Differential Equations & Applications
Volume 16, Number 1 (2024), 71–83
doi:10.7153/dea-2024-16-04

BLOW–UP SOLUTIONS FOR NON–SCALE–INVARIANT NONLINEAR SCHRÖDINGER EQUATION IN ONE DIMENSION

MASARU HAMANO*, MASAHIRO IKEDA AND SHUJI MACHIHARA

(*Communicated by C.-L. Tang*)

Abstract. In this paper, we consider the mass-critical nonlinear Schrödinger equation in one dimension. Ogawa–Tsutsumi [Proc. Amer. Math. Soc. **111** (1991), no. 2, 487–496] proved a blow-up result for negative energy solution by using a scaling argument for initial data. In general, a equation with a linear potential does not have a scale invariant, so the method by Ogawa–Tsutsumi cannot be used directly to that. In this paper, we prove a blow-up result for the equation with the linear potential by modifying the argument of Ogawa–Tsutsumi.

1. Introduction

1.1. Nonlinear Schrödinger equation

We consider the following mass-critical nonlinear Schrödinger equations:

$$i\partial_t u + \partial_x^2 u - Vu = -|u|^4 u, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{NLS$_V$}$$

In particular, we deal with the Cauchy problem of (NLS$_V$) with initial data

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}. \tag{IC}$$

DEFINITION 1.1. (Solution) Let $I \subset \mathbb{R}$ be a nonempty time interval including 0. We say that a function $u : I \times \mathbb{R} \longrightarrow \mathbb{C}$ is a solution to (NLS$_V$) with (IC) if $u \in (C_t \cap L^\infty_{t,\mathrm{loc}})(I; H^1_x(\mathbb{R}))$ and the Duhamel formula

$$u(t,x) = e^{it\partial_x^2} u_0(x) + i \int_0^t e^{i(t-s)\partial_x^2} (|u|^4 u - Vu)(s,x)\, ds$$

holds for any $t\,(\in I)$, where $H^1(\mathbb{R})$ is a usual inhomogeneous Sobolev space of order 1.

Mathematics subject classification (2020): Primary 35Q55; Secondary 35B44.
Keywords and phrases: Nonlinear Schrödinger equation, linear potential, blow-up.
* Corresponding author.

The equation (NLS$_V$) with $V=0$

$$i\partial_t u+\partial_x^2 u=-|u|^4u, \qquad (t,x)\in\mathbb{R}\times\mathbb{R} \tag{NLS$_0$}$$

is invariant by the following scaling:

$$u(t,x)\mapsto u_{[\lambda]}(t,x):=\lambda^{\frac{1}{2}}u(\lambda^2t,\lambda x), \qquad (\lambda>0). \tag{1.1}$$

From the transformation (1.1), the initial data u_0 changes to

$$u_0\mapsto u_{0,\{\lambda\}}:=\lambda^{\frac{1}{2}}u_0(\lambda x), \qquad (\lambda>0). \tag{1.2}$$

THEOREM 1.2. (Local well-posedness of (NLS$_0$), [3, 10, 17]) *Let $V=0$. For any $u_0\in H^1(\mathbb{R})$, there exist $T_{\min}\in[-\infty,0)$ and $T_{\max}\in(0,\infty]$ such that (NLS$_0$) with (IC) has a unique solution*

$$u\in(C_t\cap L^\infty_{t,loc})((T_{\min},T_{\max});H^1_x(\mathbb{R})).$$

For each compact interval $I\subset(T_{\min},T_{\max})$, the mapping $H^1(\mathbb{R})\ni u_0\mapsto u\in C_t(I;H^1_x(\mathbb{R}))$ is continuous. Moreover, the solution u has the following blow-up alternative: If $T_{\min}>-\infty$ (resp. $T_{\max}<\infty$), then

$$\lim_{t\searrow T_{\min}\,(resp. t\nearrow T_{\max})}\|u(t)\|_{H^1_x}=\infty.$$

Furthermore, the solution u preserves its mass $M[u(t)]$ and energy $E_V[u(t)]$ with respect to time t, where they are defined as follows:

$$\begin{aligned}
&\textit{(Mass)} && M[f]:=\|f\|_{L^2}^2,\\
&\textit{(Energy)} && E_V[f]:=\frac{1}{2}\|\partial_x f\|_{L^2}^2+\frac{1}{2}\int_{\mathbb{R}}V(x)|f(x)|^2dx-\frac{1}{6}\|f\|_{L^6}^6.
\end{aligned}$$

Since $\|u_{0,\{\lambda\}}\|_{L^2}=\|u_0\|_{L^2}$ holds for the scaling (1.2), (NLS$_V$) is called L^2-critical or mass-critical.

In the case $V=0$, Ogawa–Tsutsumi [20] proved the following result using the scaling (1.2):

THEOREM 1.3. *Let $V=0$ and $u_0\in H^1(\mathbb{R})$. If $E_0[u_0]<0$, then the solution u to (NLS$_0$) with (IC) blows up.*

Notice that the classic condition $xu_0\in L^2(\mathbb{R})$ (see [11]) is not required in Theorem 1.3. Since (NLS$_V$) with $V\neq0$ is not scale invariant in general, we can not apply the argument in [20] directly to (NLS$_V$). In particular, in this paper we consider the equations (NLS$_V$) with $V=\frac{\gamma}{|x|^\mu}$:

$$i\partial_t u+\partial_x^2 u-\frac{\gamma}{|x|^\mu}u=-|u|^4u, \qquad (\gamma>0,\ 0<\mu<1). \tag{NLS$_\gamma$}$$

For simplicity, we use (NLS_γ) and E_γ as ($\mathrm{NLS}_{\frac{\gamma}{|x|^\mu}}$) and $E_{\frac{\gamma}{|x|^\mu}}$ respectively. To prove a similar result for (NLS_γ) with Theorem 1.3, we give an alternative proof without the scaling argument for initial data. We note that we can see the local well-posedness of (NLS_γ) in [6]. One of our principal results is the following theorem.

THEOREM 1.4. *Let $\gamma > 0$, $0 < \mu < 1$, and let $u_0 \in H^1(\mathbb{R})$. If $E_\gamma[u_0] < 0$, then the solution u to (NLS_γ) with (IC) blows up.*

REMARK 1.5. Dinh [8] showed the blow-up result for (NLS_γ) under $\gamma > 0$, $u_0 \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}; |x|^2 dx)$, and $E_\gamma[u_0] < 0$. That is, Theorem 1.4 removes the condition $u_0 \in |x|^{-1} L^2(\mathbb{R})$ in [8].

The following corollary is also one of our principal results and holds by the same argument with Theorem 1.4. The potential $V(x) = \frac{\gamma}{|x|^\mu}$ $(\gamma > 0,\ 0 < \mu < 1)$ satisfies all of conditions in Corollary 1.6, so Theorem 1.4 is a special case of Corollary 1.6.

COROLLARY 1.6. *Let $u_0 \in H^1(\mathbb{R})$. We assume that V satisfies the following (i) $\sim$ (iii):*

(i) (NLS_V) *is locally well-posed.*

(ii) For the solution u to (NLS_V) *with* (IC), *$w(x) := \int_0^x \varphi(s)ds$, and $\varphi \in W^{3,\infty}(\mathbb{R})$, we have*

$$\frac{d^2}{dt^2}\int_{\mathbb{R}} w(x)|u(t,x)|^2 dx = 4\int_{\mathbb{R}} w''(x)|\partial_x u(t,x)|^2 dx - \frac{4}{3}\int_{\mathbb{R}} w''(x)|u(t,x)|^6 dx - \int_{\mathbb{R}} w^{(4)}(x)|u(t,x)|^2 dx - 2\int_{\mathbb{R}} w'(x)V'(x)|u(t,x)|^2 dx. \tag{1.3}$$

(iii)

$$-R\mathscr{X}'\left(\frac{x}{R}\right)V'(x) - 4V(x) \leqslant 0 \tag{1.4}$$

holds for any $R > 0$ and any $x \in \mathbb{R}$, where $\mathscr{X}$ is defined as (3.1).

If $E_V[u_0] < 0$, then the solution u to (NLS_V) *with* (IC) *blows up.*

REMARK 1.7. For example, if V is a real-valued function and $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, then ($\mathrm{NLS}_V$) is locally well-posed (see [6, Theorem 4.3.1]). If V is a real-valued function and $V,\ V' \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, then (1.3) holds (see [15, Lemma 3.1]). Since $\mathscr{X}' \leqslant 0$ for $x \leqslant 0$ and $\mathscr{X}' \geqslant 0$ for $x \geqslant 0$, if a non-negative potential V satisfies $V'(x) \leqslant 0$ $(x \leqslant 0)$ and $V'(x) \geqslant 0$ $(x \geqslant 0)$ then (1.4) holds.

In this paper, we also deal with the following equation with a delta potential:

$$i\partial_t u + \partial_x^2 u - \gamma\delta u = -|u|^4 u, \qquad (\gamma > 0). \tag{NLS$_{\gamma\delta}$}$$

The Schrödinger operator $H_{\gamma\delta} := -\partial_x^2 + \gamma\delta$ has a domain

$$\mathscr{D}(H_{\gamma\delta}) := \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : \partial_x f(0+) - \partial_x f(0-) = \gamma f(0)\}$$

and satisfies

$$H_{\gamma\delta} f = -\partial_x^2 f, \qquad f \in \mathscr{D}(H_{\gamma\delta}).$$

A local well-posedness result can be seen in [6, Theorem 3.7.1] and [9, Section 2].

THEOREM 1.8. *Let $\gamma > 0$ and let $u_0 \in H^1(\mathbb{R})$. If $E_{\gamma\delta}[u_0] < 0$, then the solution u to (NLS$_{\gamma\delta}$) with (IC) blows up, where the energy $E_{\gamma\delta}$ is defined as*

$$E_{\gamma\delta}[f] := \frac{1}{2}\|\partial_x f\|_{L^2}^2 + \frac{\gamma}{2}|f(0)|^2 - \frac{1}{6}\|f\|_{L^6}^6.$$

The blow-up result for following Schrödinger equation on the star graph $\mathscr{G}$ with J-edges can be also gotten.

$$\begin{cases} i\partial_t \boldsymbol{u} + \Delta_{\mathscr{G}} \boldsymbol{u} = -|\boldsymbol{u}|^4 \boldsymbol{u}, & (t,x) \in \mathbb{R} \times (0,\infty), \\ \boldsymbol{u}(0,x) := \boldsymbol{u_0}(x) := (u_j(0,x))_{j=1}^J, & x \in (0,\infty), \end{cases} \tag{NLS$_{\mathscr{G}}$}$$

where $J \geqslant 1$, $\boldsymbol{u}(t,x) = (u_j(t,x))_{j=1}^J : \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{C}^J$, and $|\boldsymbol{u}|^4 \boldsymbol{u} := (|u_j|^4 u_j)_{j=1}^J$. The Schrödinger operator $-\Delta_{\mathscr{G}}$ has a domain

$$\mathscr{D}(-\Delta_{\mathscr{G}}) := \{\boldsymbol{f} \in D(\mathscr{G}) : A\boldsymbol{f}(0+) + B\partial_x \boldsymbol{f}(0+) = \boldsymbol{0}\}$$

and satisfies

$$\Delta_{\mathscr{G}} \boldsymbol{f} = (\partial_x^2 f_j)_{j=1}^J, \qquad \boldsymbol{f} \in \mathscr{D}(-\Delta_{\mathscr{G}}),$$

where $D(\mathscr{G}) = \bigoplus_{j=1}^J D(0,\infty)$, $D(0,\infty)$ is a set of functions $f \in H^2(0,\infty)$ satisfying that f and $\partial_x f$ are absolutely continuous, and A, B are complex-valued $n \times n$ matrices satisfying the conditions

(A1) $J \times (2J)$ matrix (A,B) has maximal rank, that is, $\operatorname{rank}(A,B) = J$.

(A2) AB^* is self-adjoint, that is, $AB^* = (AB^*)^*$, where X^* denotes the adjoint of the matrix X and is defined as $X^* := \overline{X}^T$.

Under the assumption $(A1)$ and $(A2)$, the Laplacian $\Delta_{\mathscr{G}}$ is self-adjoint on $L^2(\mathscr{G})$ (see [18, 19]) and hence, $e^{it\Delta_{\mathscr{G}}}$ can be defined as the unitary operator on $L^2(\mathscr{G})$ by the Stone's theorem. Here, we introduce typical boundary condition.

(a) Kirchhoff boundary condition: Let A and B be

$$A = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{1.5}$$

For such A and B, $A\boldsymbol{f}(+0) + B\partial_x\boldsymbol{f}(+0) = 0$ implies that $f_i(+0) = f_j(+0)$ for any $i, j \in \{1, 2, \ldots, J\}$ and $\sum_{j=1}^{J} \partial_x f_j(+0) = 0$. This is called Kirchhoff boundary condition.

(b) Dirac delta boundary condition: Let $\gamma \neq 0$ and A, B be

$$A = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -\gamma & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{1.6}$$

For such A and B, $A\boldsymbol{f}(+0) + B\partial_x\boldsymbol{f}(+0) = 0$ implies that $f_i(+0) = f_j(+0)$ for any $i, j \in \{1, 2, \ldots, J\}$ and $\sum_{j=1}^{J} \partial_x f_j(+0) = \gamma f_1(+0)$. This is called the Dirac delta boundary condition.

(c) δ' boundary condition: Let $\gamma \in \mathbb{R}$ and A, B be

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -\gamma & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{1.7}$$

For such A and B, $A\boldsymbol{f}(+0) + B\partial_x\boldsymbol{f}(+0) = 0$ implies that $\partial_x f_i(+0) = \partial_x f_j(+0)$ for any $i, j \in \{1, 2, \ldots, J\}$ and $\sum_{j=1}^{J} f_j(+0) = \gamma \partial_x f_1(+0)$. This is called δ' boundary condition.

Lebesgue space and Sobolev space on the star graph $\mathscr{G}$ is defined respectively as

$$L^p(\mathscr{G}) := \bigoplus_{j=1}^{J} L^p(0,\infty), \quad H^s(\mathscr{G}) := \bigoplus_{j=1}^{J} H^s(0,\infty) \ \text{ for } \ s = 1, 2$$

with a norm

$$\|\boldsymbol{f}\|_{L^p(\mathscr{G})} := \begin{cases} \left(\sum_{j=1}^{J} \|f_j\|_{L^p(0,\infty)}^p\right)^{\frac{1}{p}}, & (1 \leqslant p < \infty), \\ \max_{1\leqslant j\leqslant J} \|f_j\|_{L^\infty(0,\infty)}, & (p = \infty), \end{cases}$$

$$\|\boldsymbol{f}\|_{H^s(\mathscr{G})}^2 := \sum_{j=1}^{J} \|f_j\|_{H^s(0,\infty)}^2 \text{ for } s = 1,2.$$

In addition, we set initial data space

$$H_c^1(\mathscr{G}) := \{\boldsymbol{f} \in H^1(\mathscr{G}) : f_1(0) = \ldots = f_J(0)\}$$

for ($\mathrm{NLS}_{\mathscr{G}}$) with (1.5) or (1.6). Local well-posedness results of ($\mathrm{NLS}_{\mathscr{G}}$) were cited in [1, 2, 5, 13].

THEOREM 1.9. *Let (A,B) be one of* (1.5)*,* (1.6)*, or* (1.7)*. Assume that either $\boldsymbol{u_0} \in H_c^1(\mathscr{G})$ if (A,B) satisfies* (1.5) *or* (1.6) *or $\boldsymbol{u_0} \in H^1(\mathscr{G})$ if (A,B) satisfies* (1.7)*. We suppose that $\gamma > 0$ when (A,B) satisfies* (1.6) *or* (1.7)*. If $E_{\mathscr{G}}[\boldsymbol{u_0}] < 0$, then the solution $\boldsymbol{u}$ to* ($\mathrm{NLS}_{\mathscr{G}}$) *blows up, where the energy $E_{\mathscr{G}}$ is defined as*

$$E_{\mathscr{G}}[\boldsymbol{f}] := \frac{1}{2}\|\partial_x \boldsymbol{f}\|_{L^2(\mathscr{G})}^2 - \frac{1}{6}\|\boldsymbol{f}\|_{L^6(\mathscr{G})}^6 + \frac{1}{2}P(\boldsymbol{f}),$$

where

$$P(\boldsymbol{f}) := \begin{cases} 0, & (\textit{if } (A,B) \textit{ is } (1.5).), \\ \gamma|f_1(+0)|^2, & (\textit{if } (A,B) \textit{ is } (1.6).), \\ \frac{1}{\gamma}\left|\sum_{j=1}^{J} f_j(+0)\right|^2, & (\textit{if } (A,B) \textit{ is } (1.7).). \end{cases}$$

REMARK 1.10. When $\gamma < 0$ in Theorem 1.4, the condition $E_\gamma[u_0] < 0$ does not guarantee the blow-up for the equation (NLS_γ). Indeed, the standing wave solution $u(t,x) = e^{i\omega t}Q_{\omega,\gamma}(x)$ to (NLS_γ) was gotten in [7], where $Q_{\omega,\gamma}$ satisfies

$$-\omega\phi + \partial_x^2\phi - \frac{\gamma}{|x|^\mu}\phi = -|\phi|^4\phi.$$

The standing wave solution is time global and has negative energy. For the equations ($\mathrm{NLS}_{\gamma\delta}$) and ($\mathrm{NLS}_{\mathscr{G}}$), we can say the same thing with (NLS_γ). The standing wave solution of ($\mathrm{NLS}_{\gamma\delta}$) or ($\mathrm{NLS}_{\mathscr{G}}$) can be seen in [1, 9, 12, 14].

Idea of the proof. Ogawa–Tsutsumi [20] used a localized virial identity (Proposition 2.2) with a weighted function $\mathscr{X}$ (see (3.1) below for the definition). The function $\mathscr{X}$ is equal to x^2 on $\{x : |x| \leqslant 1\}$ and Ogawa–Tsutsumi collect initial data into

$\{x : |x| \leqslant 1\}$ by the scaling (1.2). However, the equations (NLS_γ), ($\mathrm{NLS}_{\gamma\delta}$), and ($\mathrm{NLS}_{\mathscr{G}}$) are not scale invariant. So, we replace $\mathscr{X}$ with $\mathscr{X}_R$ and spread a domain (where the weighted function is equal to x^2) by taking sufficiently large R. We note that radial functions have estimate

$$\|f\|_{L^{p+1}(|x|\geqslant R)}^{p+1} \lesssim R^{-\frac{(d-1)(p-1)}{2}} \|f\|_{L^2(|x|\geqslant R)}^{\frac{p+3}{2}} \|\nabla f\|_{L^2(|x|\geqslant R)}^{\frac{p-1}{2}}$$

for spatial dimension d. Since the inequality gives us decay estimate of $\|f\|_{L^{p+1}(|x|\geqslant R)}$ as $R \to \infty$ for $d \geqslant 2$, the weighted function $\mathscr{X}_R$ is often utilized in $d \geqslant 2$. However, we cannot get the decay in $d = 1$. In this paper, we apply $\mathscr{X}_R$ to the proof and see that it performs effectively also in $d = 1$.

The organization of the rest of this paper is as follows: In Section 2, we prepare some notations and tools. In Section 3, we give an alternative proof of Theorem 1.3. In Section 4, we prove a blow-up result of (NLS_γ), ($\mathrm{NLS}_{\gamma\delta}$), and ($\mathrm{NLS}_{\mathscr{G}}$) (Theorem 1.4, 1.8, and 1.9) by using the alternative proof.

2. Preliminary

In this section, we define some notations and collect some tools.

2.1. Notations and definitions

For $1 \leqslant p \leqslant \infty$, $L^p(\mathbb{R})$ denotes the usual Lebesgue space. $H^1(\mathbb{R})$ and $W^{s,\infty}(\mathbb{R})$ ($s \in \mathbb{N}$) denote the usual Sobolev spaces. If a space domain is not specified, then x-norm is taken over R. That is, $\|f\|_{L^p} = \|f\|_{L^p(\mathbb{R})}$.

2.2. Some tools

The following lemma is given in [20, Lemma 2.1].

LEMMA 2.1. *Let $f \in H^1(\mathbb{R})$ and $g \in W^{1,\infty}(\mathbb{R})$ be a real-valued function. Then, we have*

$$\|fg\|_{L^\infty(|x|\geqslant R)} \leqslant \|f\|_{L^2(|x|\geqslant R)}^{\frac{1}{2}} \left\{2\|g^2\partial_x f\|_{L^2(|x|\geqslant R)} + \|f\partial_x(g^2)\|_{L^2(|x|\geqslant R)}\right\}^{\frac{1}{2}}$$

for any $R > 0$.

The next proposition is seen in [20, Lemma 2.2].

PROPOSITION 2.2. (Localized virial identity I) *We assume that $\varphi \in W^{3,\infty}(\mathbb{R})$ has a compact support. If we define*

$$I_w(t) := \int_{\mathbb{R}} w(x)|u(t,x)|^2 dx$$

for $w := \int_0^x \varphi(y)dy$ *and the solution* $u(t)$ *to* (NLS$_0$), *then we have*

$$I'_w(t) = 2Im\int_{\mathbb{R}} w'(x)\overline{u(t,x)}\partial_x u(t,x)dx,$$
$$I''_w(t) = 4\int_{\mathbb{R}} w''(x)|\partial_x u(t,x)|^2dx - \frac{4}{3}\int_{\mathbb{R}} w''(x)|u(t,x)|^6dx - \int_{\mathbb{R}} w^{(4)}(x)|u(t,x)|^2dx.$$

3. An alternative proof of Theorem 1.3

We define a smooth odd function ζ on $\mathbb{R}$ satisfying

$$\zeta(s) := \begin{cases} 2s & (0 \leqslant s \leqslant 1), \\ 2[s-(s-1)^3] & (1 \leqslant s \leqslant 1+1/\sqrt{3}), \\ \zeta'(s) < 0 & (1+1/\sqrt{3} < s < 2), \\ 0 & (2 \leqslant s). \end{cases}$$

For the function ζ, we set the following functions:

$$\mathscr{X}(x) := \int_0^x \zeta(s)ds, \qquad \mathscr{X}_R(x) := R^2\mathscr{X}\left(\frac{x}{R}\right). \tag{3.1}$$

PROPOSITION 3.1. *Let* $V = 0$ *and* $u_0 \in H^1(\mathbb{R})$. *If the solution* $u \in C_t([0,T_{\max});H_x^1(\mathbb{R}))$ *to* (NLS$_0$) *with* (IC) *satisfies*

$$\|u(t)\|_{L^2(|x|\geqslant R)} \leqslant \left(\frac{3}{8}\right)^{\frac{1}{4}} =: a_0, \tag{3.2}$$

then we have

$$I''_{\mathscr{X}_R}(t) \leqslant -2\widetilde{\eta} := 16E_0[u_0] + 2\eta,$$

where

$$\eta := \frac{4}{3R^2}\left(\sqrt{6} + \frac{\|\zeta''\|_{L^\infty(1+1/\sqrt{3}\leqslant|x|\leqslant 2)}}{2}\right)^2\|u_0\|_{L^2}^6 + \frac{\|\zeta^{(3)}\|_{L^\infty(1\leqslant|x|\leqslant 2)}}{2R^2}\|u_0\|_{L^2}^2.$$

Proof. Applying Proposition 2.2 with $w = \mathscr{X}_R$ given in (3.1), we have

$$I''_{\mathscr{X}_R}(t) = 16E_0[u_0] - \int_{\mathbb{R}} g_R(x)^4\left[4|\partial_x u(t,x)|^2 - \frac{4}{3}|u(t,x)|^6\right]dx \tag{3.3}$$
$$- \int_{\mathbb{R}} \frac{1}{R^2}\mathscr{X}^{(4)}\left(\frac{x}{R}\right)|u(t,x)|^2dx$$

by a simple calculation, where g_R is defined as

$$g_R(x) := \left\{2 - \mathscr{X}''\left(\frac{x}{R}\right)\right\}^{\frac{1}{4}}.$$

By Lemma 2.1, we have

$$\begin{aligned}\int_{\mathbb{R}} g_R(x)^4|u(t,x)|^6dx &= \int_{|x|\geqslant R} g_R(x)^4|u(t,x)|^6dx\\ &\leqslant \|u\|^2_{L^2(|x|\geqslant R)}\|g_R u\|^4_{L^\infty(|x|\geqslant R)}\\ &\leqslant \|u\|^4_{L^2(|x|\geqslant R)}\left\{2\|g_R^2\partial_x u\|_{L^2(|x|\geqslant R)}+\|u\partial_x(g_R^2)\|_{L^2(|x|\geqslant R)}\right\}^2\\ &\leqslant 8\|u\|^4_{L^2(|x|\geqslant R)}\|g_R^2\partial_x u\|^2_{L^2(|x|\geqslant R)}+2\|u\|^6_{L^2(|x|\geqslant R)}\|\partial_x(g_R^2)\|^2_{L^\infty(|x|\geqslant R)}.\end{aligned} \tag{3.4}$$

By the simple calculation, we have

$$|\partial_x(g_R(x)^2)|\begin{cases}=0, & (0\leqslant |x/R|\leqslant 1,\ 2\leqslant |x/R|),\\ \leqslant \sqrt{6}/R, & (1\leqslant |x/R|\leqslant 1+1/\sqrt{3}),\\ \leqslant \dfrac{1}{2R}\|\zeta''\|_{L^\infty(1+1/\sqrt{3}\leqslant|x|\leqslant 2)}, & (1+1/\sqrt{3}<|x/R|<2).\end{cases} \tag{3.5}$$

Therefore, it follows from (3.3), (3.4), and (3.5) that

$$\begin{aligned}I''_{\mathscr{X}_R}(t)\leqslant 16E_0[u_0]&-4\left\{1-\frac{8}{3}\|u\|^4_{L^2(|x|\geqslant R)}\right\}\int_{|x|\geqslant R}\left\{2-\mathscr{X}''\left(\frac{x}{R}\right)\right\}|\partial_x u(t,x)|^2dx\\ &+\frac{8}{3R^2}\left(\sqrt{6}+\frac{\|\zeta''\|_{L^\infty(1+1/\sqrt{3}\leqslant|x|\leqslant 2)}}{2}\right)^2\|u\|^6_{L^2(|x|\geqslant R)}\\ &+\frac{\|\zeta^{(3)}\|_{L^\infty(1\leqslant|x|\leqslant 2)}}{R^2}\|u\|^2_{L^2(|x|\geqslant R)},\end{aligned}$$

which completes the proof by (3.2) and the mass conservation. □

Proof. [An alternative proof of theorem 1.3] We consider only positive time. We assume for contradiction that u exists globally in positive time direction.

We take sufficiently large $R>0$ satisfying that

$$\widetilde{\eta}>0,$$
$$\frac{1}{R}\left(\int_{\mathbb{R}}\mathscr{X}_R(x)|u_0(x)|^2dx\right)^{\frac{1}{2}}\left(1+\frac{4}{\widetilde{\eta}}\|\partial_x u_0\|^2_{L^2}\right)^{\frac{1}{2}}\leqslant \frac{1}{2}a_0, \tag{3.6}$$

where $\widetilde{\eta}$ and a_0 are given in Proposition 3.1. We note that $\widetilde{\eta}\longrightarrow -8E_0[u_0]>0$ as $R\to\infty$ and

$$\frac{1}{R^2}\int_{\mathbb{R}}\mathscr{X}_R(x)|u_0(x)|^2dx=\int_{\mathbb{R}}\mathscr{X}\left(\frac{x}{R}\right)|u_0(x)|^2dx=\int_{\mathbb{R}}\int_0^{x/R}\zeta(s)ds|u_0(x)|^2dx\longrightarrow 0$$

as $R\to\infty$ by the dominated convergence theorem. We prove that $u(t)$ satisfies (3.2) for any $0\leqslant t<\infty$. We note that it follows from (3.6), $\widetilde{\eta}>0$, and

$$\mathscr{X}_R(x)\geqslant \mathscr{X}_R(R)=R^2$$

for $|x| \geqslant R$ that

$$\|u_0\|_{L^2(|x|\geqslant R)} \leqslant \frac{1}{2}a_0. \tag{3.7}$$

Here, we define t_0 as

$$t_0 := \sup\{t > 0 : \|u(s)\|_{L^2(|x|\geqslant R)} \leqslant a_0 \text{ for any } 0 \leqslant s < t\}.$$

By (3.7) and the continuity of $\|u(t)\|_{L^2}$, we note $t_0 > 0$. We recall that u is a positive time global solution. Let us prove that $t_0 = \infty (= T_{\max})$. If we assume that $0 < t_0 < \infty (= T_{\max})$, we have $\|u(t_0)\|_{L^2(|x|\geqslant R)} = a_0$ by the continuity of $\|u(t)\|_{L^2}$. Then, the solution $u(t)$ satisfies (3.2) for any $0 \leqslant t \leqslant t_0$, so it follows from Proposition 3.1 that

$$I''_{\mathscr{X}_R}(\tau) \leqslant -2\widetilde{\eta}$$

for any $0 \leqslant \tau \leqslant t_0$. Integrating this inequality over $\tau \in [0,s]$ and over $s \in [0,t]$,

$$I_{\mathscr{X}_R}(t) \leqslant I_{\mathscr{X}_R}(0) + I'_{\mathscr{X}_R}(0)t - \widetilde{\eta}t^2. \tag{3.8}$$

Combining (3.8) and $\widetilde{\eta} > 0$, we have by Proposition 2.2

$$\begin{aligned} I_{\mathscr{X}_R}(t) &\leqslant I_{\mathscr{X}_R}(0) - \widetilde{\eta}\left\{t - \frac{1}{2\widetilde{\eta}}I'_{\mathscr{X}_R}(0)\right\}^2 + \frac{1}{4\widetilde{\eta}}I'_{\mathscr{X}_R}(0)^2 \\ &\leqslant I_{\mathscr{X}_R}(0) + \frac{1}{\widetilde{\eta}}\|u_0\mathscr{X}'_R\|^2_{L^2}\|\partial_x u_0\|^2_{L^2} \end{aligned} \tag{3.9}$$

for any $0 \leqslant t \leqslant t_0$. Using $\mathscr{X}_R \geqslant R^2$ $(|x| \geqslant R)$, (3.9), $(\mathscr{X}'_R)^2 \leqslant 4\mathscr{X}_R$, and (3.6), we have

$$\|u(t)\|_{L^2(|x|\geqslant R)} \leqslant \frac{1}{R}I_{\mathscr{X}_R}(t)^{\frac{1}{2}} \leqslant \frac{1}{R}I_{\mathscr{X}_R}(0)^{\frac{1}{2}}\left(1 + \frac{4}{\widetilde{\eta}}\|\partial_x u_0\|^2_{L^2}\right)^{\frac{1}{2}} \leqslant \frac{1}{2}a_0$$

for any $0 \leqslant t \leqslant t_0$. However, this is contradiction and therefore, $t_0 = \infty (= T_{\max})$ holds.

Finally, by (3.8), we conclude that $I_{\mathscr{X}_R}(t) < 0$ for some finite time, which is absurd. Therefore, the solution u to (NLS$_0$) with (IC) blows up. □

4. Applications

To prove Theorem 1.4, we use the following localized virial identity.

PROPOSITION 4.1. (Localized virial identity II, [8, 21]) *Let* $V = \frac{\gamma}{|x|^\mu}$ *and* $0 < \mu < 1$. *We assume that* $\varphi \in W^{3,\infty}(\mathbb{R})$ *has a compact support. If we define*

$$I_{\gamma,w}(t) := \int_{\mathbb{R}} w(x)|u(t,x)|^2 dx$$

for $w := \int_0^x \varphi(y)dy$ *and the solution* $u(t)$ *to* (NLS$_\gamma$)*, then we have*

$$\begin{aligned}
I'_{\gamma,w}(t) &= 2Im \int_{\mathbb{R}} w'(x)\overline{u(t,x)}\partial_x u(t,x)dx, \\
I''_{\gamma,w}(t) &= 4\int_{\mathbb{R}} w''(x)|\partial_x u(t,x)|^2 dx - \frac{4}{3}\int_{\mathbb{R}} w''(x)|u(t,x)|^6 dx \\
&\qquad - \int_{\mathbb{R}} w^{(4)}(x)|u(t,x)|^2 dx + 2\mu \int_{\mathbb{R}} \frac{w'(x)}{x} \cdot \frac{\gamma}{|x|^\mu}|u(t,x)|^2 dx.
\end{aligned}$$

Applying Proposition 4.1 with the weighted function $\mathscr{X}_R$, we have

$$I''_{\gamma,\mathscr{X}_R}(t) \leqslant 16E_\gamma[u_0] + 2\eta + 2\int_{\mathbb{R}} \left\{\mu \frac{R}{x}\mathscr{X}'\left(\frac{x}{R}\right) - 4\right\} \frac{\gamma}{|x|^\mu}|u(t,x)|^2 dx$$

by the same argument with Proposition 3.1, where η is given in Proposition 3.1. It follows from $\mu \frac{R}{|x|}\mathscr{X}'\left(\frac{x}{R}\right) - 4 \leqslant 0$ that

$$I''_{\gamma,\mathscr{X}_R}(t) \leqslant 16E_\gamma[u_0] + 2\eta.$$

The rest of the proof of Theorem 1.4 repeats the proof of Theorem 1.3.

We turn to the nonlinear Schrödinger equation with the delta potential. To prove Theorem 1.8, we use the following localized virial identity.

PROPOSITION 4.2. (Localized virial identity III, [4, 16]) *Let* $V = \gamma\delta$. *We assume that* $\varphi \in W^{3,\infty}(\mathbb{R})$ *has a compact support and satisfies* $\varphi(0) = 0$. *If we define*

$$I_{\delta,w}(t) := \int_{\mathbb{R}} w(x)|u(t,x)|^2 dx$$

for $w := \int_0^x \varphi(y)dy$ *and the solution* $u(t)$ *to* (NLS$_{\gamma\delta}$)*, then we have*

$$\begin{aligned}
I'_{\delta,w}(t) &= 2Im \int_{\mathbb{R}} w'(x)\overline{u(t,x)}\partial_x u(t,x)dx, \\
I''_{\delta,w}(t) &= 4\int_{\mathbb{R}} w''(x)|\partial_x u(t,x)|^2 dx - \frac{4}{3}\int_{\mathbb{R}} w''(x)|u(t,x)|^6 dx \\
&\qquad - \int_{\mathbb{R}} w^{(4)}(x)|u(t,x)|^2 dx + 2\gamma w''(0)|u(t,0)|^2.
\end{aligned}$$

Applying Proposition 4.2 with the weighted function $\mathscr{X}_R$, we have

$$I''_{\delta,\mathscr{X}_R}(t) \leqslant 16E_{\gamma\delta}[u_0] + 2\eta - 4\gamma|u(t,0)|^2$$

by the same argument as in Proposition 3.1, where η is given in Proposition 3.1. It follows from $-4\gamma|u(t,0)|^2 \leqslant 0$ that

$$I''_{\delta,\mathscr{X}_R}(t) \leqslant 16E_{\gamma\delta}[u_0] + 2\eta.$$

The rest of the proof of Theorem 1.8 repeats the proof of Theorem 1.3.

To prove Theorem 1.9, we use the following localized virial identity.

PROPOSITION 4.3. (Localized virial identity IV, [12, 13]) *We assume that* $\varphi \in W^{3,\infty}(0,\infty)$ *has compact support and satisfies* $\varphi(0)=0$. *If we define*

$$I_{\mathscr{G},w}(t) := \int_{\mathscr{G}} w(x)|\boldsymbol{u}(t,x)|^2 dx := \sum_{j=1}^{J} \int_0^\infty w(x)|u_j(t,x)|^2 dx$$

for $w = \int_0^x \varphi(y)dy$ *and the solution* $\boldsymbol{u}$ *to* (NLS$_{\mathscr{G}}$)*, then we have*

$$\begin{aligned} I'_{\mathscr{G},w}(t) &= 2Im \int_{\mathscr{G}} w'(x)\overline{\boldsymbol{u}(t,x)}\partial_x \boldsymbol{u}(t,x)dx, \\ I''_{\mathscr{G},w}(t) &= 4\int_{\mathscr{G}} w''(x)|\partial_x \boldsymbol{u}(t,x)|^2 dx - \frac{4}{3}\int_{\mathscr{G}} w''(x)|\boldsymbol{u}(t,x)|^6 dx \\ &\qquad - \int_{\mathscr{G}} w^{(4)}(x)|\boldsymbol{u}(t,x)|^2 dx + 2w''(0)P(\boldsymbol{u}(t)), \end{aligned}$$

where P *is defined in Theorem* 1.9.

Applying Proposition 4.3 with the weighted function $\mathscr{X}_R$, we have

$$I''_{\mathscr{G},\mathscr{X}_R}(t) \leqslant 16E_{\mathscr{G}}[\boldsymbol{u_0}] + 2\eta_{\mathscr{G}} - 4P(u(t))$$

by the same argument with Proposition 3.1, where

$$\eta_{\mathscr{G}} := \frac{4}{3R^2}\left(\sqrt{6} + \frac{\|\zeta''\|_{L^\infty(1+1/\sqrt{3}\leqslant|x|\leqslant 2)}}{2}\right)^2 \|\boldsymbol{u_0}\|^6_{L^2(\mathscr{G})} + \frac{\|\zeta^{(3)}\|_{L^\infty(1\leqslant|x|\leqslant 2)}}{2R^2}\|\boldsymbol{u_0}\|^2_{L^2(\mathscr{G})}.$$

It follows from $-4P(u(t)) \leqslant 0$ that

$$I''_{\mathscr{G},\mathscr{X}_R}(t) \leqslant 16E_{\mathscr{G}}[\boldsymbol{u_0}] + 2\eta_{\mathscr{G}}.$$

The rest of the proof of Theorem 1.9 repeats the proof of Theorem 1.3.

REFERENCES

[1] R. ADAMI, C. CACCIAPUOTI, D. FINCO, AND D. NOJA, *Variational properties and orbital stability of standing waves for NLS equation on a star graph*, J. Differential Equations **257** (2014), no. 10, 3738–3777, MR3260240.

[2] J. P. ANGULO AND N. GOLOSHCHAPOVA, *Extension theory approach in the stability of the standing waves for the NLS equation with point interactions on a star graph*, Adv. Differential Equations **23** (2018), no. 11-12, 793–846, MR3857871.

[3] J. B. BAILLON, T. CAZENAVE, AND M. FIGUEIRA, *Équation de Schrödinger avec non-linéarité intégrale*, C. R. Acad. Sci. Paris Sér. A-B 284 (1977), no. 16, A939–A942, MR0433026.

[4] V. BANICA AND N. VISCIGLIA, *Scattering for NLS with a delta potential*, J. Differential Equations **260** (2016), no. 5, 4410–4439, MR3437592.

[5] C. CACCIAPUOTI, D. FINCO, AND D. NOJA, *Ground state and orbital stability for the NLS equation on a general starlike graph with potentials*, Nonlinearity **30** (2017), no. 8, 3271–3303, MR3685669.

[6] T. CAZENAVE, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003, xiv+323 pp, MR2002047.

[7] V. D. DINH, *On nonlinear Schrödinger equations with attractive inverse-power potentials*, Topol. Methods Nonlinear Anal. **57** (2021), no. 2, 489–523, MR4359723.

[8] V. D. DINH, *On nonlinear Schrödinger equations with repulsive inverse-power potentials*, Acta Appl. Math. **171** (2021), Paper No. 14, 52 pp, MR4198524.

[9] R. FUKUIZUMI, M. OHTA, AND T. OZAWA, *Nonlinear Schrödinger equation with a point defect*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), no. 5, 837–845, MR2457813.

[10] J. GINIBRE AND G. VELO, *On a class of nonlinear Schrödinger equations*, III. Special theories in dimensions 1, 2 and 3. Ann. Inst. H. Poincaré Sect. A (N.S.) **28** (1978), no. 3, 287–316, MR0498408.

[11] R. T. GLASSEY, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), no. 9, 1794–1797, MR0460850.

[12] N. GOLOSHCHAPOVA, *Dynamical and variational properties of the NLS-δ_s' equation on the star graph*, J. Differential Equations **310** (2022), 1–44, MR4352601.

[13] N. GOLOSHCHAPOVA AND M. OHTA, *Blow-up and strong instability of standing waves for the NLS $-\delta$ equation on a star graph*, Nonlinear Anal. **196** (2020), 111753, 23 pp, MR4066749.

[14] R. H. GOODMAN, P. J. HOLMES, AND M. I. WEINSTEIN, *Strong NLS soliton-defect interactions*, Phys. D **192** (2004), no. 3-4, 215–248, MR2065079.

[15] M. IKEDA, *Global dynamics below the ground state for the focusing semilinear Schrödinger equation with a linear potential*, J. Math. Anal. Appl. **503** (2021), no. 1, Paper No. 125291, 63 pp, MR4256194.

[16] M. IKEDA AND T. INUI, *Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential*, Anal. PDE **10** (2017), no. 2, 481–512, MR3619878.

[17] T. KATO, *On nonlinear Schödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. **46** (1987), no. 1, 113–129, MR0877998.

[18] V. KOSTRYKIN AND R. SCHRADER, *Kirchhoff's rule for quantum wires*, J. Phys. A **32** (1999), no. 4, 595–630, MR1671833.

[19] V. KOSTRYKIN AND R. SCHRADER, *Laplacians on metric graphs: eigenvalues, resolvents and semigroups*, Quantum graphs and their applications, 201–225, Contemp. Math., **415**, Amer. Math. Soc., Providence, RI, 2006, MR2277618.

[20] T. OGAWA AND Y. TSUTSUMI, *Blow-up of H^1 solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc. **111** (1991), no. 2, 487–496, MR1045145.

[21] T. TAO, M. VISAN, AND X. ZHANG, *The nonlinear Schrödinger equation with combined power-type nonlinearities*, Comm. Partial Differential Equations **32** (2007), no. 7–9, 1281–1343, MR2354495.

(Received August 14, 2023)

Masaru Hamano
Faculty of Science and Engineering Waseda University
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
e-mail: m.hamano3@kurenai.waseda.jp

Masahiro Ikeda
RIKEN Center for Advanced Intelligence Project
Nihonbashi 1-chome Mitsui Building, 15th floor, 1-4-1 Nihonbashi,
Chuo-ku, Tokyo 103-0027
and
Department of Mathematics
Faculty of Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan
e-mail: masahiro.ikeda@riken.jp
masahiro.ikeda@keio.jp

Shuji Machihara
Department of Mathematics
Graduate School of Science and Engineering Saitama University
Shimo-Okubo 255, Sakura-ku, Saitama-shi, Saitama 338-8570, Japan
e-mail: machihara@rimath.saitama-u.ac.jp