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ON THE EXISTENCE OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS ON INFINTE INTERVALS FOR NONLINEAR DISCRETE SYSTEMS

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Abstract. We provide criteria for the existence of solutions of nonlinear discrete-time boundary value problems on infinite-time intervals. The problems are formulated as nonlinear operator equations on sequence spaces and the tools of nonlinear functional analysis are employed throughout the paper.

1. Introduction

In this paper, we consider nonlinear, discrete-time, boundary value problems on infinite intervals. We establish sufficient conditions for the existence of solutions to problems of the form

$$x(k+1) = A(k)x(k) + f(k,x(k)) \tag{1}$$

subject to

$$Bx(0) + Dx(\infty) = 0. \tag{2}$$

Throughout we assume $A(k)$ is an invertible n by n matrix for each non-negative integer k, B and D are constant n by n matrices and $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous. The symbol $x(\infty)$ is used to denote the limit of $x(k)$ as k approaches infinity.

We will establish sufficient conditions for the existence of solutions to this boundary value problem based on properties of the matrices $A(k)$ and the nonlinearity f.

The boundary value problem will be formulated as a nonlinear operator equation between sequence spaces and the tools of nonlinear analysis will be used throughout. The use of functional analytic and topological methods in the study of boundary value problems is not novel. Those interested in the use of variational methods in the study of discrete boundary value problems may consult [3], [4], [5], [7].

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Fixed point theorems and topological degree theory are used in the study of boundary value problems and in the periodic behavior of dynamical systems [1], [6], [9], [10], [11], [12], [19], [22]. Problems subject to nonlocal constraints are discussed in [10], [18], [20], [23]. In [7], [13], the reader will find results pertaining to the existence of positive solutions to nonlinear boundary value problems.

Boundary value problems on infinite time intervals for discrete-time systems are analyzed in [10], [20], [21]. The connection between infinite interval boundary value problems for discrete and continuous systems can be found in [10], [14], [15].

2. Preliminaries

It is well known that if each $A(k)$ is nonsingular, then for each positive integer k, the solution of the linear initial value problem

$$\begin{gathered} x(k+1) = A(k)x(k) + w(k); \;\; k = 0,1,2,3,\ldots \\ x(0) = x_0 \end{gathered}$$

is given by the variation of parameters formula

$$x(k) = \Phi(k)x(0) + \sum_{l=0}^{k-1} \Phi(k,l+1)w(l)$$

where $\Phi(k,l)$, the fundamental matrix for the equation $x(k+1) = A(k)x(k)$, is given by

$$\Phi(k,l) = A(k-1)\cdots A(l)$$

for $k > l$ and $\Phi(k,l) = I$ if $k = l$. For the sake of notation, for $k \in \mathbb{Z}^+$ we denote $\Phi(k,0)$ as simply $\Phi(k)$. It is obvious that $\Phi(k,l) = \Phi(k)\Phi^{-1}(l)$.

For the basic results of linear systems as well as the general theory of difference equations the reader may consult [2], [8], [16].

As a matter of notation we will use the following: If u is an element of $\mathbb{R}^n$, $|u|$ will represent the Euclidian norm of u; if B is a matrix or, more generally, a bounded linear map, we will use $||B||$ to denote the operator norm. In this paper, the following sequence spaces arise in a natural way: $l_1 = \{x : x(k) \in \mathbb{R}^n$ and $\sum_{k=0}^{\infty} |x(k)| < \infty\}$, l_∞ consists of the $\mathbb{R}^p$- valued bounded sequences, and c stands for the collection of convergent sequences. We will use the standard norms in l_1 and l_∞; the norm on c is the one inherited from l_∞. It is well known that these are Banach spaces.

We assume throughout this paper that the limit of $\Phi(k)$ as k approaches infinity exists and we denote this limit by $\Phi(\infty)$. It will also be assumed that there is a constant M such that $\|\Phi(k)\| \leqslant M$ and $\|\Phi^{-1}(k)\| \leqslant M$ for all nonnegative integers k. It should be observed that the existence of such a bound for the norm of $\Phi(k)$ follows from the fact that $\Phi(\infty)$ exists. The existence of a uniform bound for $\|\Phi^{-1}(k)\|$ requires a further condition; one such condition would be the invertibility of $\Phi(\infty)$. This follows at once from the fact that $\Phi \to \Phi^{-1}$ is a continous map, actually infinitely Fréchet differentiable [17].

Observation: Consider a sequence of matrices $\{A(k)\}$ which are simultaneously diagonalizable with $A(k) = P\Lambda(k)P^{-1}$ where the diagonal matrices are given by

$$\Lambda(k) = \begin{bmatrix} \lambda_{11}(k) & 0 & \dots & 0 \\ 0 & \lambda_{22}(k) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{nn}(k) \end{bmatrix}.$$

It is obvious that $\Phi(k) = PD(k)P^{-1}$ where $D(k)$ is the diagonal matrix

$$D(k) = \begin{bmatrix} d_{11}(k) & 0 & \dots & 0 \\ 0 & d_{22}(k) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn}(k) \end{bmatrix}$$

where

$$d_{ii}(k) = \prod_{j=0}^{k-1} \lambda_{ii}(j).$$

Consequently, if there are positive constants α and β such that $\alpha \leqslant |\prod_{j=0}^{k-1} \lambda_{ii}(j)| \leqslant \beta$, for all $i = 1,2,3,\dots,n$ and all nonnegative integers k, then there is a constant M such that $\|\Phi(k)\| \leqslant M$ and $\|\Phi^{-1}(k)\| \leqslant M$ for all nonnegative integers k.

Also, if for each $i = 1,2,3,\dots,n$ the limit as k approaches infinity of $\prod_{j=0}^{k-1} \lambda_{ii}(j)$ exists and is different from zero, then $\Phi(\infty)$ exists and is invertible.

We now consider the linear boundary value problem

$$x(k+1) = A(k)x(k) + w(k) \tag{3}$$

subject to

$$Bx(0) + Dx(\infty) = 0 \tag{4}$$

where we assume w belongs to l_1 and by $x(\infty)$ we mean the limit as k approaches infinity of $x(k)$.

As part of our standing hypotheses throughout the paper we have: $A(k)$ is nonsingular for all nonnegative integers k, $\Phi(\infty)$ exists and there is a constant M such that $\|\Phi(k)\| \leqslant M$ and $\|\Phi^{-1}(k)\| \leqslant M$ for all nonnegative integers k.

PROPOSITION 1. *The only solution of*

$$x(k+1) = A(k)x(k) \tag{5}$$

subject to

$$Bx(0) + Dx(\infty) = 0 \tag{6}$$

is the trivial one, if and only if for each w *in* l_1*, the linear boundary value problem* (3)–(4) *has one and only one solution.*

Proof. Let w be an element of l_1. We see that x is a solution of the linear nonhomogenous boundary value problem if and only if

$$x(k) = \Phi(k)x(0) + \Phi(k)\sum_{l=0}^{k-1} \Phi^{-1}(l+1)\, w(l), \tag{7}$$

and

$$Bx(0) + Dx(\infty) = 0. \tag{8}$$

Since $\|\Phi^{-1}(l+1)\| \leqslant M$ for all l and w belongs to l_1, it is obvious that

$$\sum_{l=0}^{\infty} |\Phi^{-1}(l+1)\, w(l)| \tag{9}$$

is convergent. The fact that $\Phi(\infty)$ exists implies $x(\infty)$ exists and that

$$x(\infty) = \Phi(\infty)x(0) + \Phi(\infty)\sum_{l=0}^{\infty} \Phi^{-1}(l+1)\, w(l). \tag{10}$$

Therefore, x is a solution of the boundary value problem

$$x(k+1) = A(k)x(k) + w(k)$$

subject to

$$Bx(0) + Dx(\infty) = 0 \tag{11}$$

if and only if

$$Bx(0) + D[\Phi(\infty)x(0) + \Phi(\infty)\sum_{l=0}^{\infty} \Phi^{-1}(l+1)\, w(l)] = 0. \tag{12}$$

Equivalently, if and only if

$$[B + D\Phi(\infty)]x(0) = -D\Phi(\infty)\sum_{l=0}^{\infty} \Phi^{-1}(l+1)\, w(l). \tag{13}$$

Since every solution of

$$x(k+1) = A(k)x(k)$$

is given by

$$x(k) = \Phi(k)x(0)$$

it follows that the only solution of the homogenous boundary value problem is the trivial one if and only if $B + D\Phi(\infty)$ is invertible. The result now follows. $\square$

We also conclude that under the conditions of this proposition the unique solution of

$$x(k+1) = A(k)x(k) + w(k)$$

subject to

$$Bx(0) + Dx(\infty) = 0$$

is given by

$$\begin{aligned}(Kw)(k) = -\Phi(k)[B + D\Phi(\infty)]^{-1} D\Phi(\infty) \sum_{l=0}^{\infty} \Phi^{-1}(l+1)\, w(l) \\ + \Phi(k) \sum_{l=0}^{k-1} \Phi^{-1}(l+1) w(l).\end{aligned} \tag{14}$$

This formula for Kw defines a map on l_1. For each w in l_1, Kw is a sequence whose value at k is given by the above expression.

PROPOSITION 2. *K is a bounded linear map from l_1 into c.*

Proof. It is obvious that K is a linear map from l_1 into c. The fact that it is bounded is a consequence of the following:

$$\begin{aligned}|(Kw)(k)| &\leqslant |-\Phi(k)[B + D\Phi(\infty)]^{-1} D\Phi(\infty) \sum_{l=0}^{\infty} \Phi^{-1}(l+1)\, w(l)| \\ &\quad + |\Phi(k) \sum_{l=0}^{k-1} \Phi^{-1}(l+1) w(l)| \\ &\leqslant M^2 \|[B + D\Phi(\infty)]^{-1} D\Phi(\infty)\| \sum_{l=0}^{\infty} |w(l)| + M^2 \sum_{l=0}^{k-1} |w(l)| \\ &\leqslant M^2 (1 + \|[B + D\Phi(\infty)]^{-1} D\Phi(\infty)\|) \sum_{l=0}^{\infty} |w(l)|. \quad \square\end{aligned}$$

3. Main results

We now consider nonlinear boundary value problems of the form

$$x(k+1) = A(k)x(k) + f(k, x(k))$$

subject to

$$Bx(0) + Dx(\infty) = 0.$$

To streamline the statements of the next results we introduce:

H1. $\Phi(\infty)$ exists and that there is a constant M such that $\|\Phi(k)\| \leqslant M$ and $\|\Phi^{-1}(k)\| \leqslant M$ for all nonnegative integers k. The matrix $B+D\Phi(\infty)$ is invertible.

H2. It is also assumed that $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous and there is an h in l_1 and a continuous function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $|f(k,x)| \leqslant h(k)|g(x)|$ for all (k,x) in $[0,\infty) \times \mathbb{R}^n$.

We introduce the following notation: For r positive, we define $\Lambda(r) = \sup\{|g(u)| : |u| \leqslant r\}$.

THEOREM 3. *Assume H1 and H2 hold and there is a positive number r such that*

$$\frac{\Lambda(r)}{r} \leqslant \frac{1}{M^2(1+\|[B+D\Phi(\infty)]^{-1}D\Phi(\infty)\|)\sum_{l=0}^{\infty}|h(l)|}. \tag{15}$$

Then, there exists a solution of the nonlinear boundary value problem

$$x(k+1) = A(k)x(k) + f(k,x(k))$$

subject to

$$Bx(0) + Dx(\infty) = 0.$$

Proof. We will now show that the map F defined on c, by $(F(x))(k) = f(k,x(k))$ is a continuous map from c into l_1.

If x is in c,

$$|F(x(k))| = |f(k,x(k))| \leqslant h(k)|g(x(k))| \leqslant h(k)\sup\{|g(u)| : |u| \leqslant \|x\|\}.$$

Since h is in l_1 it follows that F maps c into l_1. In fact, F maps bounded subsets of c into bounded subsets of l_1.

The continuity of F on c will now be established.

Now consider a sequence $\{x_j\}$ in c which converges to x, which of course is assumed to be in c. Since f is continuous, it is obvious that for each k, $f(k,x_j(k))$ converges to $f(k,x(k))$ as j approaches infinity. We now use the Lebesgue Dominated Convergence Theorem. In order to do so, we view l_1 as L_1 with the counting measure.

Clearly, each $f(\cdot,x_j(\cdot))$ is integrable.

Obviously, there is a positive number r such that $|x_j(k)| \leqslant r$ and $|x(k)| \leqslant r$ for all j and all k. Integrating, using the counting measure we obtain

$$\lim_{j\to\infty}\int |f(\cdot,x_j(\cdot)) - f(\cdot,x(\cdot))|d\mu = 0\,. \tag{16}$$

Of course,

$$\int |f(\cdot,x_j(\cdot)) - f(\cdot,x(\cdot))|d\mu = \sum_{l=0}^{\infty}|f(l,x_j(l)) - f(l,x(l))|. \tag{17}$$

Therefore,

$$\lim_{j\to\infty}\sum_{l=0}^{\infty}|(Fx_j)(l)-(Fx)(l)|=0; \tag{18}$$

that is

$$\lim_{j\to\infty}\|Fx_j-Fx\|=0. \tag{19}$$

This establishes the continuity of $F:c\to l_1$.

It now follows that the nonlinear boundary value problem has a solution if and only if there exists an x in c such that

$$x=(K\circ F)(x). \tag{20}$$

Since K and F are continuous, it is obvious that $K\circ F$ is a continuous map from c into c.

For x in c,

$$\begin{aligned}(K\circ F)(x)(k)&=\Phi(k)\sum_{l=0}^{k-1}\Phi^{-1}(l+1)f(l,x(l))\\&\quad-\Phi(k)[(B+D\Phi(\infty))^{-1}D\Phi(\infty)]\sum_{l=0}^{\infty}\Phi^{-1}(l+1)f(l,x(l)).\end{aligned}$$

Consequently,

$$\lim_{k\to\infty}(K\circ F)(x)(k) \tag{21}$$

$$=\Phi(\infty)[I-(B+D\Phi(\infty))^{-1}D\Phi(\infty)]\sum_{l=0}^{\infty}\Phi^{-1}(l+1)\,f(l,x(l)). \tag{22}$$

As a matter of notation, we let $B(r)=\{x\in c:\|x\|\leqslant r\}$.

We will show that the closure of $(K\circ F)(B(r))$ is compact. Since completeness is evident it is sufficient to show the set is totally bounded.

Now, we observe that if $x\in B(r)$, for each nonnegative integer k

$$|(K\circ F)(x)(k)-\lim_{j\to\infty}(K\circ F)(x)(j)| \tag{23}$$

$$\begin{aligned}&=\Big|-\Phi(k)[B+D\Phi(\infty)]^{-1}D\Phi(\infty)\sum_{l=0}^{\infty}\Phi^{-1}(l+1)f(l,x(l))\\&\quad+\Phi(k)\sum_{l=0}^{k-1}\Phi^{-1}(l+1)f(l,x(l))\\&\quad+\Phi(\infty)[B+D\Phi(\infty)]^{-1}D\Phi(\infty)\sum_{l=0}^{\infty}\Phi^{-1}(l+1)f(l,x(l))\\&\quad-\Phi(\infty)\sum_{l=0}^{\infty}\Phi^{-1}(l+1)f(l,x(l))\Big|\end{aligned}$$

$$\leqslant \|\Phi(k)-\Phi(\infty)\| \Big| [B+D\Phi(\infty)]^{-1} D\Phi(\infty) \sum_{l=0}^{\infty} \Phi^{-1}(l+1) f(l,x(l)) \Big|$$

$$+ \|\Phi(k)-\Phi(\infty)\| \Big| \sum_{l=0}^{\infty} \Phi^{-1}(l+1) f(l,x(l)) \Big| + \|\Phi(k)\| \Big| \sum_{l=k}^{\infty} \Phi^{-1}(l+1) f(l,x(l)) \Big|$$

$$\leqslant \|\Phi(k)-\Phi(\infty)\| \|[B+D\Phi(\infty)]^{-1} D\Phi(\infty)\| \sum_{l=0}^{\infty} M|h(l)||g(x(l))|$$

$$+ \|\Phi(k)-\Phi(\infty)\| M \sum_{l=0}^{\infty} |h(l)||g(x(l))| + \Big| \Phi(k)\| \sum_{l=k}^{\infty} M|h(l)||g(x(l))| \Big|$$

$$\leqslant \|\Phi(k)-\Phi(\infty)\| \ (1+\|[B+D\Phi(\infty)]^{-1} D\Phi(\infty)\|) \sum_{l=0}^{\infty} M|h(l)|\Lambda(r)$$

$$+ M^2 \Lambda(r) \sum_{l=k}^{\infty} h(l).$$

From this it follows that given any $\varepsilon > 0$ there exists a positive integer N such that if $k \geqslant N$

$$|(K \circ F)(x(k)) - \lim_{j \to \infty} (K \circ F)(x(j))| < \varepsilon \tag{24}$$

for all $x \in B(r)$.

As a consequence of the fact that $(K \circ F)(B(r))$ is bounded and that

$$\bigcup_{l=0}^{N} \{x(l) : x \in B(r)\} \tag{25}$$

is contained in a finite dimensional space, it follows that the latter is totally bounded.

Using the results in (23) and (24), it is evident now that the closure of $(K \circ F)(B(r))$ is totally bounded and therefore compact.

Clearly $(K \circ F)$ maps $B(r)$ into itself. As a consequence of Schauder's Theorem, $(K \circ F)$ has a fixed point. This fixed point is a solution of the nonlinear boundary value problem. □

COROLLARY 4. *If H1 and H2 hold and*

$$\lim_{|x| \to \infty} \frac{|g(x)|}{|x|} = 0, \tag{26}$$

the boundary value problem

$$x(k+1) = A(k)x(k) + f(k,x(k))$$

$$Bx(0) + Dx(\infty) = 0$$

has a solution.

Proof. It is trivial to verify that under these conditions there exists an r such that

$$\frac{\Lambda(r)}{r} \leqslant \frac{1}{M^2(1+\|[B+D\Phi(\infty)]^{-1}D\Phi(\infty)\|)\sum_{l=0}^{\infty}|h(l)|}. \tag{27}$$

The result now follows from the previous theorem. $\square$

The example we now present illustrates how the results in this paper may be used to establish the existence of solutions of nonlinear discrete boundary value problems on infinite time intervals.

We consider problems of the form

$$x(k+1) = A(k)x(k) + f(k,x(k))$$

subject to

$$Bx(0) + Dx(\infty) = 0.$$

Each $A(k)$ is a two by two matrix, and so are the constant matrices B and D.

$$A(k) = \begin{bmatrix} (2a(k)-b(k)) & (-2a(k)+2b(k)) \\ (a(k)-b(k)) & (-a(k)+2b(k)) \end{bmatrix},$$

$f : \mathbb{R}^3 \to \mathbb{R}^2$ is continuous and there is an h in l_1 and a continuous function $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|f(k,x)| \leqslant h(k)|g(x)|$ for all (k,x) in $[0,\infty) \times \mathbb{R}^2$.

Suppose $a = \prod_{j=0}^{\infty} a(j)$ and $b = \prod_{j=0}^{\infty} b(j)$ are well defined and nonzero, and

$$\lim_{|x|\to\infty} \frac{|g(x)|}{|x|} = 0.$$

We will see that if

$$B + D\begin{bmatrix} (2a-b) & (-2a+2b) \\ (a-b) & (-a+2b) \end{bmatrix}$$

is invertible, the nonlinear boundary value problem has a solution.

It is trivial to verify that for each nonnegative integer k

$$A(k) = C\begin{bmatrix} a(k) & 0 \\ 0 & b(k) \end{bmatrix}C^{-1}$$

where

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Using the observation immediately preceeding Proposition 1, it follows that

$$\Phi(\infty) = \begin{bmatrix} 2a-b & -2a+2b \\ a-2b & -a+2b \end{bmatrix}.$$

The solvability of the boundary value problems is a direct consequence of Corollary 4.

The fundamental reasons we are able to establish the existence of solutions to this problem are the limiting behavior of the fundamental matrices and the rate of growth of the nonlinearities. The fact that the matrices $A(k)$ are simultaneously diagonalizable is not necessary, but useful in our example. It should be clear how this type of approach can be used in the analysis of nonlinear boundary value problems on infinite intervals.

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