Differential
Equations
& Applications
Volume 16, Number 1 (2024), 19–38
doi:10.7153/dea-2024-16-02

QUALITATIVE ANALYSIS OF NEUTRAL IMPLICIT FRACTIONAL q–DIFFERENCE EQUATIONS WITH DELAY

ABDELLATIF BENCHAIB, ABDELKRIM SALIM*,
SAÏD ABBAS AND MOUFFAK BENCHOHRA

(*Communicated by S. K. Ntouyas*)

Abstract. This paper explores the existence and stability of implicit neutral Caputo fractional q-difference equations within four distinct classes, incorporating various delay types such as finite, infinite, and state-dependent delays. To establish the existence of solutions, we utilize the fixed point theorem of Krasnoselskii in Banach spaces. The concluding section provides illustrative examples that highlight the obtained results.

1. Introduction

Fractional calculus have been found in several areas of engineering, mathematics, physics, and other applied sciences [2, 3, 4, 14, 15, 41]. Recently, in [3, 29]; the authors studied the existence of solutions of Caputo's fractional differential equations and inclusions. Several monographs and papers have been studied implicit fractional differential equations; see for instance [2, 3, 16] and the references therein.

The study of functional and neutral functional differential equations with has received great attention in the last years, we refer to the monographs of Hale [22], Hale and Verduyn Lunel [25], Hino *et al.* [27], Kolmanovskii and Myshkis [30], and the references therein.

The notion of q-calculus (quantum calculus) has a rich history [8, 28]. The subject of q-difference calculus has been developed over the years. For some interesting results about this subject we refer the reader to [20, 21]. The general theory of linear q-difference equations is investigated in the works of Adams [8] and Carmichael [19]. Meanwhile, Ahmad *et al.* conducted a study on several existence results for various types of nonlinear fractional q-difference equations in [10, 11]. The positive solutions of q-difference equations were examined by El-Shahed and Hassan [20]. Finally, the authors of [37] delved into the topological structure of solution sets for fractional q-difference inclusions, using Filippov's theorem.

Differential equations with infinite delay are a type of mathematical equations that describe the behavior of systems that have an infinite delay in their response to

Mathematics subject classification (2020): 26A33, 34K37.
Keywords and phrases: Fractional q-difference equation, Ulam stability, finite delay, infinite delay, sate dependent delay.

* Corresponding author.

changes in the system. The delay can be modeled by introducing an infinite delay term in the equation, which represents the time between the occurrence of an event and the system's response to that event, and it approaches infinity. The study of differential equations with infinite delay is an active area of research and it has a wide range of applications in fields such as physics, engineering, biology, and economics. Many researchers have expressed interest in the study of differential equations with infinite delay, see [7, 6, 5, 23].

In contrast to the analysis of Lyapunov and exponential stability, Ulam-Hyers stability analysis directs its focus toward the behavior of a function under perturbations, as opposed to the stability of a dynamical system or equilibrium point. Notably, the authors of [40, 39, 32, 1, 4, 7] have delved into Ulam stability concerning fractional differential problems under varying conditions. Furthermore, considerable attention has been directed towards exploring the stability of diverse functional equation types, particularly Ulam-Hyers and Ulam-Hyers-Rassias stability. This theme is pervasive in resources such as the book authored by Benchohra *et al.* [17, 18]. Research conducted by Luo *et al.* [33] and Rus [36] has also delved into the stability of operatorial equations using the Ulam-Hyers methodology.

In [38], we considered the following fractional q-difference problem:

$$\begin{cases} ({}^cD_q^\eta \chi)(\rho) = \wp(\rho, \chi(\rho)); \ \rho \in \Psi := [0, \kappa], \\ \chi(0) = \chi_0 \in \digamma, \end{cases}$$

where $q \in (0,1)$, $\eta \in (0,1]$, $\kappa > 0$, $\wp : \Psi \times \digamma \to \digamma$ is a given continuous function, $\digamma$ is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^cD_q^\eta$ is the Caputo fractional q-difference derivative of order η.

In [31], the authors proved some existence of solutions for the following problem with implicit fractional q-difference equations in Banach algebras:

$$\begin{cases} {}^cD_q^\eta \left(\frac{\chi(\rho)}{h(\rho,\chi(\rho))}\right) = \psi\left(\rho, \chi(\rho), {}^cD_q^\eta\left(\frac{\chi(\rho)}{h(\rho,\chi(\rho))}\right)\right); \ \rho \in \Psi := [0, \kappa], \\ \chi(0) = \chi_0 \in \mathbb{R}, \end{cases}$$

where $q \in (0,1)$, $\eta \in (0,1]$, $\kappa > 0$, ${}^cD_q^\eta$ is the Caputo fractional q-difference derivative of order η, $h : \Psi \times \mathbb{R} \to \mathbb{R}^*$, $\psi : \Psi \times \mathbb{R}^2 \to \mathbb{R}$ are given functions, and $\mathbb{R}^* = \mathbb{R} - \{0\}$.

In the present paper we prove some existence and Ulam stability results for the Cauchy problem of implicit neutral fractional q-difference equation with finite delay

$$\begin{cases} \chi(\rho) = \aleph(\rho); \ \rho \in [-\varepsilon, 0], \\ {}^cD_q^\eta(\chi(\rho) - \Upsilon(\rho, \chi_\rho)) = \wp(\rho, \chi(\rho), {}^cD_q^\eta(\chi(\rho) - \Upsilon(\rho, \chi_\rho))); \ \rho \in \Psi := [0, \kappa], \end{cases} \tag{1}$$

where $q \in (0,1)$, $\eta \in (0,1]$, $\kappa, \varepsilon > 0$, $\aleph \in \mho$, $\Upsilon : \Psi \times \mho \to \mathbb{R}$, $\wp : \Psi \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, ${}^cD_q^\eta$ is the Caputo fractional q-difference derivative of order η, and $\mho := C([-\varepsilon, 0], \mathbb{R})$ is the space of continuous functions on $[-\varepsilon, 0]$.

For any $\rho \in \Psi$, we define χ_ρ by

$$\chi_\rho(\vartheta) = \chi(\rho + \vartheta), \text{ for } \vartheta \in [-\varepsilon, 0].$$

Next we consider the Cauchy problem of implicit neutral fractional q-difference equation with infinite delay

$$\begin{cases} \chi(\rho) = \aleph(\rho);\ \rho \in (-\infty, 0], \\ {}^cD_q^\eta(\chi(\rho) - \Upsilon(\rho, \chi_\rho)) = \wp(\rho, \chi(\rho), {}^cD_q^\eta(\chi(\rho) - \Upsilon(\rho, \chi_\rho)));\ \rho \in \Psi, \end{cases} \tag{2}$$

where $\aleph : (-\infty, 0] \to \mathbb{R}$, $\Upsilon : \Psi \times \Bbbk \to \mathbb{R}$, $\wp : \Psi \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\Bbbk$ is a phase space.

For any $\rho \in \Psi$, we define $\chi_\rho \in \Bbbk$ by

$$\chi_\rho(\vartheta) = \chi(\rho + \vartheta);\ \text{for } \vartheta \in (-\infty, 0].$$

In the third section of this paper, we study the Cauchy problem of implicit neutral fractional q-difference equation with state-dependent delay

$$\begin{cases} \chi(\rho) = \aleph(\rho);\ \rho \in [-\varepsilon, 0], \\ {}^cD_q^\eta(\chi(\rho) - \Upsilon(\rho, \chi_{\rho(\rho,\chi_\rho)})) = \wp(\rho, \chi(\rho), {}^cD_q^\eta(\chi(\rho) - \Upsilon(\rho, \chi_{\rho(\rho,\chi_\rho)})));\ \rho \in \Psi, \end{cases} \tag{3}$$

where $\aleph \in \mho$, $\rho : \Psi \times \mho \to \mathbb{R}$, $\Upsilon : \Psi \times \mho \to \mathbb{R}$, $\wp : \Psi \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Finally, we treat the last Cauchy problem of implicit neutral fractional q-difference equation with state dependent delay

$$\begin{cases} \chi(\rho) = \aleph(\rho);\ \rho \in (-\infty, 0], \\ {}^cD_q^\eta(\chi(\rho) - \Upsilon(\rho, \chi_{\rho(\rho,\chi_\rho)})) = \wp(\rho, \chi(\rho), {}^cD_q^\eta(\chi(\rho) - \Upsilon(\rho, \chi_{\rho(\rho,\chi_\rho)})));\ \rho \in \Psi, \end{cases} \tag{4}$$

where $\aleph : (-\infty, 0] \to \mathbb{R}$, $\rho : \Psi \times \Bbbk \to \mathbb{R}$, $\Upsilon : \Psi \times \Bbbk \to \mathbb{R}$, $\wp : \Psi \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

It is important to highlight that our study draws inspiration from the publications mentioned earlier and can be seen as a natural extension and continuation of the research outlined in [38, 31]. This serves to contribute to the progress of theories pertaining to q-difference equations.

2. Preliminaries

Let $(C(\Psi), \|\cdot\|_\infty)$ be the Banach space of continuous functions $\lambda : \Psi \to \mathbb{R}$ with norm

$$\|\lambda\|_\infty := \sup_{\rho \in \Psi} |\lambda(\rho)|,$$

and let $L^1(\Psi)$ be the space of measurable functions $\lambda : \Psi \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|\lambda\|_1 = \int_\Psi |\lambda(\rho)| d\rho.$$

Set

$$[\omega_1]_q = \frac{1-q^{\omega_1}}{1-q},$$

where ω_1 is a real number.

DEFINITION 1. ([28]) The q analogue of the power $(\omega_1-\omega_2)^\beta$ is defined by

$$(\omega_1-\omega_2)^{(0)} = 1,\ (\omega_1-\omega_2)^{(\beta)} = \Pi_{j=0}^{\beta-1}(\omega_1-\omega_2 q^j);\ \omega_1,\omega_2\in\mathbb{R},\ \beta\in\mathbb{N}.$$

In general,

$$(\omega_1-\omega_2)^{(\eta)} = \omega_1{}^{\eta}\Pi_{j=0}^{\infty}\left(\frac{\omega_1-\omega_2 q^j}{\omega_1-\omega_2 q^{j+\eta}}\right);\ \omega_1,\omega_2,\eta\in\mathbb{R}.$$

DEFINITION 2. ([28]) The q-gamma function of $xi\in\mathbb{R}-\{0,-1,-2,\ldots\}$; is given by

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}}.$$

Notice that $\Gamma_q(1+\xi) = [\xi]_q\Gamma_q(\xi)$.

DEFINITION 3. ([28]) The q-derivative of order $\beta\in\mathbb{N}$ of a function $\chi:\Psi\to\mathbb{R}$ is given by $(D_q^0\chi)(\rho)=\chi(\rho)$,

$$(D_q\chi)(\rho) := (D_q^1\chi)(\rho) = \frac{\chi(\rho)-\chi(q\rho)}{(1-q)\rho};\ \rho\neq 0,\ \ (D_q\chi)(0) = \lim_{\rho\to 0}(D_q\chi)(\rho),$$

and

$$(D_q^\beta\chi)(\rho) = (D_q D_q^{\beta-1}\chi)(\rho);\ \rho\in\Psi,\ \beta\in\{1,2,\ldots\}.$$

Set $I_\rho := \{tq^\beta : \beta\in\mathbb{N}\}\cup\{0\}$.

DEFINITION 4. ([28]) The q-integral of a function $\chi: I_\rho\to\mathbb{R}$ is given by

$$(I_q\chi)(\rho) = \int_0^\rho \chi(\vartheta)d_qs = \sum_{\beta=0}^{\infty}\rho(1-q)q^\beta\wp(tq^\beta),$$

provided that the series converges.

Notice that $(D_qI_q\chi)(\rho)=\chi(\rho)$, and if χ is continuous at 0, then

$$\chi(\rho) = \chi(0) + (I_qD_q\chi)(\rho).$$

DEFINITION 5. ([9]) The Riemann-Liouville fractional q-integral of order $\eta\in\mathbb{R}_+ := [0,\infty)$ of a function $\chi:\Psi\to\mathbb{R}$ is given by $(I_q^0\chi)(\rho)=\chi(\rho)$, and

$$(I_q^\eta\chi)(\rho) = \int_0^\rho \frac{(\rho-q\vartheta)^{(\eta-1)}}{\Gamma_q(\eta)}\chi(\vartheta)d_qs;\ \rho\in\Psi.$$

LEMMA 1. ([34]) *For $\eta \in \mathbb{R}_+$ and $\varkappa \in (-1,\infty)$, we have*

$$(I_q^{\eta}(\rho-a)^{(\varkappa)})(\rho) = \frac{\Gamma_q(1+\varkappa)}{\Gamma(1+\varkappa+\eta)}(\rho-a)^{(\varkappa+\eta)};\ 0<a<\rho<\kappa.$$

In particular,

$$(I_q^{\eta}1)(\rho) = \frac{\rho^{(\eta)}}{\Gamma_q(1+\eta)} = \frac{\rho^{\eta}}{\Gamma_q(1+\eta)}.$$

DEFINITION 6. ([35]) The Riemann-Liouville fractional q-derivative of order $\eta \in \mathbb{R}_+$ of a function $\chi : \Psi \to \mathbb{R}$ is given by $(D_q^0\chi)(\rho) = \chi(\rho)$, and

$$(D_q^{\eta}\chi)(\rho) = (D_q^n I_q^{n-\eta}\chi)(\rho);\ \rho \in \Psi,$$

where n denotes the integer part of $\eta+1$.

DEFINITION 7. ([35]) The Caputo fractional q-derivative of order $\eta \in \mathbb{R}_+$ of a function $\chi : \Psi \to \mathbb{R}$ is given by $({}^C D_q^0\chi)(\rho) = \chi(\rho)$, and

$$({}^C D_q^{\eta}\chi)(\rho) = (I_q^{n-\eta} D_q^n\chi)(\rho);\ \rho \in \Psi.$$

LEMMA 2. ([35]) *Let $\eta \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^{\eta}\,{}^C D_q^{\eta}\chi)(\rho) = \chi(\rho) - \sum_{j=0}^{n-1}\frac{\rho^j}{\Gamma_q(1+j)}(D_q^j\chi)(0).$$

In particular, if $\eta \in (0,1)$, then

$$(I_q^{\eta}\,{}^C D_q^{\eta}\chi)(\rho) = \chi(\rho) - \chi(0).$$

LEMMA 3. *Let $\Upsilon : \Psi \times \mho \to \mathbb{R}$, $\wp : \Psi \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\Upsilon(\cdot,w) \in C(\Psi)$ and $\wp(\cdot,\chi,\lambda) \in C(\Psi)$, for each $w \in \mho$, and $\chi,\lambda \in \mathbb{R}$. Then, (1) is equivalent to the problem of obtaining the solutions of the integral equation*

$$\begin{cases} \chi(\rho) = \aleph(\rho);\ \rho \in [-\varepsilon,0], \\ \Phi(\rho) = \wp(\rho,\Upsilon(\rho,\chi_\rho) + \aleph(0) - \Upsilon(0,\chi_0) + (I_q^{\eta}\Phi)(\rho),\Phi(\rho));\ \rho \in \Psi, \end{cases}$$

and if $\Phi(\cdot) \in C(\Psi)$, is the solution of this equation, then

$$\begin{cases} \chi(\rho) = \aleph(\rho);\ \rho \in [-\varepsilon,0], \\ \chi(\rho) = \Upsilon(\rho,\chi_\rho) + \aleph(0) - \Upsilon(0,\chi_0) + (I_q^{\eta}\Phi)(\rho);\ \rho \in \Psi. \end{cases}$$

From Lemma 3, we conclude the following corollary.

COROLLARY 1. *The solutions of the problem (1) are the fixed points of the operator* $\digamma : C([-\varepsilon,\kappa]) \to C([-\varepsilon,\kappa])$ *given by*

$$\begin{cases} (\digamma\chi)(\rho) = \aleph(\rho);\ \rho \in [-\varepsilon, 0], \\ (\digamma\chi)(\rho) = \Upsilon(\rho,\chi_\rho) + \aleph(0) - \Upsilon(0,\chi_0) + (I_q^\eta \Phi)(\rho);\ \rho \in \Psi, \end{cases} \tag{5}$$

where $\Phi \in C(\Psi)$ *such that*

$$\Phi(\rho) = \wp(\rho,\chi(\rho),\Phi(\rho)),$$

or

$$\Phi(\rho) = \wp(\rho,\Upsilon(\rho,\chi_\rho) + \aleph(0) - \Upsilon(0,\chi_0) + (I_q^\eta\Phi)(\rho),\Phi(\rho)).$$

Let $\varpi > 0$ and $\daleth : \Psi \to \mathbb{R}$ be a continuous and positive function. We put the following inequalities

$$|(\digamma\chi)(\rho) - \chi(\rho)| \leqslant \varpi;\ \rho \in \Psi. \tag{6}$$

$$|(\digamma\chi)(\rho) - \chi(\rho)| \leqslant \daleth(\rho);\ \rho \in \Psi. \tag{7}$$

$$|(\digamma\chi)(\rho) - \chi(\rho)| \leqslant \varpi\daleth(\rho);\ \rho \in \Psi. \tag{8}$$

DEFINITION 8. ([3, 36]) The problem (1) is Ulam-Hyers stable if there exists $c_\digamma > 0$ where for each $\varpi > 0$ and for each solution $\chi \in C(\Psi)$ of (6) there exists a solution $\lambda \in C(\Psi)$ of (1) with

$$|\chi(\rho) - \lambda(\rho)| \leqslant \varpi c_\digamma;\ \rho \in \Psi.$$

DEFINITION 9. ([3, 36]) The problem (1) is generalized Ulam-Hyers stable if there exists $c_\digamma : C(\mathbb{R}_+,\mathbb{R}_+)$ with $c_\digamma(0) = 0$ such that for each $\varpi > 0$ and for each solution $\chi \in C(\Psi)$ of the inequality (6) there exists a solution $\lambda \in C(\Psi)$ of (1) with

$$|\chi(\rho) - \lambda(\rho)| \leqslant c_\digamma(\varpi);\ \rho \in \Psi.$$

DEFINITION 10. ([3, 36]) The problem (1) is Ulam-Hyers-Rassias stable with respect to $\daleth$ if there exists $c_{\digamma,\daleth} > 0$where for each $\varpi > 0$ and for each solution $\chi \in C(\Psi)$ of (8) there exists a solution $\lambda \in C(\Psi)$ of (1) with

$$|\chi(\rho) - \lambda(\rho)| \leqslant \varpi c_{\digamma,\daleth}\daleth(\rho);\ \rho \in \Psi.$$

DEFINITION 11. ([3, 36]) The problem (1) is generalized Ulam-Hyers-Rassias stable with respect to $\daleth$ if there exists $c_{\digamma,\daleth} > 0$ where for each solution $\chi \in C(\Psi)$ of (7) there exists a solution $\lambda \in C(\Psi)$ of (1) with

$$|\chi(\rho) - \lambda(\rho)| \leqslant c_{\digamma,\daleth}\daleth(\rho);\ \rho \in \Psi.$$

Let $(\Bbbk, \|\cdot\|_\Bbbk)$ be a phase space. It is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following [23]:

(A_1) If $z:(-\infty,\kappa]\rightarrow\mathbb{R}$ continuous on Ψ and $z_\rho\in\Bbbk$, for all $\rho\in(-\infty,0]$, then there are constants $\Im_1,\Im_2,\Im_3>0$ such that for any $\rho\in\Psi$, the following conditions hold:

(i) z_ρ is in $\Bbbk$;

(ii) $\|z(\rho)\|\leqslant\Im_1\|z_\rho\|_{\Bbbk}$,

(iii) $\|z_\rho\|_{\mathscr{B}}\leqslant\Im_2\sup_{\vartheta\in[0,\rho]}\|z(\vartheta)\|+\Im_3\sup_{\vartheta\in(-\infty,0]}\|z_\vartheta\|_{\Bbbk}$,

(A_2) For the function $z(\cdot)$ in (A_1), z_ρ is a $\Bbbk$-valued continuous function on Ψ.

(A_3) The space $\Bbbk$ is complete.

EXAMPLE 1. Let $\Bbbk$ be the set of all functions $\varsigma:(-\infty,0]\rightarrow\mathbb{R}$ which are continuous on $[-\varepsilon,0]$, $\varepsilon\geqslant0$, with the semi-norm

$$\|\varsigma\|_{\Bbbk}=\sup_{\rho\in[-\varepsilon,0]}\|\varsigma(\rho)\|.$$

Then we have $\Im_1=\Im_2=\Im_3=1$. The quotient space $\widehat{\Bbbk}=\Bbbk/\|\cdot\|_{\Bbbk}$ is isometric to the space $C([-\varepsilon,0],\mathbb{R})$ of all continuous functions from $[-\varepsilon,0]$ into $\mathbb{R}$ with the supremum norm, this means that functional differential equations with finite delay are included in our axiomatic model.

3. Existence and stability results with finite delay

Let $C:=C([-\varepsilon,\kappa],\mathbb{R})$ be the Banach space of continuous functions from $[-\varepsilon,\kappa]$ into $\mathbb{R}$ with the norm

$$\|\chi\|_C:=\sup_{\rho\in[-\varepsilon,\kappa]}|\chi(\rho)|.$$

DEFINITION 12. A solution of the problem (1) is a function $\chi\in C$ that satisfies the initial condition $\chi(\rho)=\aleph(\rho)$ on $[-\varepsilon,0]$, and the equation ${}^cD_q^\eta(\chi(\rho)-\Upsilon(\rho,\chi(\rho))=\wp(\rho,\chi_\rho,({}^cD_q^\eta\chi)(\rho))$ on Ψ.

Consider the following hypotheses:

(H_1) The function Υ satisfies:

$$|\Upsilon(\rho,\chi)-\Upsilon(\rho,\lambda)|\leqslant\varsigma\|\chi-\lambda\|_{\mho},$$

for $\rho\in\Psi$ and $\chi,\lambda\in\mho$, where $0<\varsigma<1$.

(H_2) There exist continuous functions $\delta_1,\delta_2,\delta_3:\Psi\rightarrow\mathbb{R}_+$ with $\delta_3(\rho)<1$ such that

$$|\wp(\rho,\chi,\lambda)|\leqslant\delta_1(\rho)+\delta_2(\rho)|\chi|+\delta_3(\rho)|\lambda|,\ \text{for each }\rho\in\Psi\text{ and }\chi,\lambda\in\mathbb{R}.$$

Set

$$\aleph^* = \sup_{\rho\in[-\varepsilon,0]} |\aleph(\rho)|,\ \delta_1{}^* = \sup_{\rho\in\Psi} \delta_1(\rho),\ \delta_2{}^* = \sup_{\rho\in\Psi} \delta_2(\rho),$$

$$\delta_3{}^* = \sup_{\rho\in\Psi} \delta_3(\rho),\ \Upsilon^* := \sup_{\rho\in\Psi} |\Upsilon(\rho,0)|.$$

THEOREM 1. *Suppose that the hypotheses* (H_1), (H_2), *and the condition*

$$2\varsigma + \frac{\kappa^\eta \delta_2{}^*}{(1-\delta_3{}^*)\Gamma_q(1+\eta)} < 1,$$

hold. Then the problem (1) has at least one solution on $[-\varepsilon,\kappa]$.

Proof. Consider the operators $A,B : C([-\varepsilon,\kappa]) \to C([-\varepsilon,\kappa])$ defined by

$$\begin{cases} (A\chi)(\rho) = 0;\ \rho\in[-\varepsilon,0], \\ (A\chi)(\rho) = \aleph(0) - \Upsilon(0,\chi_0) + (I_q^\eta \Phi)(\rho);\ \rho\in\Psi, \end{cases} \tag{9}$$

where $\Phi \in C(\Psi)$ with $\Phi(\rho) = \wp(\rho,\chi(\rho),\Phi(\rho))$, and

$$\begin{cases} (B\chi)(\rho) = \aleph(\rho);\ \rho\in[-\varepsilon,0], \\ (B\chi)(\rho) = \Upsilon(\rho,\chi_\rho);\ \rho\in\Psi. \end{cases} \tag{10}$$

Set

$$\mu \geqslant \max\left\{ \aleph^*, \frac{2h^* + \aleph^* + \frac{\kappa^\eta(\delta_1{}^*+\delta_2{}^*\mu)}{(1-\delta_3{}^*)\Gamma_q(1+\eta)}}{1-2\varsigma-\frac{\kappa^\eta\delta_2{}^*}{(1-\delta_3{}^*)\Gamma_q(1+\eta)}} \right\},$$

and let $\Delta_\mu = \{\chi \in C([-\varepsilon,\kappa]) : \|\chi\|_C \leqslant \mu\}$ be the closed and convex ball in C.

We shall prove in three steps that A and B satisfy the conditions of Krasnoselskii's fixed point theorem [12, 13].

Step 1. $A\chi + B\lambda \in \Delta_\mu$ whenever $\chi,\lambda\in\Delta_\mu$.
Let $\chi,\lambda\in\Delta_\mu$. Then, for each $\rho\in[-\varepsilon,0]$ we have

$$|A\chi(\rho) + B\lambda(\rho)| = \aleph(\rho) \leqslant \aleph^* \leqslant \mu,$$

and for each $\rho\in\Psi$, we have

$$|(A\chi)(\rho)+(B\lambda)(\rho)| \leqslant |\Upsilon(\rho,\lambda_\rho)| + |\aleph(0)| + |\Upsilon(0,\chi_0)| + \int_0^\rho \frac{(\rho-q\vartheta)^{(\eta-1)}}{\Gamma_q(\eta)} |\Phi(\vartheta)| d_q s,$$

where $\Phi\in C(\Psi)$ with

$$\Phi(\rho) = \wp(\rho,\chi(\rho),\Phi(\rho)).$$

By using (H_2), for each $\rho \in \Psi$ we have

$$\begin{aligned}|\Phi(\rho)| &\leqslant \delta_1(\rho)+\delta_2(\rho)|\chi(\rho)|+\delta_3(\rho)|\Phi(\rho)|\\ &\leqslant {\delta_1}^*+{\delta_2}^*\mu+{\delta_3}^*|\Phi(\rho)|.\end{aligned}$$

This gives

$$|\Phi(\rho)| \leqslant \frac{{\delta_1}^*+{\delta_2}^*\mu}{1-{\delta_3}^*}.$$

Thus

$$\begin{aligned}\|A(\chi)+B(\lambda)\|_\infty &\leqslant |\aleph(0)|+|\Upsilon(0,0)|+|\Upsilon(0,\chi_0)-\Upsilon(0,0)|+\frac{\kappa^\eta({\delta_1}^*+{\delta_2}^*\mu)}{(1-{\delta_3}^*)\Gamma_q(1+\eta)}\\ &\quad+|\Upsilon(\rho,\lambda_\rho)-\Upsilon(\rho,0)|+|\Upsilon(\rho,0)|\\ &\leqslant \aleph^*+\Upsilon^*+\varsigma\|\chi_0\|_\mho+\frac{\kappa^\eta({\delta_1}^*+{\delta_2}^*\mu)}{(1-{\delta_3}^*)\Gamma_q(1+\eta)}+\varsigma\|\lambda_\rho\|_\mho+\Upsilon^*\\ &\leqslant \aleph^*+\Upsilon^*+\varsigma\mu+\frac{\kappa^\eta({\delta_1}^*+{\delta_2}^*\mu)}{(1-{\delta_3}^*)\Gamma_q(1+\eta)}+\varsigma\mu+\Upsilon^*\\ &= 2h^*+\aleph^*+\frac{\kappa^\eta({\delta_1}^*+{\delta_2}^*\mu)}{(1-{\delta_3}^*)\Gamma_q(1+\eta)}+\mu\left(2\varsigma+\frac{\kappa^\eta{\delta_2}^*}{(1-{\delta_3}^*)\Gamma_q(1+\eta)}\right)\\ &\leqslant \mu.\end{aligned}$$

Hence, we get

$$\|A(\chi)+B(\lambda)\|_C \leqslant \mu.$$

This proves that $A\chi+B\lambda \in \Delta_\mu$ whenever $\chi,\lambda \in \Delta_\mu$.

Step 2. $A:\Delta_\mu \to \Delta_\mu$ is compact and continuous.

Claim 1. A is continuous.
Let $\{\chi_\beta\}_{\beta\in\mathbb{N}}$ be a sequence such that $\chi_\beta \to \chi$ in Δ_μ. Then we have

$$|(A\chi_\beta)(\rho)-(A\chi)(\rho)| \leqslant \int_0^\rho \frac{(\rho-q\vartheta)^{(\eta-1)}}{\Gamma_q(\eta)}|(\Phi_\beta(\vartheta)-\Phi(\vartheta))|d_qs;\ \rho\in\Psi,$$

where $\Phi_\beta,\Phi \in C(\Psi)$ such that

$$\Phi_\beta(\rho)=\wp(\rho,\chi_\beta(\rho),\Phi_\beta(\rho)),$$

and

$$\Phi(\rho)=\wp(\rho,\chi(\rho),\Phi(\rho)).$$

Since $\chi_\beta \to \chi$ as $\beta\to\infty$ and $\wp$ is continuous, we get

$$\Phi_\beta(\rho)\to\Phi(\rho) \text{ as } \beta\to\infty, \text{ for each } \rho\in\Psi.$$

Hence

$$\|A(\chi_\beta)-A(\chi)\|_\infty \leqslant \frac{{\delta_1}^*+{\delta_2}^*\mu}{1-{\delta_3}^*}\|\Phi_\beta-\Phi\|_\infty \to 0 \text{ as } \beta\to\infty.$$

Claim 2. $A(\Delta_\mu)$ is bounded and equicontinuous.

We have $A(\Delta_\mu) \subset \Delta_\mu$ and Δ_μ is bounded, thus $A(\Delta_\mu)$ is bounded. Next, let $\rho_1, \rho_2 \in \Psi$, such that $\rho_1 < \rho_2$ and let $\chi \in \Delta_\mu$. Then, there exists $\Phi \in C(\Psi)$ with $\Phi(\rho) = \wp(\rho, \chi(\rho), \Phi(\rho))$, such that

$$\begin{aligned}|(A\chi)(\rho_1) - (A\chi)(\rho_2)| \leqslant & \int_0^{\rho_1} \frac{|(\rho_2 - q\vartheta)^{(\eta-1)} - (\rho_1 - q\vartheta)^{(\eta-1)}|}{\Gamma_q(\eta)} |\Phi(\vartheta)| d_q s \\ & + \int_{\rho_1}^{\rho_2} \frac{|(\rho_2 - q\vartheta)^{(\eta-1)}|}{\Gamma_q(\eta)} |\Phi(\vartheta)| d_q s.\end{aligned}$$

Hence

$$\begin{aligned}|(A\chi)(\rho_1) - (A\chi)(\rho_2)| \leqslant & \frac{\delta_1{}^* + \delta_2{}^* \mu}{1 - \delta_3{}^*} \int_0^{\rho_1} \frac{|(\rho_2 - q\vartheta)^{(\eta-1)} - (\rho_1 - q\vartheta)^{(\eta-1)}|}{\Gamma_q(\eta)} d_q s \\ & + \frac{\delta_1{}^* + \delta_2{}^* \mu}{1 - \delta_3{}^*} \int_{\rho_1}^{\rho_2} \frac{|(\rho_2 - q\vartheta)^{(\eta-1)}|}{\Gamma_q(\eta)} d_q s \\ & \to 0 \text{ as } \rho_1 \to \rho_2.\end{aligned}$$

As a consequence of the above claims, the Arzelá-Ascoli theorem implies that $A : \Delta_\mu \to \Delta_\mu$ is continuous and compact.

Step 3. B is a contraction mapping.

Let $\chi, \lambda \in \Delta_\mu$. From (H_1), for each $\rho \in \Psi$, we have

$$\begin{aligned}|(B\chi)(\rho) - (B\lambda)(\rho)| & \leqslant |\Upsilon(\rho, \chi_\rho) - \Upsilon(\rho, \lambda_\rho)| \\ & \leqslant \varsigma \|\chi_\rho - \lambda_\rho\|_{\mho}.\end{aligned}$$

Thus

$$\|B(\chi) - B(\lambda)\|_\infty \leqslant \varsigma \|\chi - \lambda\|_\infty.$$

Hence

$$\|B(\chi) - B(\lambda)\|_C \leqslant \varsigma \|\chi - \lambda\|_C,$$

which implies that the operator B is a contraction.

As a consequence of the three above steps, from Krasnoselskii's fixed point theorem [12, 13], the operator equation $(A + B)(\chi) = \chi$ has at least a solution. □

Now, we prove a result about the generalized Ulam-Hyers-Rassias stability of the problem (1).

The hypotheses:

(H_3) There exist functions $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \in C(\Psi, [0, \infty))$ with $\Lambda_3(\rho) < 1$ such that

$$(1 + |\chi|)|\wp(\rho, \chi, \lambda)| \leqslant \Lambda_1(\rho)\daleth(\rho) + \Lambda_2(\rho)\daleth(\rho)|\chi| + \Lambda_3(\rho)|\lambda|,$$

for each $\rho \in \Psi$ and $\chi, \lambda \in \mathbb{R}$, and

$$(1 + \|w - z\|_{\mho})|\Upsilon(\rho, w) - \Upsilon(\rho, z)| \leqslant \Lambda_4(\rho)\daleth(\rho)\|w - z\|_{\mho},$$

for each $\rho \in \Psi$ and $w, z \in \mho$,

(H_4) There exists $\varkappa_{\beth} > 0$ such that for each $\rho \in \Psi$, we have

$$(I_q^\eta \beth)(\rho) \leqslant \varkappa_{\beth} \beth(\rho).$$

Set $\beth^* = \sup_{\rho\in\Psi} \beth(\rho)$ and

$$\Lambda_i^* = \sup_{\rho\in\Psi} \Lambda_i(\rho),\ i \in \{1,2,3,4\}.$$

THEOREM 2. *Suppose that the hypotheses* (H_3), (H_4) *and the conditions* $\Lambda_4^*\beth^* < 1$, *and*

$$\Lambda_3^* + 2\Lambda_4^*\beth^* + \frac{\kappa^\eta \Lambda_2^*\beth^*}{\Gamma_q(1+\eta)} - 2\Lambda_3^*\Lambda_4^*\beth^* < 1,$$

hold. Then the problem (1) is generalized Ulam-Hyers-Rassias stable.

Proof. Let $\digamma$ be the operator defined in (5). It's clear that (H_3) implies (H_1) with $\varsigma = \Lambda_4^*\beth^*$, and; (H_3) implies (H_2) with $\delta_1 \equiv \Lambda_1\beth$, $\delta_2 \equiv \Lambda_2\beth$ and $\delta_3 \equiv \Lambda_3$.

Let χ be a solution of (7), and λ is a solution of (1). Thus, we have $\lambda(\rho) = \aleph(\rho)$; $\rho \in [-\varepsilon, 0]$, and

$$\lambda(\rho) = \Upsilon(\rho, \lambda_\rho) + \aleph(0) - \Upsilon(0, \lambda_0) + (I_q^\eta z)(\rho);\ \rho \in \Psi,$$

where $z \in C(\Psi)$ such that $z(\rho) = \wp(\rho, \lambda(\rho), z(\rho))$.

From the inequality (7) for each $\rho \in \Psi$, we have

$$|\chi(\rho) - \Upsilon(\rho, \chi_\rho) - \aleph(0) + \Upsilon(0, \chi_0) - (I_q^\eta \Phi)(\rho)| \leqslant (I_q^\eta \beth)(\rho),$$

where $\Phi \in C(\Psi)$ such that $\Phi(\rho) = \wp(\rho, \chi(\rho), \Phi(\rho))$.

From (H_3) and (H_4), for each $\rho \in \Psi$, we get

$$\begin{aligned}
|\chi(\rho) - \lambda(\rho)| &\leqslant |\chi(\rho) - \Upsilon(\rho, \chi_\rho) - \aleph(0) + \Upsilon(0, \chi_0) - (I_q^\eta \Phi)(\rho)| \\
&\quad + |\Upsilon(\rho, \chi_\rho) - \Upsilon(\rho, \lambda_\rho) + |\Upsilon(\rho, \chi_0) - \Upsilon(\rho, \lambda_0)| + (I_q^\eta(\Phi - z))(\rho)| \\
&\leqslant (I_q^\eta \beth)(\rho) + 2\Lambda_4^*\beth(\rho) + \int_0^\rho \frac{(\rho - q\vartheta)^{(\eta-1)}}{\Gamma_q(\eta)} (|(\Phi(\vartheta)| + |z(\vartheta))|) d_q s \\
&\leqslant (I_q^\eta \beth)(\rho) + 2\Lambda_4^*\beth(\rho) + \frac{\Lambda_1^* + \Lambda_2^*}{1 - \Lambda_3^*}(I_q^\eta \beth)(\rho) \\
&\leqslant \varkappa_\varsigma \beth(\rho) + 2\Lambda_4^*\beth(\rho) + \varkappa_\varsigma \frac{\Lambda_1^* + \frac{\Lambda_2^*|\chi(\rho)|}{1+|\chi(\rho)|}}{1 - \Lambda_3^*} \beth(\rho) \\
&\leqslant \left[2\Lambda_4^* + \varkappa_\varsigma \left(1 + \frac{\Lambda_1^* + \Lambda_2^*}{1 - \Lambda_3^*} \right) \right] \beth(\rho) \\
&:= c_{\wp,\Upsilon,\beth} \beth(\rho).
\end{aligned}$$

Hence, we conclude the generalized Ulam-Hyers-Rassias stability of problem (1). □

4. Existence and stability results with infinite delay

Consider the space

$$\Omega := \{\chi : (-\infty,\kappa] \to \mathbb{R} : \chi_\rho \in \Bbbk \text{ for } \rho \in \Psi \text{ and } \chi|_\Psi \in C(\Psi)\}.$$

In the present section, we are concerned with the problem (2).

The hypotheses:

(H_{01}) The function Υ satisfies the Lipschitz condition:

$$|\Upsilon(\rho,\chi)-\Upsilon(\rho,\lambda)| \leqslant \varsigma\|\chi-\lambda\|_\Bbbk,$$

for $\rho \in \Psi$ and $\chi,\lambda \in \Bbbk$, where $0<\varsigma<1$.

(H_{02}) There exist functions $\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4 \in C(\Psi,\mathbb{R}_+)$ with $\Lambda_3(\rho)<1$ such that

$$(1+|\chi|)|\wp(\rho,\chi,\lambda)| \leqslant \Lambda_1(\rho)\daleth(\rho)+\Lambda_2(\rho)\daleth(\rho)|\chi|+\Lambda_3(\rho)|\lambda|,$$

for each $\rho \in \Psi$ and $\chi,\lambda \in \mathbb{R}$, and

$$(1+\|w-z\|_\Bbbk)|\Upsilon(\rho,w)-\Upsilon(\rho,z)| \leqslant \Lambda_4(\rho)\daleth(\rho)\|w-z\|_\Bbbk,$$

for each $\rho \in \Psi$ and $w,z \in \Bbbk$,

THEOREM 3. *From the hypotheses (H_{01}), (H_2) and the condition*

$$\frac{\kappa^\eta {\delta_2}^*}{\Gamma_q(1+\eta)}+{\delta_3}^*+\varsigma-\varsigma{\delta_3}^*<1,$$

the problem (2) has at least one solution defined on $(-\infty,\kappa]$.

Proof. Define the operators $A,B:\Omega\to\Omega$ by

$$\begin{cases}(A\chi)(\rho)=0;\ \rho\in(-\infty,0],\\ (A\chi)(\rho)=\chi_0-\Upsilon(0,\chi_0)+(I_q^\eta\Phi)(\rho);\ \rho\in\Psi,\end{cases} \tag{11}$$

where $\Phi\in C(\Psi)$ with $\Phi(\rho)=\wp(\rho,\chi(\rho),\Phi(\rho))$, and

$$\begin{cases}(B\chi)(\rho)=\aleph(\rho);\ \rho\in(-\infty,0],\\ (B\chi)(\rho)=\Upsilon(\rho,\chi_\rho);\ \rho\in\Psi.\end{cases} \tag{12}$$

Let $\lambda(\cdot):(-\infty,\kappa]\to\mathbb{R}$ be a function defined by,

$$\lambda(\rho)=\begin{cases}\aleph(\rho), & \rho\in(-\infty,0],\\ 0; & \rho\in\Psi.\end{cases}$$

Then $\lambda_\rho = \aleph(\rho)$ for all $\rho \in (-\infty, 0]$. For each $w \in C(\Psi)$ with $w(\rho) = 0$ for each $\rho \in (-\infty, 0]$, we denote by $\overline{w}$ the function defined by

$$\overline{w}(\rho) = \begin{cases} 0, & \rho \in (-\infty, 0], \\ w(\rho) & \rho \in \Psi. \end{cases}$$

If $\chi(\cdot)$ satisfies,

$$\chi(\rho) = \Upsilon(\rho, \chi_\rho),$$

then, $\chi(\rho) = \overline{w}(\rho) + \lambda(\rho)$; $\rho \in \Psi$, and then $\chi_\rho = \overline{w}_\rho + \lambda_\rho$, for every $\rho \in \Psi$. Thus, the function $w(\cdot)$ satisfies

$$w(\rho) = \Upsilon(\rho, \chi_\rho).$$

Let

$$C_0 = \{w \in \Omega : \ w(\rho) = 0 \ \text{ for } \rho \in (-\infty, 0]\},$$

be the Banach space with norm $\|\cdot\|_\kappa$, with

$$\|w\|_\kappa = \sup_{\rho \in (-\infty,0]} \|w_\rho\|_{\mathbb{k}} + \sup_{\rho \in \Psi} \|w(\rho)\| = \sup_{\rho \in \Psi} \|w(\rho)\|, \ w \in C_0.$$

Consider the operator $P : C_0 \to C_0$ be defined by

$$(Pw)(\rho) = \Upsilon(\rho, \chi_\rho). \tag{13}$$

Then the operators $A+B$ and $A+P$ have the same fixed points. Set

$$\mu \geqslant \frac{(1-\delta_3{}^*)[2h^* + |\chi_0|(1+\varsigma)] + \frac{\kappa^\eta \delta_1{}^*}{\Gamma_q(1+\eta)}}{1 - \delta_3{}^* - \varsigma + \varsigma\delta_3{}^* - \frac{\kappa^\eta \delta_2{}^*}{\Gamma_q(1+\eta)}},$$

and define the ball $\Delta_\mu = \{\chi \in \Omega : \|\chi\|_T \leqslant \mu\}$ in Ω. We can prove as in Theorem 1 that the operators P and B satisfy the conditions of Krasnoselskii's fixed point theorem [12, 13]. This implies that the operator $A+B$ has at least a fixed point which is a solution of problem (2).

From Theorem 3, we can conclude the following result about the generalized Ulam-Hyers-Rassias stability of problem (2). $\square$

THEOREM 4. *Assume that the hypotheses* (H_{02}) *and* (H_4) *hold. If* $\Lambda_4^* \daleth^* < 1$, *and*

$$\Lambda_3^* + 2\Lambda_4^* \daleth^* + \frac{\kappa^\eta \Lambda_2^* \daleth^*}{\Gamma_q(1+\eta)} - 2\Lambda_3^*\Lambda_4^*\daleth^* < 1,$$

then the problem (2) *has a solution and it is generalized Ulam-Hyers-Rassias stable.*

5. Existence and stability results with state dependent delay

5.1. The finite delay case

Set $\mathscr{R} := \mathscr{R}_{\rho^-} = \{\rho(\rho,\chi) : (\rho,\chi) \in \Psi \times C(\Psi),\ \rho(\rho,\chi) \leqslant 0\}$. We always assume that $\rho : \Psi \times C(\Psi) \to \mathbb{R}$ is continuous and the function $\rho \longmapsto \chi_\rho$ is continuous from $\mathscr{R}$ into $C(\Psi)$.

As in Theorems 1 and 2, we conclude the following results.

THEOREM 5. *Assume that the hypotheses (H_1) and (H_2) hold. If*

$$2\varsigma + \frac{\kappa^\eta \delta_2{}^*}{(1-\delta_3{}^*)\Gamma_q(1+\eta)} < 1,$$

then the problem (3) has at least one solution defined on $[-\varepsilon,\kappa]$.

THEOREM 6. *Assume that the hypotheses (H_3) and (H_4) hold. If $\Lambda_4^* \daleth^* < 1$, and*

$$\Lambda_3^* + 2\Lambda_4^* \daleth^* + \frac{\kappa^\eta \Lambda_2^* \daleth^*}{\Gamma_q(1+\eta)} - 2\Lambda_3^* \Lambda_4^* \daleth^* < 1,$$

then (3) has at least a solution and it is generalized Ulam-Hyers-Rassias stable.

5.2. The infinite delay case

Set $\mathscr{R}' := \mathscr{R}'_{\rho^-} = \{\rho(\rho,\chi) : (\rho,\chi) \in \Psi \times \Bbbk\ \rho(\rho,\chi) \leqslant 0\}$. We always assume that the functions $\rho : \Psi \times \Bbbk \to \mathbb{R}$ and $\rho \in \mathscr{R}' \longmapsto \chi_\rho \in \Bbbk$ are continuous.

In the sequel we will make use of the following hypothesis:

$(C_\aleph)$ There exists a continuous bounded function $L : \mathscr{R}'_{\rho^-} \to (0,\infty)$ such that

$$\|\aleph_\rho\|_\Bbbk \leqslant L(\rho)\|\aleph\|_\Bbbk, \text{ for any } \rho \in \mathscr{R}'.$$

LEMMA 4. ([26]) *If $\chi \in \Omega$, then*

$$\|\chi_\rho\|_\Bbbk = (\Im_3 + L')\|\aleph\|_\Bbbk + \Im_2 \sup_{\tau \in [0,\max\{0,\rho\}]} \|\chi(\tau)\|,$$

where

$$L' = \sup_{\rho \in \mathscr{R}'} L(\rho).$$

As in Theorems 3 and 4, we conclude the following result:

THEOREM 7. *Assume that the hypotheses $(C_\aleph)$, (H_{01}) and (H_2) hold. If*

$$\frac{\kappa^\eta \delta_2{}^*}{\Gamma_q(1+\eta)} + \delta_3{}^* + \varsigma - \varsigma\delta_3{}^* < 1,$$

then the problem (4) has at least one solution defined on $(-\infty,\kappa]$.

THEOREM 8. *Assume that the hypotheses* $(C_{\aleph})$, (H_{02}) *and* (H_4) *hold. If* $\Lambda_4^*\gimel^* < 1$, *and*

$$\Lambda_3^* + 2\Lambda_4^*\gimel^* + \frac{\kappa^\eta \Lambda_2^*\gimel^*}{\Gamma_q(1+\eta)} - 2\Lambda_3^*\Lambda_4^*\gimel^* < 1,$$

then (4) *has a solution and it is generalized Ulam-Hyers-Rassias stable.*

6. Some examples

EXAMPLE 2. Consider the implicit fractional $\frac{1}{4}$-difference equations

$$\begin{cases} {}^cD_{\frac{1}{4}}^{\frac{1}{2}}(\chi(\rho) - \Upsilon(\rho,\chi_\rho) = \wp(\rho,\chi(\rho), {}^cD_{\frac{1}{4}}^{\frac{1}{2}}(\chi(\rho) - \Upsilon(\rho,\chi_\rho)); \ \rho \in [0,1], \\ \chi(\rho) = 2 + \rho^2; \ \rho \in [-2,0], \end{cases} \tag{14}$$

where

$$\wp(\rho,x,y) = \frac{\rho^2}{1+|x|+|y|}\left(e^{-7} + \frac{1}{e^{\rho+5}}\right)(\rho^2 + xt^2 + y); \ \rho \in [0,1], \ x,y \in \mathbb{R},$$

and

$$\Upsilon(\rho,z) = \frac{\rho^4}{1+|z-2|}\left(e^{-7} + \frac{1}{e^{\rho+5}}\right); \ \rho \in [0,1], \ z \in C([-2,0]),$$

The hypothesis (H_1) is satisfied with $\varsigma = 2e^{-5}$. Also, the hypothesis (H_2) is satisfied with $\gimel(\rho) = \rho^2$ and $\delta_1(\rho) = \delta_2(\rho) = \delta_3(\rho) = \left(e^{-7} + \frac{1}{e^{\rho+5}}\right)\rho$. A simple computation show that all conditions of Theorems 1 and 2 are satisfied. Hence, our problem (14) has at least a solution defined on $[-2,1]$, and it is generalized Ulam-Hyers-Rassias stable.

EXAMPLE 3. Consider now the following problem

$$\begin{cases} {}^cD_{\frac{1}{4}}^{\frac{1}{2}}(\chi(\rho) - \Upsilon(\rho,\chi_\rho) = \wp(\rho,\chi(\rho), {}^cD_{\frac{1}{4}}^{\frac{1}{2}}(\chi(\rho) - \Upsilon(\rho,\chi_\rho)); \ \rho \in [0,1], \\ \chi(\rho) = 1 + \rho^2; \ \rho \in (-\infty,0], \end{cases} \tag{15}$$

where

$$\wp(\rho,x,y) = \frac{\rho^2}{1+|x|+|y|}\left(e^{-7} + \frac{1}{e^{\rho+5}}\right)(\rho^2 + xt^2 + y); \ \rho \in [0,1], \ x,y \in \mathbb{R},$$

and

$$\Upsilon(\rho,z) = \frac{\rho^4}{1+z_\rho}\left(e^{-7} + \frac{1}{e^{\rho+5}}\right); \ \rho \in [0,1], \ z \in \Bbbk_\gamma,$$

where

$$\Bbbk_\gamma = \{\chi \in C((-\infty,0],\mathbb{R}) : \lim_{\|\tau\|\to\infty} e^{\gamma\tau}\chi(\tau) \text{ exists in } \mathbb{R}\}.$$

The norm of $\mathbb{k}_\gamma$ is given by

$$\|\chi\|_\gamma = \sup_{\tau\in(-\infty,0]} e^{\gamma\tau}|\chi(\tau)|.$$

Let $\chi : (-\infty,1] \to \mathbb{R}$ such that $\chi_\rho \in \mathbb{k}_\gamma$ for $\rho \in (-\infty,0]$, then

$$\begin{aligned}\lim_{\|\tau\|\to\infty} e^{\gamma\tau}\chi_\rho(\tau) &= \lim_{\|\tau\|\to\infty} e^{\gamma(\tau-\rho)}\chi(\tau)\\ &= e^{-\gamma\rho}\lim_{\|\tau\|\to\infty} e^{\gamma\tau}\chi(\tau) < \infty.\end{aligned}$$

Hence $\chi_\rho \in \mathbb{k}_\gamma$. Finally we prove that

$$\|\chi_\rho\|_\gamma = \Im_2 \sup\{|\chi(\vartheta)| : \vartheta \in [0,\rho]\} + \Im_3 \sup\{\|\chi_\vartheta\|_\gamma : \vartheta \in (-\infty,0]\},$$

where $\Im_2 = \Im_3 = 1$ and $\Im_1 = 1$.

If $\rho+\tau \leqslant 0$ we get

$$\|\chi_\rho\|_\gamma = \sup\{|\chi(\rho)| : \rho \in (-\infty,0]\},$$

and if $\rho+\tau \geqslant 0$, then we have

$$\|\chi_\rho\|_\gamma = \sup\{|\chi(\vartheta)| : \vartheta \in [0,\rho]\}.$$

Thus for all $\rho+\tau \in [0,1]$, we get

$$\|\chi_\rho\|_\gamma = \sup\{|\chi(\vartheta)| : \vartheta \in (-\infty,0]\} + \sup\{|\chi(\vartheta)| : \vartheta \in [0,\rho]\}.$$

Then

$$\|\chi_\rho\|_\gamma = \sup\{\|\chi_\vartheta\|_\gamma : \vartheta \in (-\infty,0]\} + \sup\{|\chi(\vartheta)| : \vartheta \in [0,\rho]\}.$$

$(\mathbb{k}_\gamma, \|\cdot\|_\gamma)$ is a Banach space. We conclude that $\mathbb{k}_\gamma$ is a phase space. Simple computations show that all conditions of Theorems 3 and 4 are satisfied.

EXAMPLE 4. In this example, we consider the following problem

$$\begin{cases} {}^cD^{\frac{1}{2}}_{\frac{1}{4}}(\chi(\rho)-\Upsilon(\rho,\chi_{\rho(\rho,\chi_\rho)})) = \wp(\rho,\chi(\rho), {}^cD^{\frac{1}{2}}_{\frac{1}{4}}(\chi(\rho)-\Upsilon(\rho,\chi_{\rho(\rho,\chi_\rho)})); \ \rho\in[0,1],\\ \chi(\rho) = 2+\rho^2; \ \rho\in[-2,0], \end{cases} \tag{16}$$

where

$$\wp(\rho,x,y) = \frac{\rho^2}{1+|x|+|y|}\left(e^{-7}+\frac{1}{e^{\rho+5}}\right)(\rho^2+xt^2+y); \ \rho\in[0,1],\ x,y\in\mathbb{R},$$

and

$$\Upsilon(\rho,z) = \frac{\rho^4}{1+\|z-\sigma(z(\rho))\|}\left(e^{-7}+\frac{1}{e^{\rho+5}}\right); \ \rho\in[0,1],\ z\in C([-2,0]),$$

where $\sigma \in C(\mathbb{R},[0,2])$,

$$\rho(\rho,\aleph)=\rho-\sigma(\aleph(0)),\ \ (\rho,\aleph)\in\Psi\times C([-2,0],\mathbb{R}).$$

The hypothesis (H_1) is satisfied with $\varsigma = 2e^{-5}$. Also, the hypothesis (H_2) is satisfied with

$$\beth(\rho)=\rho^2 \quad \delta_1(\rho)=\delta_2(\rho)=\delta_3(\rho)=\left(e^{-7}+\frac{1}{e^{\rho+5}}\right)\rho.$$

A simple computation show that all conditions of Theorems 5 and 6 are satisfied.

EXAMPLE 5. Now, we treat the following implicit fractional $\frac{1}{4}$-difference equations

$$\begin{cases} {}^cD^{\frac{1}{2}}_{\frac{1}{4}}(\chi(\rho)-\Upsilon(\rho,\chi_\rho)=\wp(\rho,\chi(\rho),{}^cD^{\frac{1}{2}}_{\frac{1}{4}}(\chi(\rho)-\Upsilon(\rho,\chi_\rho));\ \rho\in[0,1], \\ \chi(\rho)=1+\rho^2;\ \rho\in(-\infty,0], \end{cases} \tag{17}$$

where

$$\wp(\rho,x,y)=\frac{\rho^2}{1+|x|+|y|}\left(e^{-7}+\frac{1}{e^{\rho+5}}\right)(\rho^2+xt^2+y);\ \rho\in[0,1],\ x,y\in\mathbb{R},$$

and

$$\Upsilon(\rho,z)=\frac{\rho^4}{1+|z(\rho-\sigma(\chi(\rho)))|}\left(e^{-7}+\frac{1}{e^{\rho+5}}\right);\ \rho\in[0,1],\ z\in\Bbbk,$$

where $\sigma\in C(\mathbb{R},[0,\infty))$ and $\Bbbk_\gamma$ is the phase space defined in Example 2.

Simple computations show that from the Theorem 3, the problem (17) has at least one solution on $(-\infty,1]$, and the Theorem 4 implies the generalized Ulam-Hyers-Rassias stability.

Conclusion

In this paper, we have presented an analysis of the existence of solutions for a class of implicit neutral Caputo fractional q-difference equations with finite, infinite, and state-dependent delays. Our approach utilizes Krasnoselskii's fixed point theorem to obtain the results. Furthermore, we have illustrated the practical applications of our results through specific examples. We hope that our analysis can inspire further research in this area and contribute to the development of more complex q-difference systems. In our future work, we aim to expand the study to higher order differential equations, with different types of conditions and impulsive effects.

Declarations

Ethical approval. This article does not contain any studies with human participants or animals performed by any of the authors.

Competing interests. It is declared that authors has no competing interests.

Author's contributions. The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

Funding. Not available.

Availability of data and materials. Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

REFERENCES

[1] S. ABBAS, W. A. ALBARAKATI, M. BENCHOHRA AND S. SIVASUNDARAM, *Dynamics and stability of Fredholm type fractional order Hadamard integral equations*, J. Nonlinear Stud. **22** (4) (2015), 673–686.

[2] S. ABBAS, M. BENCHOHRA, J. R. GRAEF AND J. HENDERSON, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.

[3] S. ABBAS, M. BENCHOHRA AND G. M. N'GUÉRÉKATA, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[4] S. ABBAS, M. BENCHOHRA AND G. M. N'GUÉRÉKATA, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[5] M. S. ABDO, *Boundary value problem for fractional neutral differential equations with infinite delay*, Abhath Journal of Basic and Applied Sciences. **1** (1) (2022), 1–18.

[6] M. S. ABDO, S. K. PANCHAL, *Weighted fractional neutral functional differential equations*, J. Sib. Fed. Univ. Math. Phys. **11** (5) (2018), 535–549, https://doi.org/10.17516/1997-1397-2018-11-5-535-549.

[7] M. S. ABDO, S. K. PANCHAL, H. A. WAHASH, *Ulam-Hyers-Mittag-Leffler stability for a ψ-Hilfer problem with fractional order and infinite delay*, Results Appl. Math. **7** (2020), 100115, https://doi.org/10.1016/j.rinam.2020.100115

[8] C. R. ADAMS, *On the linear ordinary q-difference equation*, Annals Math. **30** (1928), 195–205.

[9] R. AGARWAL, *Certain fractional q-integrals and q-derivatives*, Proc. Camb. Philos. Soc. **66** (1969), 365–370.

[10] B. AHMAD, *Boundary value problem for nonlinear third order q-difference equations*, Electron. J. Differential Equations **2011** (2011), no. 94, pp 1–7.

[11] B. AHMAD, S. K. NTOUYAS AND L. K. PURNARAS, *Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations*, Adv. Difference Equ. **2012**, 2012:140.

[12] B. AHMAD, J. J. NIETO, A. ALSAEDI, *Existence of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions*, Acta Math. Sci. **31** (2011) 2122–2130.

[13] A. ANGURAJ, P. KARTHIKEYAN, J. J. TRUJILLO, *Existence of solutions to fractional mixed integrodifferential equations with nonlocal initial condition*, Adv. Differential Equations (2011) 1–12, ID690653.

[14] D. BALEANU, K. DIETHELM, E. SCALAS AND J. J. TRUJILLO, *Fractional Calculus Models and Numerical Methods*, World Scientific, Singapore, 2012.

[15] D. BALEANU, Z. B. GUVENC, J. A. TENREIRO MACHADO (eds.), *New Trends in Nanotechnology and Fractional Calculus*, Applications, Springer, Dordrecht, 2010.

[16] M. BENCHOHRA, F. BERHOUN AND G. M. N'GUÉRÉKATA, *Bounded solutions for fractional order differential equations on the half-line*, Bull. Math. Anal. Appl. **146** (4) (2012), 62–71.

[17] M. BENCHOHRA, E. KARAPINAR, J. E. LAZREG AND A. SALIM, *Advanced Topics in Fractional Differential Equations: A Fixed Point Approach*, Springer, Cham, 2023.

[18] M. BENCHOHRA, E. KARAPINAR, J. E. LAZREG AND A. SALIM, *Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives*, Springer, Cham, 2023.

[19] R. D. CARMICHAEL, *The general theory of linear q-difference equations*, American J. Math. **34** (1912), 147–168.

[20] M. EL-SHAHED, H. A. HASSAN, *Positive solutions of q-difference equation*, Proc. Amer. Math. Soc. **138** (2010), 1733–1738.

[21] S. ETEMAD, S. K. NTOUYAS AND B. AHMAD, *Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders*, Mathematics **7** 659 (2019), 1–15.

[22] J. K. HALE, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[23] J. HALE AND J. KATO, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac., **21** (1978), 11–41.

[24] J. K. HALE, K. R. MEYER, *A class of functional equations of neutral type*, Mem. Amer. Math. Soc. **76** (1967), 1–65.

[25] J. K. HALE AND S. M. VERDUYN LUNEL, *Introduction to Functional Differential Equation*, Applied Mathematical Sciences 99, Springer-Verlag, New York, 1993.

[26] E. HERNÁNDEZ, A. PROKOPCZYK AND L. LADEIRA, *A note on partial functional differential equations with state-dependent delay*, Nonlinear Anal. Real World Appl. **7** (2006), 510–519.

[27] Y. HINO, S. MURAKAMI AND T. NAITO, *Functional Differential Equations with Unbounded Delay*, Springer-Verlag, Berlin, 1991.

[28] V. KAC AND P. CHEUNG, *Quantum Calculus*, Springer, New York, 2002.

[29] A. A. KILBAS, *Hadamard-type fractional calculus*, J. Korean Math. Soc. **38** (6) (2001) 1191–1204.

[30] V. KOLMANOVSKII AND A. MYSHKIS, *Introduction to the Theory and Application of Functional-Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[31] N. LALEDJ, A. SALIM, J. E. LAZREG, S. ABBAS, B. AHMAD AND M. BENCHOHRA, *On implicit fractional q-difference equations: Analysis and stability*, Math. Methods Appl. Sci. **45** (17) (2022), 10775-10797, https://doi.org/10.1002/mma.8417.

[32] K. LIU, J. WANG AND D. O'REGAN, *Ulam-Hyers-Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations*, Adv Differ Equ. **2019** (2019), 50.

[33] D. LUO, Z. LUO, H. QIU, *Existence and Hyers-Ulam stability of solutions for a mixed fractional-order nonlinear delay difference equation with parameters*, Math. Probl. Eng. **2020**, 9372406 (2020).

[34] P. M. RAJKOVIC, S. D. MARINKOVIC AND M. S. STANKOVIC, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math. **1** (2007), 311–323.

[35] P. M. RAJKOVIC, S. D. MARINKOVIC AND M. S. STANKOVIC, *On q-analogues of Caputo derivative and Mittag-Leffler function*, Fract. Calc. Appl. Anal., **10** (2007), 359–373.

[36] I. A. RUS, *Ulam stability of ordinary differential equations*, Studia Univ. Babes-Bolyai, Math. **LIV** (4) (2009), 125–133.

[37] A. SALIM, S. ABBAS, M. BENCHOHRA AND E. KARAPINAR, *A Filippov's theorem and topological structure of solution sets for fractional q-difference inclusions*, Dynam. Systems Appl. **31** (2022), 17–34, https://doi.org/10.46719/dsa202231.01.02.

[38] A. SALIM, S. ABBAS, M. BENCHOHRA AND J. E. LAZREG, *Caputo fractional q-difference equations in Banach spaces*, J. Innov. Appl. Math. Comput. Sci. **3** (1) (2023), 1–14, https://doi.org/10.58205/jiamcs.v3i1.67.

[39] A. SALIM, J. E. LAZREG, B. AHMAD, M. BENCHOHRA AND J. J. NIETO, *A study on k-generalized ψ-Hilfer derivative operator*, Vietnam J. Math. (2022).

[40] A. SALIM, M. BENCHOHRA, J. E. LAZREG AND G. N'GUÉRÉKATA, *Existence and k-Mittag-Leffler-Ulam-Hyers stability results of k-generalized ψ-Hilfer boundary value problem*, Nonlinear Studies. **29** (2022), 359–379.

[41] Y. ZHOU, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

(Received April 5, 2023)

Abdellatif Benchaib
Department of Mathematics
University of Tlemcen
Algeria
e-mail: abdelben0402@gmail.com

Abdelkrim Salim
Faculty of Technology
Hassiba Benbouali University of Chlef
P.O. Box 151 Chlef 02000, Algeria
e-mail: salim.abdelkrim@yahoo.com

Saïd Abbas
Department of Electronics
Tahar Moulay University of Saïda
P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria
e-mail: abbasmsaid@yahoo.fr

Mouffak Benchohra
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbès
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
e-mail: benchohra@yahoo.com