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CANONICAL REPRESENTATION OF THIRD–ORDER DELAY DYNAMIC EQUATIONS ON TIME SCALES

JOHN R. GRAEF* AND IRENA JADLOVSKÁ

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Abstract. The authors show that any semicanonical or noncanonical third-order linear delay dynamic equation on a time scale can be written in canonical form without imposing any additional conditions on the coefficient functions. Since this is true for any time scale, this means it holds for differential and difference equations. The implication of this is the significant result that any set of conditions which show that a related equation in canonical form is oscillatory will guarantee that the semicanonical or noncanonical equation is also oscillatory. Several examples of the application of the results are incorporated into the paper.

1. Introduction

Consider the third-order delay dynamic equation

$$\left(a_2(t)\left(a_1(t)y^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+q(t)y(\tau(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},\tag{E}$$

where $\mathbb{T}$ is a time scale with $t_0\geqslant 0$, $\sup\mathbb{T}=\infty$, $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$, a_1, a_2, $q\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $\tau\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\tau(t)\leqslant t$, and $\lim_{t\to\infty}\tau(t)=\infty$. The usual notation and terminology for time scales as can be found in Bohner and Peterson [7] will be used throughout. A solution y of (E) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and it is said to be *nonoscillatory* otherwise. An equation is said to be oscillatory if all of its solutions are oscillatory.

A wide literature has been devoted to the investigation of asymptotic and oscillatory properties of solutions to (E), often accomplished by dividing the set of all nonoscillatory solutions into particular classes. To introduce such a classification, it is convenient to describe the operator

$$Ly=\left(a_2\left(a_1y^{\Delta}\right)^{\Delta}\right)^{\Delta}$$

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* Corresponding author.

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on the basis of convergence or divergence of the improper integrals

$$A_i(t_0) := \int_{t_0}^{\infty} \frac{1}{a_i(s)} \Delta s, \quad i = 1, 2.$$

We say that the operator L (and consequently, the equation (E)) is in canonical form if

$$A_1(t_0) = A_2(t_0) = \infty \tag{C}$$

and it is in noncanonical form if

$$A_1(t_0) < \infty, \quad A_2(t_0) < \infty. \tag{N}$$

If either

$$A_1(t_0) < \infty, \quad A_2(t_0) = \infty \tag{S_1}$$

or

$$A_1(t_0) = \infty, \quad A_2(t_0) < \infty, \tag{S_2}$$

then L is in semicanonical form. Generally, the set $\mathscr{N}$ of all nontrivial nonoscillatory solutions of (E) can be divided into the following four classes for $t \geqslant T_y$ for some $T_y \geqslant t_0$:

$$\begin{aligned}
\mathscr{N}_0 &= \left\{ y \in \mathscr{N} : yy^\Delta < 0, y\left(a_1 y^\Delta\right)^\Delta > 0 \right\}, \\
\mathscr{N}_2 &= \left\{ y \in \mathscr{N} : yy^\Delta > 0, y\left(a_1 y^\Delta\right)^\Delta > 0 \right\}, \\
\mathscr{N}_* &= \left\{ y \in \mathscr{N} : yy^\Delta > 0, y\left(a_1 y^\Delta\right)^\Delta < 0 \right\}, \\
\mathscr{N}_{**} &= \left\{ y \in \mathscr{N} : yy^\Delta < 0, y\left(a_1 y^\Delta\right)^\Delta < 0 \right\}.
\end{aligned}$$

This means:

$$\begin{aligned}
(C) &\implies \mathscr{N} = \mathscr{N}_0 \cup \mathscr{N}_2; \\
(N) &\implies \mathscr{N} = \mathscr{N}_0 \cup \mathscr{N}_2 \cup \mathscr{N}_* \cup N_{**}; \\
(S_1) &\implies \mathscr{N} = \mathscr{N}_0 \cup \mathscr{N}_2 \cup \mathscr{N}_{**}; \\
(S_2) &\implies \mathscr{N} = \mathscr{N}_0 \cup \mathscr{N}_2 \cup \mathscr{N}_*.
\end{aligned} \tag{1.1}$$

In view of the above, the obvious advantage of examining equations in canonical form is that it results in the smallest possible number of classes of nonoscillatory solutions that result from applying the famous Kiguradze lemma [16, Lemma 1]. Recall that in the case where $\mathbb{T} = \mathbb{R}$, the operator L can be written in an essentially unique canonical form due to the classical result of Trench [19, Theorem 1]. For a discrete analogue for the case $\mathbb{T} = \mathbb{Z}$, we refer the reader to [14, Theorem 1]. Explicit closed-form canonical representations of third-order semicanonical and noncanonical differential operators resulting from the application of Trench's Lemma 1 and Lemma 2 in [19] can be found in [8]. Employing useful identities, less complicated canonical representations of noncanonical differential operators were presented in [3].

In the recent paper [12] by the first author, it was shown that the semicanonical dynamic equation (E) with either (S_1) or (S_2) holding can be rewritten, under certain integral conditions, in an equivalent canonical form. This work reflected a growing interest in establishing relations between canonical, semicanonical, and noncanonical equations separately for the case $\mathbb{T}=\mathbb{R}$ corresponding to differential equations [3, 8, 10, 17] and for the case $\mathbb{T}=\mathbb{Z}$ corresponding to difference equations [2, 18].

The purpose of the present work is to complement the results in [12] by providing an equivalent canonical representation of L in noncanonical form (N) and both semicanonical forms (S_1) and (S_2), without requiring any additional conditions. The newly obtained classification scheme enables us to apply known oscillation criteria for the canonical case (C) to give oscillation criteria for equations in the noncanonical and semicanonical cases, and thus contributes to the general study of oscillation theory of canonical dynamic equations on time scales, and by default for ordinary differential and difference equations.

2. Notation and auxiliary results

We will use the usual time scale notation that for a function f, by f^σ we mean $f(\sigma(t))$ for $t\in\mathbb{T}$, where σ is the usual forward jump operator. Let

$$\begin{aligned} A_i(t) &= \int_t^\infty \frac{1}{a_i(s)}\Delta s, \quad i=1,2,\\ A_{12}(t) &= \int_t^\infty \frac{1}{a_1(s)}A_2(s)\Delta s,\\ A_{21}(t) &= \int_t^\infty \frac{1}{a_2(s)}A_1^\sigma(s)\Delta s, \end{aligned}$$

provided that the improper integrals are well defined.

In the sequel, we will use the following extended notation:

$$M[b_0,b_1,b_2]y = b_2\left(b_1\left(b_0y\right)^\Delta\right)^\Delta$$

and

$$M[b_0,b_1,b_2,b_3]y = b_3\left(b_2\left(b_1\left(b_0y\right)^\Delta\right)^\Delta\right)^\Delta, \tag{2.1}$$

i.e.,

$$M[b_0,b_1,b_2,b_3]y = \begin{cases} b_3\left(M[b_0,b_1,b_2]y\right)^\Delta,\\ M[b_1,b_2,b_3]\left(b_0y\right)^\Delta,\end{cases}$$

where $b_i\in C_{rd}([t_0,\infty)_\mathbb{T},\mathbb{R})$, $i=0,1,2,3$. Therefore, we can rewrite L as

$$L = M[1,a_1,a_2,1].$$

The following auxiliary result will be repeatedly used in our proofs.

LEMMA 1. *The second-order noncanonical dynamic operator* $M[b_0, b_1, b_2]$ *with*

$$B_1(t_0) := \int_{t_0}^{\infty} \frac{1}{b_1(s)} \Delta s < \infty$$

can be rewritten in canonical form

$$M[b_0, b_1, b_2] = M\left[\frac{b_0}{B_1}, b_1 B_1 B_1^{\sigma}, \frac{b_2}{B_1^{\sigma}}\right], \tag{2.2}$$

such that

$$\int_{t_0}^{\infty} \frac{1}{b_1(s) B_1(s) B_1^{\sigma}(s)} \Delta s = \infty.$$

Proof. The equivalence in (2.2) can be shown by straightforward differentiation as in the proof of [12, Theorem 2.1]. A simple computation yields

$$\int_{t_0}^{\infty} \frac{1}{b_1(s) B_1(s) B_1^{\sigma}(s)} \Delta s = \int_{t_0}^{\infty} \left(\frac{1}{B_1(s)}\right)^{\Delta} \Delta s = \lim_{t \to \infty} \frac{1}{B_1(t)} - \frac{1}{B_1(t_0)} = \infty,$$

which completes the proof of the lemma. □

REMARK 1. In the case $\mathbb{T} = \mathbb{R}$, Lemma 1 coincides with Trench's original result [19, Lemma 1].

Next, we present a couple of useful identities holding in the noncanonical case (N).

LEMMA 2. *Let* (N) *hold. Then*

$$A_1 A_2 = A_{12} + A_{21} \tag{2.3}$$

and

$$\int_{t}^{\infty} \frac{A_{21}(s)}{a_1(s) A_1(s) A_1^{\sigma}(s)} \Delta s = \frac{A_{12}(t)}{A_1(t)}. \tag{2.4}$$

Proof. Integrating the identity

$$(A_1 A_2)^{\Delta} = -\frac{A_2}{a_1} - \frac{A_1^{\sigma}}{a_2} \tag{2.5}$$

from t to ∞ gives

$$A_1(t) A_2(t) = \int_{t}^{\infty} \frac{1}{a_1(s)} A_2(s) \Delta s + \int_{t}^{\infty} \frac{1}{a_2(s)} A_1^{\sigma}(s) \Delta s = A_{12}(t) + A_{21}(t),$$

which proves (2.3).

To prove (2.4), we first show that the integral on the left-hand side is well-defined. First, notice that by l'Hospital's rule, we have

$$\lim_{t\to\infty}\frac{A_{12}(t)}{A_1(t)}=\lim_{t\to\infty}A_2(t)=0. \tag{2.6}$$

From (2.3) and (2.6), we see that

$$\lim_{t\to\infty}\frac{A_{21}(t)}{A_1(t)}=\lim_{t\to\infty}\frac{A_1(t)A_2(t)-A_{12}(t)}{A_1(t)} \tag{2.7}$$

$$=\lim_{t\to\infty}A_2(t)-\frac{A_{12}(t)}{A_1(t)}=\lim_{t\to\infty}A_2(t)-A_2(t)=0. \tag{2.8}$$

Using the integration by parts formula on time scales ([7, Theorem 1.77]), (N), and (2.7), we obtain

$$\int_{t_0}^{\infty}\frac{A_{21}(s)}{a_1(s)A_1(s)A_1^{\sigma}(s)}\Delta s=\int_{t_0}^{\infty}A_{21}(s)\left(\frac{1}{A_1(s)}\right)^{\Delta}\Delta s$$

$$=\left.\frac{A_{21}(t)}{A_1(t)}\right|_{t_0}^{\infty}+\int_{t_0}^{\infty}\frac{1}{a_2(s)}\Delta s<\infty. \tag{2.9}$$

Now, from (2.3), we have

$$\int_{t}^{\infty}\frac{A_{21}(s)}{a_1(s)A_1(s)A_1^{\sigma}(s)}\Delta s=-\frac{A_{21}(t)}{A_1(t)}+A_2(t)$$

$$=\frac{A_{12}(t)-A_1(t)A_2(t)}{A_1(t)}+A_2(t)=\frac{A_{12}(t)}{A_1(t)},$$

so (2.4) holds, and this completes the proof of the lemma. □

3. Canonical representation of L

In this section, we provide closed-form canononical representations for the operator L in the cases where (S_1), (S_2), and (N) hold, respectively. Our first result applies to the operator L in the semicanonical form (S_1), and it extends [12, Theorem 2.1] to the case $A_{21}(t_0)<\infty$.

THEOREM 1. *Let (S_1) hold. The operator Ly can be written in the canonical form*

$$Ly=\begin{cases}\left(\dfrac{a_2}{A_1^{\sigma}}\left(a_1A_1A_1^{\sigma}\left(\dfrac{y}{A_1}\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}, & \textit{if } A_{21}(t_0)=\infty,\\[2ex] \dfrac{1}{A_{21}^{\sigma}}\left(\dfrac{a_2}{A_1^{\sigma}}A_{21}A_{21}^{\sigma}\left(\dfrac{a_1A_1A_1^{\sigma}}{A_{21}}\left(\dfrac{y}{A_1}\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}, & \textit{if } A_{21}(t_0)<\infty.\end{cases} \tag{3.1}$$

Proof. Since $A_1(t_0) < \infty$, applying Lemma 1, we can rewrite the operator L as

$$\begin{aligned} Ly &= M[1,a_1,a_2,1]y \\ &= (M[1,a_1,a_2]y)^{\Delta} \\ &= \left(M\left[\frac{1}{A_1},a_1A_1A_1^{\sigma},\frac{a_2}{A_1^{\sigma}}\right]y\right)^{\Delta} \\ &= M\left[\frac{1}{A_1},a_1A_1A_1^{\sigma},\frac{a_2}{A_1^{\sigma}},1\right]y \end{aligned} \tag{3.2}$$

with

$$\int_{t_0}^{\infty} \frac{1}{a_1(s)A_1(s)A_1^{\sigma}(s)}\Delta s = \int_{t_0}^{\infty} \left(\frac{1}{A_1(s)}\right)^{\Delta} \Delta s = \infty.$$

Hence, if $A_{21}(t_0) = \infty$, then the operator (3.1) is in canonical form. If $A_{21}(t_0) < \infty$, we again apply Lemma 1 to obtain

$$\begin{aligned} M\left[\frac{1}{A_1},a_1A_1A_1^{\sigma},\frac{a_2}{A_1^{\sigma}},1\right]y &= M\left[a_1A_1A_1^{\sigma},\frac{a_2}{A_1^{\sigma}},1\right]\left(\frac{y}{A_1}\right)^{\Delta} \\ &= M\left[\frac{a_1A_1A_1^{\sigma}}{A_{21}},\frac{a_2A_{21}A_{21}^{\sigma}}{A_1^{\sigma}},\frac{1}{A_{21}^{\sigma}}\right]\left(\frac{y}{A_1}\right)^{\Delta} \\ &= M\left[\frac{1}{A_1},\frac{a_1A_1A_1^{\sigma}}{A_{21}},\frac{a_2A_{21}A_{21}^{\sigma}}{A_1^{\sigma}},\frac{1}{A_{21}^{\sigma}}\right]y \end{aligned} \tag{3.3}$$

with

$$\int_{t_0}^{\infty} \frac{1}{A_{21}(s)A_{21}^{\sigma}(s)}\frac{A_1^{\sigma}(s)}{a_2(s)}\Delta s = \int_{t_0}^{\infty} \left(\frac{1}{A_{21}(s)}\right)^{\Delta} \Delta s = \infty.$$

Since

$$\begin{aligned} \int_{t_0}^{\infty} \frac{A_{21}(s)}{a_1(s)A_1(s)A_1^{\sigma}(s)}\Delta s &= \int_{t_0}^{\infty} A_{21}(s)\left(\frac{1}{A_1(s)}\right)^{\Delta}\Delta s \\ &= \frac{A_{21}(t)}{A_1(t)}\bigg|_{t_0}^{\infty} + \int_{t_0}^{\infty} \frac{1}{a_2(s)}A_1^{\sigma}(s)\frac{1}{A_1^{\sigma}(s)}\Delta s \\ &\geqslant \int_{t_0}^{\infty} \frac{1}{a_2(s)}\Delta s = \infty, \end{aligned}$$

the operator (3.1) is in canonical form. The proof is now complete. $\square$

Our second result applies to the operator L in the semicanonical form (S_2); it extends [12, Theorem 3.1] to the case $A_{12}(t_0) < \infty$.

THEOREM 2. *Let* (S_2) *hold. The operator* Ly *can be written in the canonical form*

$$Ly = \begin{cases} \dfrac{1}{A_2^\sigma}\left(a_2 A_2 A_2^\sigma \left(\dfrac{a_1}{A_2} y^\Delta\right)^\Delta\right)^\Delta, & \text{if } A_{12}(t_0) = \infty, \\ \dfrac{1}{A_2^\sigma}\left(\dfrac{a_2 A_2 A_2^\sigma}{A_{12}^\sigma}\left(\dfrac{a_1}{A_2} A_{12} A_{12}^\sigma \left(\dfrac{y}{A_{12}}\right)^\Delta\right)^\Delta\right)^\Delta, & \text{if } A_{12}(t_0) < \infty. \end{cases} \tag{3.4}$$

Proof. Since $A_2(t_0) < \infty$, we can rewrite Ly using Lemma 1 as

$$\begin{aligned} Ly &= M[1, a_1, a_2, 1]y \\ &= M[a_1, a_2, 1]y^\Delta \\ &= M\left[\frac{a_1}{A_2}, a_2 A_2 A_2^\sigma, \frac{1}{A_2^\sigma}\right] y^\Delta \\ &= M\left[1, \frac{a_1}{A_2}, a_2 A_2 A_2^\sigma, \frac{1}{A_2^\sigma}\right] y \end{aligned}$$

with

$$\int_{t_0}^{\infty} \frac{1}{a_2(s) A_2(s) A_2^\sigma(s)} \Delta s = \infty.$$

Hence, if $A_{12}(t_0) = \infty$, then the operator (3.4) is in canonical form.

If $A_{12}(t_0) < \infty$, we again apply Lemma 1 to obtain

$$\begin{aligned} M\left[1, \frac{a_1}{A_2}, a_2 A_2 A_2^\sigma, \frac{1}{A_2^\sigma}\right] y &= \frac{1}{A_2^\sigma}\left(M\left[1, \frac{a_1}{A_2}, a_2 A_2 A_2^\sigma\right] y\right)^\Delta \\ &= \frac{1}{A_2^\sigma}\left(M\left[\frac{1}{A_{12}}, \frac{a_1}{A_2} A_{12} A_{12}^\sigma, \frac{a_2 A_2 A_2^\sigma}{A_{12}}\right] y\right)^\Delta \\ &= M\left[\frac{1}{A_{12}}, \frac{a_1}{A_2} A_{12} A_{12}^\sigma, \frac{a_2 A_2 A_2^\sigma}{A_{12}}, \frac{1}{A_2^\sigma}\right] y \end{aligned}$$

with

$$\int_{t_0}^{\infty} \frac{A_2(s)}{a_1(s) A_{12}(s) A_{12}^\sigma(s)} \Delta s = \infty.$$

Finally,

$$\begin{aligned} \int_{t_0}^{\infty} \frac{A_{12}^\sigma(s)}{a_2(s) A_2(s) A_2^\sigma(s)} \Delta s &= \int_{t_0}^{\infty} A_{12}^\sigma(s) \left(\frac{1}{A_2(s)}\right)^\Delta \Delta s \\ &= \left.\frac{A_{12}(t)}{A_2(t)}\right|_{t_0}^{\infty} + \int_{t_0}^{\infty} \frac{1}{a_1(s)} \Delta s = \infty \end{aligned}$$

by (S_2), so (3.4) is in canonical form, and this completes the proof. □

Our third result applies to the operator L if it is in noncanonical form (N). The particular case $\mathbb{T} = \mathbb{R}$ can be found in [3, Theorem 2.1].

THEOREM 3. *Let* (N) *hold. The operator* Ly *can be written in the canonical form*

$$Ly = \frac{1}{A_{21}^{\sigma}}\left(\frac{a_2 A_{21} A_{21}^{\sigma}}{A_{12}^{\sigma}}\left(\frac{a_1 A_{12} A_{12}^{\sigma}}{A_{21}}\left(\frac{y}{A_{12}}\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}. \tag{3.5}$$

Proof. Since $A_1(t_0) < \infty$, by (3.2),

$$Ly = M\left[\frac{1}{A_1}, a_1 A_1 A_1^{\sigma}, \frac{a_2}{A_1^{\sigma}}, 1\right] y.$$

In view of (N), clearly $A_{21}(t_0) < \infty$, and so we arrive at (3.3), i.e.,

$$Ly = M\left[\frac{1}{A_1}, \frac{a_1 A_1 A_1^{\sigma}}{A_{21}}, \frac{a_2 A_{21} A_{21}^{\sigma}}{A_1^{\sigma}}, \frac{1}{A_{21}^{\sigma}}\right] y$$

with

$$\int_{t_0}^{\infty} \frac{1}{A_{21}(s) A_{21}^{\sigma}(s)} \frac{A_1^{\sigma}(s)}{a_2(s)} \Delta s = \infty.$$

Since by (2.4),

$$\frac{A_{12}(t_0)}{A_1(t_0)} = \int_{t_0}^{\infty} \frac{A_{21}(s)}{a_1(s) A_1(s) A_1^{\sigma}(s)} \Delta s < \infty$$

(see (2.9)), we then apply Lemma 1 to obtain

$$\begin{aligned}
M&\left[\frac{1}{A_1}, \frac{a_1 A_1 A_1^{\sigma}}{A_{21}}, \frac{a_2 A_{21} A_{21}^{\sigma}}{A_1^{\sigma}}, \frac{1}{A_{21}^{\sigma}}\right] y \\
&= \frac{1}{A_{21}^{\sigma}}\left(M\left[\frac{1}{A_1}, \frac{a_1 A_1 A_1^{\sigma}}{A_{21}}, \frac{a_2 A_{21} A_{21}^{\sigma}}{A_1^{\sigma}}\right] y\right)^{\Delta} \\
&= \frac{1}{A_{21}^{\sigma}}\left(M\left[\frac{1}{A_{12}}, \frac{a_1 A_{12} A_{12}^{\sigma}}{A_{21}}, \frac{a_2 A_{21} A_{21}^{\sigma}}{A_{12}^{\sigma}}\right] y\right)^{\Delta} \\
&= M\left[\frac{1}{A_{12}}, \frac{a_1 A_{12} A_{12}^{\sigma}}{A_{21}}, \frac{a_2 A_{21} A_{21}^{\sigma}}{A_{12}^{\sigma}}, \frac{1}{A_{21}^{\sigma}}\right] y
\end{aligned}$$

with

$$\int_{t_0}^{\infty} \frac{A_{21}(s)}{a_1(s) A_{12}(s) A_{12}^{\sigma}(s)} \Delta s = \infty.$$

Using (2.3) and integration by parts ([7, Theorem 1.77]), we see that

$$\begin{aligned}
\int_{t_0}^{\infty} \frac{A_{12}^{\sigma}(s)}{a_2(s) A_{21}(s) A_{21}^{\sigma}(s)} \Delta s &= \int_{t_0}^{\infty} \frac{A_1^{\sigma}(s) A_2^{\sigma}(s) - A_{21}^{\sigma}(s)}{a_2(s) A_{21}(s) A_{21}^{\sigma}(s)} \Delta s \\
&= \int_{t_0}^{\infty} \left(\frac{1}{A_{21}(s)}\right)^{\Delta} A_2^{\sigma}(s) - \frac{1}{a_2(s) A_{21}(s)} \Delta s \\
&= \frac{A_2(t)}{A_{21}(t)}\bigg|_{t_0}^{\infty} = \infty,
\end{aligned}$$

where by l'Hospital's rule,

$$\lim_{t\to\infty}\frac{A_2(t)}{A_{21}(t)}=\lim_{t\to\infty}\frac{1}{A_1^\sigma(t)}=\infty.$$

since $A_1(t)\to 0$ as $t\to\infty$. This proves the theorem. $\square$

In a recent work [11], the authors pointed out that in the continuous case $\mathbb{T}=\mathbb{R}$, the noncanonical operator can be rewritten in a simplified canonical form without needing to compute the integrals A_{12} and A_{21} if the functions a_i are of the same type, e.g., they are both of the form t^α or $\exp(\alpha t)$. In the sequel, we provide an analogue of this observation on a time scale $\mathbb{T}$.

COROLLARY 1. *Let* (N) *hold and*

$$\frac{a_2(t)A_2(t)}{a_1(t)A_1^\sigma(t)}=\ell \quad \textit{for some } \ell\in\mathbb{R}. \tag{3.6}$$

Then the operator Ly can be written in the canonical form

$$Ly=\frac{1}{A_1^\sigma A_2^\sigma}\left(a_2A_1A_2\left(a_1A_1^\sigma A_2^\sigma\left(\frac{y}{A_1A_2}\right)^\Delta\right)^\Delta\right)^\Delta.$$

Proof. It suffices to note that using (3.6) in (2.5) and integrating gives

$$A_{12}=\frac{\ell}{1+\ell}A_1A_2 \text{ and } A_{21}=\frac{1}{1+\ell}A_1A_2.$$

In view of Theorem 3, the conclusion is immediate. $\square$

4. Applications

The simplest way to illustrate our results is to begin with the continuous case $\mathbb{T}=\mathbb{R}$, where $\sigma(t)=t$, $y^\Delta(t)=y'(t)$ and $\int_a^b f(t)\Delta t=\int_a^b f(t)dt$.

EXAMPLE 1. Consider the third-order differential operator

$$Ly(t)=\left(e^{\beta t}\left(e^{\alpha t}y'(t)\right)'\right)', \quad t\in\mathbb{R}, \quad \alpha,\beta\in\mathbb{R}. \tag{4.1}$$

Notice that if $\alpha\leqslant 0$ and $\beta\leqslant 0$, then L is already in canonical form. Hence, it makes sense to consider the following three cases.

If $\alpha>0$ and $\beta\leqslant 0$, then L is in the semicanonical form (S_1), and by Theorem 1, it can be written in the canonical form

$$Ly(t)=\begin{cases}\left(e^{(\alpha+\beta)t}\left(e^{-\alpha t}\left(e^{\alpha t}y(t)\right)'\right)'\right)', & \text{if } \alpha+\beta\leqslant 0,\\ e^{(\alpha+\beta)t}\left(e^{-(\alpha+\beta)t}\left(e^{\beta t}\left(e^{\alpha t}y(t)\right)'\right)'\right)', & \text{if } \alpha+\beta>0.\end{cases}$$

If $\alpha \leqslant 0$ and $\beta > 0$, then L is in the semicanonical form (S_2), and by Theorem 2, it can be written in the canonical form

$$Ly(t) = \begin{cases} e^{\beta t}\left(e^{-\beta t}\left(e^{(\alpha+\beta)t}y'(t)\right)'\right)', & \text{if } \alpha+\beta \leqslant 0, \\ e^{\beta t}\left(e^{\alpha t}\left(e^{-(\alpha+\beta)t}\left(e^{(\alpha+\beta)t}y(t)\right)'\right)'\right)', & \text{if } \alpha+\beta > 0. \end{cases}$$

If $\alpha > 0$ and $\beta > 0$, then L is in noncanonical form (N), and by Theorem 3, it can be written in the canonical form

$$Ly(t) = e^{(\alpha+\beta)t}\left(e^{-\alpha t}\left(e^{-\beta t}\left(e^{(\alpha+\beta)t}y(t)\right)'\right)'\right)'.$$

It should be noted that the semicanonical cases with $\alpha+\beta > 0$ as well as the noncanonical case are not covered by the previous results in [12].

The following example, which can be seen as a discrete analogue of Example 1, shows how Theorems 1–3 apply in case $\mathbb{T} = \mathbb{N}$. In this situation, $\sigma(t) = t+1$, $y^{\Delta}(t) = \Delta y(t) = y(t+1) - y(t)$, and $\int_a^b f(t)\Delta t = \sum_{i=a}^{b-1} f(i)$.

EXAMPLE 2. Consider the third-order difference operator

$$Ly(t) = \Delta\left(\beta^t \Delta\left(\alpha^t \Delta y(t)\right)\right), \quad t \in \mathbb{N}, \quad \alpha,\beta > 0. \tag{4.2}$$

First note that L is in canonical form if $\alpha \leqslant 1$ and $\beta \leqslant 1$.

If $\alpha > 1$ and $\beta \leqslant 1$, then L is in the semicanonical form (S_1), and by Theorem 1, it can be written in the canonical form

$$Ly(t) = \begin{cases} \Delta((\alpha\beta)^t \Delta(\alpha^{-t}\Delta(\alpha^t y(t)))), & \text{if } \alpha\beta \leqslant 1, \\ (\alpha\beta)^t \Delta((\alpha\beta)^{-t}\Delta(\beta^t \Delta(\alpha^t y(t)))), & \text{if } \alpha\beta > 1. \end{cases}$$

If $\alpha \leqslant 1$ and $\beta > 1$, then L is in the semicanonical form (S_2), and by Theorem 2, it can be written in the canonical form

$$Ly(t) = \begin{cases} \beta^t \Delta(\beta^{-t}\Delta((\alpha\beta)^t \Delta y(t))), & \text{if } \alpha\beta \leqslant 1, \\ \beta^t \Delta(\alpha^t \Delta((\alpha\beta)^{-t}\Delta((\alpha\beta)^t y(t)))), & \text{if } \alpha\beta > 1. \end{cases}$$

Finally, if $\alpha > 1$ and $\beta > 1$, then L is in noncanonical form (N), and by Theorem 3, it can be written in the canonical form

$$Ly(t) = (\alpha\beta)^t \Delta\left(\alpha^{-t}\Delta\left(\beta^{-t}\Delta\left((\alpha\beta)^t y(t)\right)\right)\right).$$

Again, we stress that the previous results in [12] do not apply in the semicanonical cases with $\alpha\beta > 1$ nor in the noncanonical case.

Now, let us focus on the noncanonical case (N). It is easy to see that both of the operators (4.1) and (4.2) satisfy condition (3.6) with $\ell = \alpha/\beta$ and $\ell = \beta(\alpha-1)/(\beta-1)$, respectively, and this allows us to apply Corollary 1 as in the following example.

EXAMPLE 3. Consider the third-order dynamic operator

$$Ly = \left(t\sigma(t)\left(t\sigma(t)y^{\Delta}\right)^{\Delta}\right)^{\Delta}, \quad t \in \mathbb{T}.$$

Clearly, L is in noncanonical form since

$$A_1(t) = A_2(t) = \int_t^{\infty} \frac{1}{s\sigma(s)}\Delta s = \frac{1}{t}.$$

Although the integrals

$$A_{12}(t) = \int_t^{\infty} \frac{1}{s^2\sigma(s)}\Delta s \quad \text{and} \quad A_{21}(t) = \int_t^{\infty} \frac{1}{s\sigma^2(s)}\Delta s$$

cannot be analytically computed on an arbitrary time scale $\mathbb{T}$, if (3.6) holds, i.e.,

$$\frac{\sigma(t)}{t} = \ell \quad \text{for some } \ell \in \mathbb{R}, \tag{4.3}$$

then L can be explicitly written in the canonical form

$$Ly(t) = \frac{1}{\sigma^2(t)}\left(t^2y(t)\right)^{\Delta\Delta\Delta}.$$

Notice that condition (4.3) is satisfied for $\mathbb{T} = \mathbb{R}$ $(\ell = 1)$ as well as for the quantum time scale $\mathbb{T} = q^{\mathbb{N}} = \{t : t = q^k, k \in \mathbb{N}, q > 1\}$ $(\ell = q)$, but it is not satisfied for many other time scales such as $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = h\mathbb{Z}_+$, $h > 0$, or $\mathbb{T} = \mathbb{N}_0^2$.

5. Main results

Based on Theorems 1–3, we can now formulate our main results that we obtain by rewriting (E) when it is in the semicanonical forms (S_1) or (S_2), or in the noncanonical form (N) into an equation in canonical form

$$\left(b_2(t)\left(b_1(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} + \widetilde{q}(t)x(\tau(t)) = 0. \tag{E_C}$$

THEOREM 4. *Let (S_1) hold. Equation (E) has a solution $y(t)$ if and only if the canonical equation (E_C) with*

$$b_1(t) = \begin{cases} a_1(t)A_1(t)A_1^{\sigma}(t), & \text{if } A_{21}(t_0) = \infty, \\ \dfrac{a_1(t)A_1(t)A_1^{\sigma}(t)}{A_{21}(t)}, & \text{if } A_{21}(t_0) < \infty, \end{cases}$$

$$b_2(t) = \begin{cases} \dfrac{a_2(t)}{A_1^{\sigma}(t)}, & \text{if } A_{21}(t_0) = \infty, \\ \dfrac{a_2(t)A_{21}(t)A_{21}^{\sigma}(t)}{A_1^{\sigma}(t)}, & \text{if } A_{21}(t_0) < \infty, \end{cases}$$

$$\widetilde{q}(t) = \begin{cases} q(t)A_1(\tau(t)), & \text{if } A_{21}(t_0) = \infty, \\ q(t)A_1(\tau(t))A_{21}^{\sigma}(t), & \text{if } A_{21}(t_0) < \infty, \end{cases}$$

has a solution

$$x(t) = \frac{y(t)}{A_1(t)}.$$

THEOREM 5. *Let* (S_2) *hold. Equation* (E) *has a solution* $y(t)$ *if and only if the canonical equation* (E_C) *with*

$$b_1(t) = \begin{cases} \dfrac{a_1(t)}{A_2(t)}, & \text{if } A_{12}(t_0) = \infty, \\ \dfrac{a_1(t)A_{12}(t)A_{12}^{\sigma}(t)}{A_2(t)}, & \text{if } A_{12}(t_0) < \infty, \end{cases}$$

$$b_2(t) = \begin{cases} a_2(t)A_2(t)A_2^{\sigma}(t), & \text{if } A_{12}(t_0) = \infty, \\ \dfrac{a_2(t)A_2(t)A_2^{\sigma}(t)}{A_{12}^{\sigma}(t)}, & \text{if } A_{12}(t_0) < \infty, \end{cases}$$

$$\widetilde{q}(t) = \begin{cases} q(t)A_2^{\sigma}(t), & \text{if } A_{12}(t_0) = \infty, \\ q(t)A_{12}(\tau(t))A_2^{\sigma}(t), & \text{if } A_{12}(t_0) < \infty, \end{cases}$$

has a solution

$$x(t) = \begin{cases} y(t), & \text{if } A_{12}(t_0) = \infty, \\ \dfrac{y(t)}{A_{12}(t)}, & \text{if } A_{12}(t_0) < \infty. \end{cases}$$

THEOREM 6. *Let* (N) *hold. Equation* (E) *has a solution* $y(t)$ *if and only if the canonical equation* (E_C) *with*

$$b_1(t) = \frac{a_1(t)A_{12}(t)A_{12}^{\sigma}(t)}{A_{21}(t)},$$

$$b_2(t) = \frac{a_2(t)A_{21}(t)A_{21}^{\sigma}(t)}{A_{12}^{\sigma}(t)},$$

$$\widetilde{q}(t) = q(t)A_{12}(\tau(t))A_{21}^{\sigma}(t),$$

has a solution

$$x(t) = \frac{y(t)}{A_{12}(t)}.$$

In the noncanonical case (N), we can obtain a simplified version of Theorem 6 that results from applying Corollary 1.

COROLLARY 2. *Let* (N) *and* (3.6) *hold. Equation* (E) *has a solution* $y(t)$ *if and only if the canonical equation* (E_C) *with*

$$b_1(t) = a_1(t)A_1^{\sigma}(t)A_2^{\sigma}(t),$$
$$b_2(t) = a_2(t)A_1(t)A_2(t),$$
$$\widetilde{q}(t) = q(t)A_1(\tau(t))A_2(\tau(t))A_1^{\sigma}(t)A_2^{\sigma}(t),$$

has a solution

$$x(t) = \frac{y(t)}{A_1(t)A_2(t)}.$$

REMARK 2. The obvious advantage of Corollary 2 over Theorem 6 is that we do not need to compute the integrals A_{12} and A_{21}.

It is important to point out that, as an immediate consequence of Theorems 4–6, any set of conditions that implies the oscillation of all solutions of the canonical equation (E_C) guarantees the same property for (E) in semicanonical or noncanonical form. This is true on any unbounded time scale $\mathbb{T}$, including the classical ones $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$. In our opinion, this is a highly significant result.

To demonstrate the applicability of our results in this direction, let us recall an oscillation criterion for the third-order delay differential equation

$$x'''(t) + \tilde{q}(t)x(\tau(t)) = 0, \quad t \geqslant 1, \tag{$\tilde{E}_C$}$$

which is a particular case of (E_C) with $\mathbb{T} = \mathbb{R}$ and $b_1 = b_2 = 1$.

THEOREM 7. *Let* $\lambda_* := \liminf_{t\to\infty} \frac{t}{\tau(t)}$ *and*

$$M_2 = \max\left\{-p(p-1)(p-2)\lambda_*^{p-2} : 1 < p < 2\right\}.$$

If

$$\liminf_{t\to\infty} \tau^2(t)t\tilde{q}(t) > \begin{cases} 0, & \text{for } \lambda_* = \infty, \\ M_2, & \text{for } \lambda_* < \infty, \end{cases} \tag{5.1}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \int_{u}^{t} \int_{x}^{t} \tilde{q}(s)\mathrm{d}s\mathrm{d}x\mathrm{d}u > 1, \tag{5.2}$$

then ($\tilde{E}_C$) *is oscillatory.*

Note that condition (5.1) (see [13, Theorem 2.1])) eliminates the solutions belonging to the class $\mathscr{N}_2$, and condition (5.2) ensures that the class $\mathscr{N}_0$ is empty (see [9, Theorem 9]). An important observation here is that (5.1) remains sharp for the delay Euler differential equation

$$x'''(t) + \frac{q_0}{t^3}x(\lambda t) = 0, \quad q_0 > 0, \quad \lambda \in (0,1], \quad t \geqslant 1, \tag{5.3}$$

as shown in [13, Corollary 2.1]. To the best of our knowledge, a similarly sharp criterion for the nonexistence of $\mathscr{N}_0$-type solutions has not yet been obtained (see [6] for more details).

Now, let us consider the case of the simple noncanonical differential equation

$$\left(t^2(t^2y'(t))'\right)' + q(t)y(\tau(t)) = 0, \quad t \in \mathbb{R}, \quad t \geqslant 1. \tag{$\tilde{E}$}$$

By Theorem 6, equation ($\tilde{E}$) can be written as the canonical equation ($\tilde{E}_C$) with $\tilde{q}(t) = \tau^{-2}(t)t^{-2}q(t)$, so that $y(t)$ is a solution of ($\tilde{E}$) if and only if $x(t) = t^2y(t)$ is a solution of ($\tilde{E}_C$). In view of Theorem 7, the following result is immediate.

THEOREM 8. *Let λ_* and M_2 be as in Theorem 7. If*

$$\liminf_{t\to\infty} \frac{q(t)}{t} > \begin{cases} 0, & \textit{for } \lambda_* = \infty, \\ M_2, & \textit{for } \lambda_* < \infty, \end{cases} \tag{5.4}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \int_{u}^{t} \int_{x}^{t} \frac{q(s)}{\tau^2(s)s^2} \mathrm{d}s\mathrm{d}x\mathrm{d}u > 1, \tag{5.5}$$

then ($\tilde{E}$) *is oscillatory.*

Notice that the noncanonical delay Euler differential equation

$$(t^2(t^2y'(t))')' + q_0 t y(\lambda t) = 0, \quad q_0 > 0, \quad \lambda \in (0,1], \quad t \geqslant 1,$$

has a nonoscillatory solution $y \in \mathcal{N}_{**}$ if $q_0 = M_2$; this shows that condition (5.4) cannot be improved.

Next, we formulate a variant of the oscillation criterion for (E_C), which can be viewed as an extension of Theorem 7. We will use the notation:

$$B_i(t) = \int_{t_0}^{t} \frac{1}{b_i(s)} ds, \quad i = 1,2, \quad \text{and} \quad B_{12}(t) = \int_{t_0}^{t} \frac{B_2(s)}{b_1(s)} ds.$$

THEOREM 9. *Let*

$$\begin{aligned} \beta_* &:= \liminf_{t\to\infty} B_2(t)B_{12}(\tau(t))\tilde{q}(t)b_2(t), \\ \lambda_* &:= \liminf_{t\to\infty} \frac{t}{\tau(t)}, \\ k_* &= \liminf_{t\to\infty} \frac{B_2^{\beta_*}(t)\int_{t_0}^{t} \frac{B_2^{1-\beta_*}(s)}{b_1(s)} ds}{B_{12}(t)}, \end{aligned}$$

and

$$M = \max\left\{ p(1-p)\lambda_*^{1/k_f - 1} : 0 < p < 1 \right\},$$

where

$$k_f = \liminf_{t\to\infty} \frac{B_2^{p}(t)\int_{t_0}^{t} \frac{B_2^{1-p}(s)}{b_1(s)} ds}{B_{12}(t)}.$$

If

$$\beta_* > \begin{cases} 0, & \textit{for } \lambda_* = \infty \textit{ or } k_* = \infty, \\ M, & \textit{for } \lambda_* < \infty \textit{ and } k_* < \infty, \end{cases} \tag{5.6}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \frac{1}{b_1(u)} \int_{u}^{t} \frac{1}{b_2(x)} \int_{x}^{t} \tilde{\tilde{q}}(s) \mathrm{d}s\mathrm{d}x\mathrm{d}u > 1, \tag{5.7}$$

then equation (E_C) *is oscillatory.*

For condition (5.6), we refer the reader to [15, Corollary 3, Corollary 4], while condition (5.7) is due to [9, Theorem 9]. Notice that again, (5.6) is optimal in that it remains sharp when applied to the delay Euler differential equation

$$\left(t^{\beta}\left(t^{\alpha}y'(t)\right)'\right)' + q_0 t^{\alpha+\beta-3}y(\lambda t) = 0, \quad q_0 > 0, \quad \lambda \in (0,1], \quad t \geqslant 1.$$

For our last example in this paper we illustrate our results in this section by considering the third-order delay differential equation whose coefficients are powers of t, namely,

$$\left(t^{\beta}\left(t^{\alpha}y'(t)\right)'\right)' + q(t)y(\tau(t)) = 0, \quad t \in \mathbb{R}, \ t \geqslant t_0 \geqslant 1. \tag{E_p}$$

THEOREM 10. *Equation (E_p) is oscillatory if and only if (E_C) is oscillatory in each of the following cases.*

1. $\alpha > 1$ *and* $\beta \leqslant 1$:
   - *(i)* $\alpha+\beta \leqslant 2$ *and* $b_1(t) = t^{2-\alpha}$, $b_2(t) = t^{\alpha+\beta-1}$, $\widetilde{q}(t) = \tau^{1-\alpha}(t)q(t)$;
   - *(ii)* $\alpha+\beta > 2$ *and* $b_1(t) = t^{\beta}$, $b_2(t) = t^{3-\alpha-\beta}$, $\widetilde{q}(t) = \tau^{1-\alpha}(t)t^{2-\alpha-\beta}q(t)$.
2. $\alpha \leqslant 1$ *and* $\beta > 1$:
   - *(i)* $\alpha+\beta \leqslant 2$ *and* $b_1(t) = t^{\alpha+\beta-1}$, $b_2(t) = t^{2-\beta}$, $\widetilde{q}(t) = t^{1-\beta}q(t)$;
   - *(ii)* $\alpha+\beta > 2$ *and* $b_1(t) = t^{3-\alpha-\beta}$, $b_2(t) = t^{\alpha}$, $\widetilde{q}(t) = \tau^{2-\alpha-\beta}(t)t^{1-\beta}q(t)$.
3. $\alpha > 1$ *and* $\beta > 1$:

   $$b_1(t) = t^{2-\beta}, \ b_2(t) = t^{2-\alpha}, \ \widetilde{q}(t) = \tau^{2-\alpha-\beta}(t)t^{2-\alpha-\beta}q(t).$$

Proof. We will prove the first case and leave the remaining ones to the reader.

1. If $\alpha > 1$ and $\beta \leqslant 1$, then (E_p) is in semicanonical form (S_1), i.e., we have

   $$A_1(t) = \frac{t^{1-\alpha}}{\alpha-1} \quad \text{and} \quad A_2(t_0) = \infty.$$

   Consequently, two subcases can occur.

   (i) If $\alpha+\beta \leqslant 2$, then $A_{21}(t_0) = \infty$. Clearly,

   $$b_1(t) = \frac{t^{2-\alpha}}{(\alpha-1)^2}, \quad b_2(t) = (\alpha-1)t^{\alpha+\beta-1}, \quad \widetilde{q}(t) = \frac{\tau^{1-\alpha}(t)}{(\alpha-1)}q(t).$$

   By Theorem 4, (E_p) has a solution $y(t)$ if and only if the canonical equation

   $$\left(t^{\alpha+\beta-1}\left(t^{2-\alpha}x'(t)\right)'\right)' + \tau^{1-\alpha}(t)q(t)x(\tau(t)) = 0,$$

   has a solution $x(t) = t^{\alpha-1}y(t)$.

(ii) If $\alpha+\beta>2$, then

$$A_{21}(t)=\frac{t^{2-\alpha-\beta}}{(\alpha-1)(\alpha+\beta-2)}$$

and clearly,

$$b_1(t)=\left(\frac{\alpha+\beta-2}{\alpha-1}\right)t^{\beta},\quad b_2(t)=\frac{t^{3-\alpha-\beta}}{(\alpha+\beta-2)^2(\alpha-1)},$$
$$\widetilde{q}(t)=\frac{\tau^{1-\alpha}(t)t^{2-\alpha-\beta}}{(\alpha-1)^2(\alpha+\beta-2)}q(t).$$

Again by Theorem 4, (E_p) has a solution $y(t)$ if and only if the canonical equation

$$\left(t^{3-\alpha-\beta}\left(t^{\beta}x'(t)\right)'\right)'+\tau^{1-\alpha}(t)t^{2-\alpha-\beta}q(t)x(\tau(t))=0,$$

has a solution $x(t)=t^{\alpha-1}y(t)$.

The proofs of the remaining cases are similar and as such we leave the details to the reader. □

Combining Theorems 9 and 10, one can easily establish an oscillation criterion for (E_p).

THEOREM 11. *If* (5.6) *and* (5.7) *hold with* $b_1(t)$, $b_2(t)$, *and* $\tilde{q}(t)$ *as in Theorem* 10, *then* (E_p) *is oscillatory.*

Here, we would like to stress that the results in [12] do not cover the semicanonical cases with $\alpha+\beta>2$ nor the noncanonical case discussed in the above example.

REMARK 3. Many other analogous results can be established for equation (E) using Theorems 4–6 and existing oscillation results for (E_C). The details are left to the reader.

REMARK 4. It should be clear that we can apply our results here, in particular, Theorems 4–6, to nonlinear equations such as

$$\left(a_2(t)\left(a_1(t)y^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+q(t)y^{\gamma}(\tau(t))=0,$$

where γ is the ratio of odd positive integers, as was done in [12].

REMARK 5. To the best of our knowledge, there is no oscillation result for the delay dynamic equation (E) that would involve sharp oscillation constants known in the continuous case. This is left as an interesting research problem.

6. Conclusion

In this paper we show that any semicanonical or noncanonical third-order linear delay dynamic equation on a time scale can be written in canonical form without imposing any additional conditions on the coefficient functions or any restrictions on the time scale. As a consequence, this is also true for differential and difference equations. Applications to equations whose coefficients are exponential functions or powers of t are also given.

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John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA
e-mail: john-graef@utc.edu

Irena Jadlovská
Mathematical Institute
Slovak Academy of Sciences
Grešákova 6, 040 01 Košice, Slovakia
e-mail: jadlovska@saske.sk