_2(0)=\alpha_0$, and inequality $a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\geqslant 0$, we find

$$\tfrac{1}{2}\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\leqslant C\int_0^t(\lambda_1(s)-\lambda_2(s),\alpha_1(s)-\alpha_2(s))_{L^2(\Omega)}ds,$$

which implies

$$\begin{aligned}&\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\\&\quad\leqslant C\left(\int_0^t\|\lambda_1(s)-\lambda_2(s)\|^2_{L^2(\Omega)}\,ds+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}\,ds\right).\end{aligned}$$

This inequality combined with the Gronwall inequality leads to

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\leqslant C\int_0^t\|\lambda_1(s)-\lambda_2(s)\|^2_{L^2(\Omega)}\,ds\,,\forall t\in[0,T].\tag{94}$$

Form the previous inequality and estimates (94), (93), (79) and (64) it follows now that

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\lambda_1)(t)-\Lambda(\boldsymbol{\eta}_2,\lambda_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)}\\&\qquad\leqslant C\int_0^T\|(\boldsymbol{\eta}_1,\lambda_1)(s)-(\boldsymbol{\eta}_2,\lambda_2)(s)\|^2_{\mathscr{H}\times L^2(\Omega)}\,ds.\end{aligned}\tag{95}$$

Reiterating this inequality m times we obtain

$$\begin{aligned}&\|\Lambda^m(\boldsymbol{\eta}_1,\lambda_1)-\Lambda^m(\boldsymbol{\eta}_2,\lambda_2)\|^2_{C(0,T;\mathscr{H}\times L^2(\Omega))}\\&\qquad\leqslant\frac{C^mT^m}{m!}\|(\boldsymbol{\eta}_1,\lambda_1)-(\boldsymbol{\eta}_2,\lambda_2)\|^2_{C(0,T;\mathscr{H}\times L^2(\Omega))}.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathscr{H}\times L^2(\Omega))$, and so Λ has a unique fixed point. □

Now we have every thing that is required to prove Theorem 1.

Proof. Let $(\boldsymbol{\eta}^*,\lambda^*)\in C(0,T;\mathscr{H}\times L^2(\Omega))$ be the fixed point of Λ and

$$\boldsymbol{u}=\boldsymbol{u}_{\eta^*},\quad \theta=\theta_{\eta^*},\quad \varphi_{\eta^*}=\varphi,\quad \alpha=\alpha_{\lambda^*} \tag{96}$$

$$\boldsymbol{\sigma}=\mathscr{A}\varepsilon(\dot{\boldsymbol{u}})+\mathscr{E}^*\nabla\varphi(t)+\boldsymbol{\sigma}_{\eta^*\lambda^*}, \tag{97}$$

$$\boldsymbol{D}=\mathscr{E}\varepsilon(\boldsymbol{u})+\boldsymbol{B}\nabla(\varphi). \tag{98}$$

We prove that $(\boldsymbol{u},\boldsymbol{\sigma},\theta,\varphi,\alpha,\boldsymbol{D})$ satisfies (49)–(55) and (56)–(61). Indeed, we write (82) for $\boldsymbol{\eta}^*=\boldsymbol{\eta}$, $\lambda^*=\lambda$ and use (96)–(97) to obtain that (49) is satisfied. Now we consider (62) for $\boldsymbol{\eta}^*=\boldsymbol{\eta}$, $\lambda^*=\lambda$ and use (96) to find

$$\begin{gathered}(\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}(t)),\varepsilon(\boldsymbol{v}-\dot{\boldsymbol{u}}(t)))_{\mathscr{H}}+(\mathscr{B}(\varepsilon(\boldsymbol{u}(t)),\alpha(t)),\varepsilon(\boldsymbol{v})-\varepsilon(\dot{\boldsymbol{u}}(t)))_{\mathscr{H}}\\ +(\boldsymbol{\eta}^*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_{\mathscr{H}}+j(\boldsymbol{v})-j(\dot{\boldsymbol{u}}(t))\geqslant(\mathbf{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_V\\ \forall\boldsymbol{v}\in V,\ t\in[0,T].\end{gathered} \tag{99}$$

The equalities $\Lambda^1(\boldsymbol{\eta}^*,\lambda^*)=\boldsymbol{\eta}^*$ and $\Lambda^2(\boldsymbol{\eta}^*,\lambda^*)=\lambda^*$. combined with (88)–(89), (96) and (97) show that for all $\boldsymbol{v}\in V$,

$$\begin{aligned}(\boldsymbol{\eta}^*(t),\boldsymbol{v})_{\mathscr{H}\times V}&=(\mathscr{E}^*\nabla\varphi(t),\varepsilon(\boldsymbol{v}))_{\mathscr{H}}-(C_e\theta(t),\varepsilon(\boldsymbol{v}))_{\mathscr{H}},\\ &+\left(\int_0^t\mathscr{G}(\boldsymbol{\sigma}(s)-\mathscr{A}\varepsilon(\dot{\boldsymbol{u}}(s))-\mathscr{E}^*\nabla\varphi(t),\varepsilon(\boldsymbol{u}(s)))ds,\varepsilon(\boldsymbol{v})\right)_{\mathscr{H}},\end{aligned} \tag{100}$$

$$\lambda^*(t)=S(\varepsilon(\boldsymbol{u}(t)),\alpha(t)). \tag{101}$$

We substitute (100) in (99)) and use (49) to see that (51) is satisfied.

We write now (73) for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ and use (96) to find (52). From (77) and (96) we see that (53) is satisfied.

We write (80) for $\lambda=\lambda^*$ and use (96) and (101) to find that (54) is satisfied

Next, (55), The regularities (56), (57), (59) and (60) follow from Lemmas 1, 2, 3 and 4. The regularity $\boldsymbol{\sigma}\in C(0,T;\mathscr{H})$ follows from Lemmas 5.

Let now $t_1,t_2\in[0,T]$, from (21), (31), (32) and (98), we conclude that there exists a positive constant $C>0$ verifying

$$\|\boldsymbol{D}(t_1)-\boldsymbol{D}(t_2)\|_H\leqslant C(\|\varphi(t_1)-\varphi(t_2)\|_W+\|\boldsymbol{u}(t_1)-\boldsymbol{u}(t_2)\|_V).$$

The regularity of $\boldsymbol{u}$ and φ given by (56) and (57) implies

$$\boldsymbol{D}\in C(0,T;H). \tag{102}$$

We choose $\phi\in D(\Omega)^d$ in (52) and using (44) we find

$$\operatorname{div}\boldsymbol{D}(t)=q_0(t),\quad \forall t\in[0,T], \tag{103}$$

Property (61) follows from (38),(102) and (103) which concludes the existence part the Theorem 1. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ, and the unique solvability of the Problems $\mathscr{P}_\eta^1$, $\mathscr{P}_\eta^2$, $\mathscr{P}_\eta^3$, $\mathscr{P}_\lambda$ and $\mathscr{P}_{\eta,\lambda}$ which completes the proof. □

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(Received August 15, 2022)

Ahmed Hamidat
University of El Oued, Fac, Exact Sciences
Lab Laboratory of Operator Theory and PDE:
Foundations and Applications
39000, El Oued, Algeria
e-mail: hamidat-ahmed@univ-eloued.dz

Adel Aissaoui
University of El Oued, Department of Mathematics, Fac, Exact Sciences
Lab Laboratory of Operator Theory and PDE:
Foundations and Applications
39000, El Oued, Algeria
e-mail: aissaoui-adel@univ-eloued.dz