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RESULTS ON NON–INSTANTANEOUS IMPULSIVE φ–CAPUTO FRACTIONAL DIFFERENTIAL SYSTEMS: STABILITY AND CONTROLLABILITY

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Abstract. The main objective of this work is to investigate a class of φ-Caputo fractional differential systems with impulsive effects and nonlocal conditions. We used Banach fixed point theorem, fractional calculus, and semigroup theory to study the existence of piecewise continuous mild solution for the proposed system. Moreover, we proved the novel stability criteria for the considered system. Further, we investigated the exact and trajectory controllability of the proposed system. Finally, the main results are validated with the aid of an example.

1. Introduction

The fractional differential equations (FDEs for short) have acquired considerable significance due to lots of practical applications such as in multi-agent systems, epidemiological models, electric circuits, fluid mechanics, neural networks, viscoelasticity, and control theory. For the basic theory of FDEs and their applications, see [19, 22, 27, 28, 39]. Fractional derivatives are important tools for the description of the hereditary characteristic of many materials and processes. Several definitions of fractional derivative exist for special kinds of kernel dependency, for example, Caputo, Caputo-Hadamard, Hadamard, Caputo Erdélyi-Kober, and Riemann-Liouville. In [3], Almeida derived a new fractional derivative corresponding to another function, which is known as φ-Caputo derivative. The φ-Caputo derivative depends on a kernel φ and fractional derivatives like Caputo, Caputo Erdélyi-Kober and Caputo-Hadamard are obtained by choosing the particular kernel. The φ-Caputo fractional systems are considered by many researchers, see [4, 5, 18, 35] and the references therein. Suechoei and Sa Ngiamsunthorn [31] investigated the existence and stability of fractional differential system with φ-Caputo derivative. Many real-life problems have some sudden changes in their states and these sudden changes are called the impulsive effects. These impulsive effects are classified into two types namely, instantaneous impulsive effects, and another is non-instantaneous impulsive effects. Impulsive effects begin at any arbitrary fixed point and continue with a finite time interval, which is known as non-instantaneous

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impulses. For more details on non-instantaneous impulses, see [2,6,10–13,16,36–38] and the references therein.

There are only few papers that deal with the stability of the non-instantaneous impulsive fractional system [1,15,23] and the references therein. Kumar et al. [21] studied the stability and controllability of fractional differential system with impulsive effect. The controllability problem has drawn the attention of many scientists and researchers since it plays a crucial role in engineering and control theory. Exact controllability is the possibility to transfer the system from any initial state to any target state by choosing a control function. Trajectory controllability is the possibility to transfer the system from any initial state to any target state along a prescribed trajectory, rather than a suitable control function steering a given initial state to any target state. For more details on the exact and trajectory controllability, see [7–9,14,24–26,29,32,34]. Si et al. [30] studied the controllability of a system governed by Stieltjes differential equations. Venkatesan and George [33] analyzed the trajectory controllability for fractional differential equations in Hilbert spaces. To the author's knowledge, no manuscript exists on the existence, stability, exact, and trajectory controllability for φ-Caputo fractional systems with impulsive effects and nonlocal conditions in the literature which is the key inspiration to our research work in this manuscript.

Motivated by the above research, we establish the existence and stability for the following φ-Caputo fractional system defined as follows:

$$\begin{cases} {}_0^C\mathrm{D}_\varphi^\chi z(\gamma) = \mathscr{H} z(\gamma) + \Delta(\gamma, z(\gamma)), \;\; \gamma \in \cup_{l=0}^{\psi} (\sigma_l, \gamma_{l+1}], \\ z(\gamma) = \mathscr{P}_l(\gamma, z(\gamma_l^-)), \;\; \gamma \in \cup_{l=1}^{\psi} (\gamma_l, \sigma_l], \\ z(0) + \mathscr{Q}(z) = z_0, \end{cases} \tag{1.1}$$

and for the controllability results, we consider the following φ-Caputo fractional control system:

$$\begin{cases} {}_0^C\mathrm{D}_\varphi^\chi z(\gamma) = \mathscr{H} z(\gamma) + \mathscr{G} v(\gamma) + \Delta(\gamma, z(\gamma)), \;\; \gamma \in \cup_{l=0}^{\psi} (\sigma_l, \gamma_{l+1}], \\ z(\gamma) = \mathscr{P}_l(\gamma, z(\gamma_l^-)), \;\; \gamma \in \cup_{l=1}^{\psi} (\gamma_l, \sigma_l], \\ z(0) + \mathscr{Q}(z) = z_0, \end{cases} \tag{1.2}$$

where the state $z(\cdot)$ takes its values in Hilbert space $\mathscr{Z}$ and ${}_0^C D_\varphi^\chi$ denotes the φ-Caputo derivative of order $\chi \in (0,1)$. $0 = \sigma_0 = \gamma_0 < \gamma_1 < \sigma_1 < \gamma_2 < \cdots < \gamma_\psi < \sigma_\psi < \gamma_{\psi+1} = b < \infty$, $\mathscr{J}_1 = [0,b]$ and $\mathscr{H}$ is the generator of a C_0-semigroup $\{\mathscr{T}(\gamma)\}_{\gamma \geqslant 0}$ on $\mathscr{Z}$. The functions $\mathscr{P}_l(\gamma, z(\gamma_l^-))$ represents non-instantaneous impulses on $(\gamma_l, \sigma_l]$, $l = 1,2,\ldots,\psi$. The function $v(\cdot)$ in $L^2(\mathscr{J}_1, \mathscr{S})$ is the control and $\mathscr{G}$ is an operator from Banach space $\mathscr{S}$ into $\mathscr{Z}$, which is linear and bounded. The functions $\Delta : \mathscr{J}_1 \times \mathscr{Z} \to \mathscr{Z}$, $\mathscr{Q} : C(\mathscr{J}_1, \mathscr{Z}) \to \mathscr{Z}$, and $\mathscr{P}_l : (\gamma_l, \sigma_l] \times \mathscr{Z} \to \mathscr{Z}$, $l = 1,2,\ldots,\psi$, are fulfilled some suitable assumptions that will be specified later.

The work is arranged as follows. In section 2, we recall some useful results. In section 3 and section 4, we proved the existence of mild solution and stability for the considered system, respectively. Moreover, in section 5 and section 6, we investigated the exact and trajectory controllability of the above system. In section 7, an example is given to demonstrate the obtained results.

2. Preliminaries

Let $\mathscr{J}_2=[a,b]$ and $\varphi\in C^m(\mathscr{J}_2,\mathbb{R})$ an increasing function such that $\varphi'(\gamma)\neq 0,\ \forall\,\gamma\in\ \mathscr{J}_2$.

DEFINITION 1. [31] The φ-Caputo fractional derivative of the function $\mathscr{R}$ of order χ $(m-1<\chi<m,\ m\in\mathbb{N})$, is defined as

$$({}^C_a\mathrm{D}^{\chi}_{\varphi}\mathscr{R})(\gamma)=({}_aI^{m-\chi}_{\varphi}\mathscr{R}^{[m]})(\gamma)=\frac{1}{\Gamma(m-\chi)}\int_a^{\gamma}(\varphi(\gamma)-\varphi(e))^{m-\chi-1}\mathscr{R}^{[m]}(e)\varphi'(e)de,$$

where $m=[\chi]+1$ and $\mathscr{R}^{[m]}(\gamma)=\left(\frac{1}{\varphi'(\gamma)}\frac{d}{d\gamma}\right)^m\mathscr{R}(\gamma)$.

LEMMA 1. [31] *Let $\mathscr{R}\in C^m([a,b])$ and $\chi>0$. Then we have*

$${}_aI^{\chi}_{\varphi}\,{}^C_a\mathrm{D}^{\chi}_{\varphi}\mathscr{R}(\gamma)=\mathscr{R}(\gamma)-\sum_{k=0}^{n-1}\frac{\mathscr{R}^{[k]}(a^+)}{k!}(\varphi(\gamma)-\varphi(a))^k.$$

In particular, for $\chi\in(0,1)$, we obtain

$${}_aI^{\chi}_{\varphi}\,{}^C_a\mathrm{D}^{\chi}_{\varphi}\mathscr{R}(\gamma)=\mathscr{R}(\gamma)-\mathscr{R}(a).$$

LEMMA 2. [17] *Let $\chi>0$ and $\Lambda>0$, then*

1. ${}_aI^{\chi}_{\varphi}(\varphi(\gamma)-\varphi(a))^{\Lambda-1}(\gamma)=\frac{\Gamma(\Lambda)}{\Gamma(\Lambda+\chi)}(\varphi(\gamma)-\varphi(a))^{\Lambda+\chi-1}$.
2. ${}_a\mathrm{D}^{\chi}_{\varphi}(\varphi(\gamma)-\varphi(a))^{\Lambda-1}(t)=\frac{\Gamma(\Lambda)}{\Gamma(\Lambda-\chi)}(\varphi(\gamma)-\varphi(a))^{\Lambda+\chi-1}$.

LEMMA 3. *The φ-Caputo fractional Cauchy problem:*

$$\begin{cases}{}^C_0D^{\chi}_{\varphi}z(\gamma)=\mathscr{H}z(\gamma)+\mathscr{D}(\gamma),\ \ \gamma\in(0,b],\\ z(0)=z_0,\end{cases}\tag{2.1}$$

has a solution which is defined as follows:

$$z(\gamma)=\mathscr{S}^{\chi}_{\varphi}(\gamma,0)z_0+\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}^{\chi}_{\varphi}(\gamma,e)\mathscr{D}(e)\varphi'(e)de,\tag{2.2}$$

where

$$\mathscr{S}^{\chi}_{\varphi}(\gamma,e)z=\int_0^{\infty}\phi_{\chi}(\theta)\mathscr{T}((\varphi(\gamma)-\varphi(e))^{\chi}\theta)zd\theta,$$
$$\mathscr{T}^{\chi}_{\varphi}(\gamma,e)z=\chi\int_0^{\infty}\theta\phi_{\chi}(\theta)\mathscr{T}((\varphi(\gamma)-\varphi(e))^{\chi}\theta)zd\theta,\ \ 0\leqslant e\leqslant\gamma\leqslant b,$$

where $\phi_\chi(\theta) = \dfrac{1}{\chi}\theta(-1-(1/\chi))\rho_\chi(\theta^{-1/\chi})$ *is the probability density function defined on* $(0,\infty)$, *i.e.*

$$\phi_\chi(\theta) \geqslant 0, \quad \theta \in (0,\infty) \ \ \textit{and} \ \ \int_0^\infty \phi_\chi(\theta)d\theta = 1.$$

Proof. Eq. (2.1) is rewritten in the form of the following integral equation

$$z(\gamma) = z_0 + \frac{1}{\Gamma(\chi)}\int_0^\gamma (\varphi(\gamma)-\varphi(e))^{\chi-1}[\mathscr{H}z(e)+\mathscr{D}(e)]\varphi'(e)de, \tag{2.3}$$

provided that the Eq. (2.3) exists. Let $\Lambda > 0$. By using the Laplace transform, we obtain

$$Z(\Lambda) = \frac{z_0}{\Lambda} + \frac{1}{\Lambda^\chi}\left(\mathscr{H}Z(\Lambda)+\hat{\mathscr{D}}(\Lambda)\right),$$

where

$$\begin{aligned} Z(\Lambda) &= \int_0^\infty e^{-\Lambda(\varphi(\xi)-\varphi(0))}z(\xi)\varphi'(\xi)d\xi, \\ \hat{\mathscr{D}}(\Lambda) &= \int_0^\infty e^{-\Lambda(\varphi(\xi)-\varphi(0))}\mathscr{D}(\xi)\varphi'(\xi)d\xi. \end{aligned}$$

It follows that

$$\begin{aligned} Z(\Lambda) &= \Lambda^{\chi-1}(\Lambda^\chi I-\mathscr{H})^{-1}z_0 + (\Lambda^\chi I-\mathscr{H})^{-1}\hat{\mathscr{D}}(\Lambda) \\ &= \Lambda^{\chi-1}\int_0^\infty e^{-\Lambda^\chi e}\mathscr{T}(e)z_0de + \int_0^\infty e^{-\Lambda^\chi e}\mathscr{T}(e)\hat{\mathscr{D}}(\Lambda)de. \end{aligned}$$

Taking $e = \hat{\gamma}^\chi$, we obtain

$$\begin{aligned} Z(\Lambda) &= \chi\int_0^\infty (\Lambda\hat{\gamma})^{\chi-1}e^{-(\Lambda\hat{\gamma})^\chi}\mathscr{T}(\hat{\gamma}^\chi)z_0d\hat{\gamma} + \chi\int_0^\infty \hat{\gamma}^{\chi-1}e^{-(\Lambda\hat{\gamma})^\chi}\mathscr{T}(\hat{\gamma}^\chi)\hat{\mathscr{D}}(\Lambda)d\hat{\gamma} \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \chi\int_0^\infty (\Lambda\hat{\gamma})^{\chi-1}e^{-(\Lambda\hat{\gamma})^\chi}\mathscr{T}(\hat{\gamma}^\chi)z_0d\hat{\gamma}, \\ I_2 &= \chi\int_0^\infty \hat{\gamma}^{\chi-1}e^{-(\Lambda\hat{\gamma})^\chi}\mathscr{T}(\hat{\gamma}^\chi)\hat{\mathscr{D}}(\Lambda)d\hat{\gamma}. \end{aligned}$$

Taking $\hat{\gamma} = \varphi(\gamma)-\varphi(0)$, we obtain

$$\begin{aligned} I_1 &= \chi\int_0^\infty \Lambda^{\chi-1}(\varphi(\gamma)-\varphi(0))^{\chi-1}e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))^\chi}\mathscr{T}((\varphi(\gamma)-\varphi(0))^\chi)z_0\varphi'(\gamma)d\gamma \\ &= \int_0^\infty \frac{-1}{\Lambda}\frac{d}{d\gamma}\left(e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))^\chi}\right)\mathscr{T}((\varphi(\gamma)-\varphi(0))^\chi)z_0d\gamma. \end{aligned}$$

$$
\begin{aligned}
I_2 &= \chi \int_0^\infty (\varphi(\gamma)-\varphi(0))^{\chi-1} e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))^\chi} \mathscr{T}((\varphi(\gamma)-\varphi(0))^\chi)\hat{\mathscr{D}}(\Lambda)\varphi'(\gamma)d\gamma \\
&= \int_0^\infty \int_0^\infty \chi(\varphi(\gamma)-\varphi(0))^{\chi-1} e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))^\chi} \mathscr{T}((\varphi(\gamma)-\varphi(0))^\chi) \\
&\quad \times e^{-(\Lambda(\varphi(e)-\varphi(0)))}\mathscr{D}(e)\varphi'(e)\varphi'(\gamma)de d\gamma.
\end{aligned}
$$

Now, we take following one-sided stable probability density

$$
\rho_\chi(\theta) = \frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\theta^{-\chi i-1}\frac{\Gamma(\chi i+1)}{i!}\sin(i\pi\chi), \ \ \theta \in (0,\infty),
$$

whose integration is defined as follows

$$
\int_0^\infty e^{-\Lambda\theta}\rho_\chi(\theta)d\theta = e^{-\Lambda^\chi}, \ \ \chi\in(0,1). \tag{2.4}
$$

Using Eq. (2.4), we obtain

$$
\begin{aligned}
I_1 &= \int_0^\infty \frac{-1}{\Lambda}\frac{d}{d\gamma}\left(\int_0^\infty e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))\theta}\rho_\chi(\theta)d\theta\right)\mathscr{T}((\varphi(\gamma)-\varphi(0))^\chi)z_0 d\gamma \\
&= \int_0^\infty\int_0^\infty \theta\rho_\chi(\theta)e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))\theta}\mathscr{T}((\varphi(\gamma)-\varphi(0))^\chi)z_0\varphi'(\gamma)d\theta d\gamma \\
&= \int_0^\infty e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))}\left(\int_0^\infty \rho_\chi(\theta)\mathscr{T}\left(\frac{(\varphi(\gamma)-\varphi(0))^\chi}{\theta^\chi}\right)d\theta\right)z_0\varphi'(\gamma)d\gamma,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^\infty\int_0^\infty\int_0^\infty \chi(\varphi(\gamma)-\varphi(0))^{\chi-1}\rho_\chi(\theta)e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))\theta}\mathscr{T}((\varphi(\gamma)-\varphi(0))^\chi) \\
&\quad \times e^{-(\Lambda(\varphi(e)-\varphi(0)))}\mathscr{D}(e)\varphi'(e)\varphi'(\gamma)d\theta de d\gamma \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \chi e^{-(\Lambda(\varphi(\gamma)+\varphi(e)-2\varphi(0)))}\frac{(\varphi(\gamma)-\varphi(0))^{\chi-1}}{\theta^\chi}\rho_\chi(\theta) \\
&\quad \times\mathscr{T}\left(\frac{(\varphi(\gamma)-\varphi(0))^\chi}{\theta^\chi}\right)\mathscr{D}(e)\varphi'(e)\varphi'(\gamma)d\theta de d\gamma \\
&= \int_0^\infty\int_0^\xi\int_0^\infty \chi e^{-(\Lambda(\varphi(\xi)-\varphi(0)))}\rho_\chi(\theta)\frac{(\varphi(\gamma)-\varphi(0))^{\chi-1}}{\theta^\chi}\mathscr{T}\left(\frac{(\varphi(\gamma)-\varphi(0))^\chi}{\theta^\chi}\right) \\
&\quad \times\mathscr{D}(\varphi^{-1}(\varphi(\xi)-\varphi(\gamma)+\varphi(0))))\varphi'(\xi)\varphi'(\gamma)d\theta d\gamma d\xi \\
&= \int_0^\infty e^{-(\Lambda(\varphi(\xi)-\varphi(0)))}\left(\int_0^\xi\int_0^\infty \chi\rho_\chi(\theta)\frac{(\varphi(\xi)-\varphi(e))^{\chi-1}}{\theta^\chi}\right. \\
&\quad \left.\times\mathscr{T}\left(\frac{(\varphi(\xi)-\varphi(e))^\chi}{\theta^\chi}\right)\mathscr{D}(e)\varphi'(e)d\theta de\right)\varphi'(\xi)d\xi.
\end{aligned}
$$

Hence, we get

$$Z(\Lambda) = \int_0^{\infty} e^{-(\Lambda(\varphi(\gamma)-\varphi(0)))} \left(\int_0^{\infty} \rho_\chi(\theta) \mathscr{T}\left(\frac{(\varphi(\gamma)-\varphi(0))^\chi}{\theta^\chi} \right) z_0 d\theta \right) \varphi'(\gamma) d\gamma$$
$$+ \int_0^{\infty} e^{-(\Lambda(\varphi(\xi)-\varphi(0)))} \left(\int_0^{\xi} \int_0^{\infty} \chi \rho_\chi(\theta) \frac{(\varphi(\xi)-\varphi(e))^{\chi-1}}{\theta^\chi} \right.$$
$$\left. \times \mathscr{T}\left(\frac{(\varphi(\xi)-\varphi(e))^\chi}{\theta^\chi} \right) \mathscr{D}(e)\varphi'(e) d\theta de \right) \varphi'(\xi) d\xi.$$

By inverse Laplace transform, we get

$$z(\gamma) = \int_0^{\infty} \rho_\chi(\theta) \mathscr{T}\left(\frac{(\varphi(\gamma)-\varphi(0))^\chi}{\theta^\chi} \right) z_0 d\theta$$
$$+ \int_0^{\gamma} \int_0^{\infty} \chi \rho_\chi(\theta) \frac{(\varphi(\gamma)-\varphi(e))^{\chi-1}}{\theta^\chi} \mathscr{T}\left(\frac{(\varphi(\gamma)-\varphi(e))^\chi}{\theta^\chi} \right) \mathscr{D}(e)\varphi'(e) d\theta de.$$

Thus, we obtain

$$z(\gamma) = \int_0^{\infty} \phi_\chi(\theta) \mathscr{T}\left((\varphi(\gamma)-\varphi(0))^\chi \theta \right) z_0 d\theta$$
$$+\chi \int_0^{\gamma} \int_0^{\infty} \theta \phi_\chi(\theta) (\varphi(\gamma)-\varphi(e))^{\chi-1} \mathscr{T}\left((\varphi(\gamma)-\varphi(e))^\chi \theta \right) \mathscr{D}(e)\varphi'(e) d\theta de,$$

where $\phi_\chi(\theta) = \frac{1}{\chi}\theta^{-1-\frac{1}{\chi}} \rho_\chi(\theta^{-\frac{1}{\chi}})$. For any $z \in \mathscr{Z}$, the operators $\mathscr{S}_\varphi^\chi(\gamma,e)$ and $\mathscr{T}_\varphi^\chi(\gamma,e)$ defined as

$$\mathscr{S}_\varphi^\chi(\gamma,e)z = \int_0^{\infty} \phi_\chi(\theta) \mathscr{T}((\varphi(\gamma)-\varphi(e))^\chi \theta) z d\theta,$$

and

$$\mathscr{T}_\varphi^\chi(\gamma,e)z = \chi \int_0^{\infty} \theta \phi_\chi(\theta) \mathscr{T}((\varphi(\gamma)-\varphi(e))^\chi \theta) z d\theta, \ \ 0 \leqslant e \leqslant \gamma \leqslant b.$$

Hence, we obtain

$$z(\gamma) = \mathscr{S}_\varphi^\chi(\gamma,0)z_0 + \int_0^{\gamma} (\varphi(\gamma)-\varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma,e)\mathscr{D}(e)\varphi'(e) de.$$

DEFINITION 2. A function $z : \mathscr{J}_1 \to \mathscr{Z}$ is a mild solution of (1.1) if for every $\gamma \in \mathscr{J}_1$, $z(\gamma)$ fulfills $z(0) = z_0$, and $z(\gamma) = \mathscr{P}_l(\gamma, z(\gamma_l^-))$, $\gamma \in (\gamma_l, \sigma_l]$, $l = 1,2,\ldots,\psi$, and

$$z(\gamma) = \mathscr{S}_\varphi^\chi(\gamma,0)[z_0 - \mathscr{Q}(z)] + \int_0^{\gamma} (\varphi(\gamma)-\varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma,e)\Delta(e,z(e))\varphi'(e) de,$$

for all $\gamma \in [0,\gamma_1]$, $l = 0$ and

$$z(\gamma) = \mathscr{S}_\varphi^\chi(\gamma,\sigma_l)\mathscr{P}_l(\sigma_l, z(\gamma_l^-))$$
$$+ \int_{\sigma_l}^{\gamma} (\varphi(\gamma)-\varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma,e)\Delta(e,z(e))\varphi'(e) de, \tag{2.5}$$

for all $\gamma \in (\sigma_l, \gamma_{l+1}]$, $l = 1,2,\ldots,\psi$.

Define the space $\mathscr{PC}(\mathscr{Z})$ formed by functions $\{z(\gamma) : \gamma \in [0,b]\}$ such that z is continuous at $\gamma \neq \gamma_l$, $z(\gamma_l^-) = z(\gamma_l)$ and $z(\gamma_l^+)$ exists for all $l = 1,2,\ldots,\psi$ with norm

$$\|z\|_{\mathscr{PC}} = \sup_{0\leqslant\gamma\leqslant b} \|z(\gamma)\|.$$

Then $(\mathscr{PC}(\mathscr{Z}), \|\cdot\|_{\mathscr{PC}})$ is Banach space. □

3. Solvability of φ-Caputo fractional systems

We assume the subsequent hypotheses:

[A1]: The function $\Delta : \mathscr{J}_1 \times \mathscr{Z} \to \mathscr{Z}$ is continuous and there exist constants $\hat{\mathscr{K}}_\Delta$, $\hat{\mathscr{M}}_\Delta > 0$ such that

$$\|\Delta(\gamma,z)\| \leqslant \hat{\mathscr{K}}_\Delta(1+\|z\|), \ \forall z \in \mathscr{Z},$$
$$\|\Delta(\gamma,z_1)-\Delta(\gamma,z_2)\| \leqslant \hat{\mathscr{M}}_\Delta \|z_1 - z_2\|, \ \forall z_1,z_2 \in \mathscr{Z}.$$

[A2]: The functions $\mathscr{P}_l : (\gamma_l, \sigma_l] \times \mathscr{Z} \to \mathscr{Z}$, $l = 1,2,\ldots,\psi$, are continuous and there exist constants $\hat{\mathscr{K}}_{\mathscr{P}_l}$, $\hat{\mathscr{M}}_{\mathscr{P}_l} > 0$, $l = 1,2,\ldots,\psi$, such that

$$\|\mathscr{P}_l(\gamma,z)\| \leqslant \hat{\mathscr{K}}_{\mathscr{P}_l}(1+\|z\|), \ \forall z \in \mathscr{Z},$$
$$\|\mathscr{P}_l(\gamma,z_1)-\mathscr{P}_l(\gamma,z_2)\| \leqslant \hat{\mathscr{M}}_{\mathscr{P}_l} \|z_1 - z_2\|, \ \forall z_1,z_2 \in \mathscr{Z}.$$

[A3]: The function $\mathscr{Q} : C(\mathscr{J}_1,\mathscr{Z}) \to \mathscr{Z}$ is Lipschitz continuous, i.e., there exists a constant $\hat{\mathscr{K}}_{\mathscr{Q}} > 0$ such that

$$\|\mathscr{Q}(z_1)-\mathscr{Q}(z_2)\| \leqslant \hat{\mathscr{K}}_{\mathscr{Q}}\|z_1 - z_2\|, \ \forall z_1,z_2 \in \mathscr{Z}.$$

[A4]: The following inequalities hold

$$\max_{1\leqslant l\leqslant\psi}\left\{\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}} + \hat{\mathscr{K}}_{\mathscr{P}_l} + \mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l} + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_\Delta(\varphi(b)-\varphi(0))^\chi\right\} < 1.$$

For simplicity

$$h_0 = \mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}, \ \ h_l = \mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}, \ \ l = 1,2,\ldots,\psi,$$

and

$$o_l = \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_\Delta(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^\chi, \ \ l = 0,1,\ldots,\psi.$$

LEMMA 4. [31] *For any fixed* $\gamma \geqslant e \geqslant 0$, $\mathscr{S}_\varphi^\chi(\gamma,e)$ *and* $\mathscr{T}_\varphi^\chi(\gamma,e)$ *are bounded linear operators and*

$$\|\mathscr{S}_\varphi^\chi(\gamma,e)(z)\| \leqslant \mathscr{M}\|z\|, \ \ \|\mathscr{T}_\varphi^\chi(\gamma,e)(z)\| \leqslant \frac{\chi\mathscr{M}}{\Gamma(1+\chi)}\|z\| = \frac{\mathscr{M}}{\Gamma(\chi)}\|z\|.$$

THEOREM 1. *If the hypotheses* [A1]–[A4] *are fulfilled. Then the* φ*-Caputo fractional system* (1.1) *has a unique mild solution on* $\mathscr{J}_1$ *provided*

$$\mathscr{O} = \max_{1\leqslant l\leqslant\psi}\left[\delta_0, \hat{\mathscr{M}}_{\mathscr{P}_l}, \delta_l\right] < 1,$$

where $\delta_l = h_l + o_l,\ l = 0,1,\ldots,\psi.$

Proof. For $\tau > 0,$ we define

$$\mathscr{W}_\tau = \{z \in \mathscr{PC}(\mathscr{Z}) : \|z\|_{\mathscr{PC}} \leqslant \tau\}.$$

Clearly, $\mathscr{W}_\tau$ is a bounded and closed subset of $\mathscr{PC}(\mathscr{Z})$. We define the operator $\mathfrak{F}$ on $\mathscr{W}_\tau$ as follows

$$(\mathfrak{F}z)(\gamma) = \begin{cases} \mathscr{S}^\chi_\varphi(\gamma,0)[z_0 - \mathscr{Q}(z)] & \\ +\int_0^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}^\chi_\varphi(\gamma,e)\Delta(e,z(e))\varphi'(e)de & \gamma\in[0,\gamma_1],\ l=0,\\ \mathscr{P}_l(\gamma,z(\gamma_l^-)), & \gamma\in(\gamma_l,\sigma_l],\ l\geqslant 1,\\ \mathscr{S}^\chi_\varphi(\gamma,\sigma_l)\mathscr{P}_l(\sigma_l,z(\gamma_l^-)) & \\ +\int_{\sigma_l}^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}^\chi_\varphi(\gamma,e)\Delta(e,z(e))\varphi'(e)de & \gamma\in(\sigma_l,\gamma_{l+1}],\ l\geqslant 1. \end{cases}$$

Step 1. There exists $\tau > 0$ such that $\mathfrak{F}(\mathscr{W}_\tau) \subset \mathscr{W}_\tau$. If we assume that this assertion is false, then for any $\tau > 0$, we can choose $\gamma \in \mathscr{J}_1$ and $z^\tau \in \mathscr{W}_\tau$ such that $\|\mathfrak{F}(z^\tau)(\gamma)\| > \tau$. For any $\gamma \in [0,\gamma_1],$ we obtain

$$\begin{aligned}\tau &< \|\mathfrak{F}(z^\tau)(\gamma)\|\\ &\leqslant \|\mathscr{S}^\chi_\varphi(\gamma,0)[z_0-\mathscr{Q}(z^\tau)]\|\\ &\quad + \left\|\int_0^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}^\chi_\varphi(\gamma,e)\Delta(e,z^\tau(e))\varphi'(e)de\right\|\\ &\leqslant \mathscr{M}\|z_0\| + \mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\tau + \mathscr{M}\|\mathscr{Q}(0)\|\\ &\quad + \frac{\mathscr{M}}{\Gamma(\chi)}\hat{\mathscr{K}}_\Delta(1+\tau)\int_0^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\varphi'(e)de\\ &\leqslant \mathscr{M}\|z_0\| + \mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\tau + \mathscr{M}\|\mathscr{Q}(0)\|\\ &\quad + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_\Delta(1+\tau)(\varphi(\gamma_1)-\varphi(0))^\chi.\end{aligned}$$

If $\gamma \in (\gamma_l,\sigma_l],\ l = 1,2,\ldots,\psi,$ then we obtain

$$\tau < \|\mathfrak{F}(z^\tau)(\gamma)\| = \|\mathscr{P}_l(\gamma,z^\tau(\gamma_l^-))\|^2 \leqslant \hat{\mathscr{K}}_{\mathscr{P}_l}(1+\tau). \tag{3.1}$$

Similarly, if $\gamma \in (\sigma_l, \gamma_{l+1}]$, $l = 1, 2, \ldots, \psi$, then we obtain

$$\begin{aligned}
\tau &< \|\mathfrak{F}(z^\tau)(\gamma)\| \\
&\leqslant \|\mathscr{S}_\varphi^\chi(\gamma, \sigma_l)\mathscr{P}_l(\sigma_l, z^\tau(\gamma_l^-))\| \\
&\quad + \left\| \int_{\sigma_l}^{\gamma} (\varphi(\gamma) - \varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma, e)\Delta(e, z^\tau(e))\varphi'(e)de \right\| \\
&\leqslant \mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l}(1+\tau) + \frac{\mathscr{M}}{\Gamma(\chi)}\hat{\mathscr{K}}_\Delta(1+\tau)\int_{\sigma_l}^{\gamma}(\varphi(\gamma) - \varphi(e))^{\chi-1}\varphi'(e)de \\
&\leqslant \mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l}(1+\tau) + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_\Delta(1+\tau)(\varphi(\gamma_{l+1}) - \varphi(\sigma_l))^\chi.
\end{aligned}$$

For every $\gamma \in \mathscr{J}_1$, we obtain

$$\begin{aligned}
\tau &< \|\mathfrak{F}(z^\tau)(\gamma)\| \\
&\leqslant \mathscr{Y}^* + \mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\tau + \hat{\mathscr{K}}_{\mathscr{P}_l}\tau + \mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l}\tau + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_\Delta\tau(\varphi(b) - \varphi(0))^\chi,
\end{aligned} \tag{3.2}$$

where

$$\mathscr{Y}^* = \max_{1\leqslant l\leqslant\psi}\left\{\mathscr{M}\|z_0\| + \mathscr{M}\|\mathscr{Q}(0)\| + \hat{\mathscr{K}}_{\mathscr{P}_l} + \mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l} + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_\Delta(\varphi(b) - \varphi(0))^\chi\right\}.$$

Here, $\mathscr{Y}^*$ is independent of τ, both sides of Eq. (3.2) are dividing by τ and taking $\tau \to \infty$, we obtain

$$1 < \mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}} + \hat{\mathscr{K}}_{\mathscr{P}_l} + \mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l} + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_\Delta(\varphi(b) - \varphi(0))^\chi,$$

which contradicts to [A4]. Hence, we obtain $\mathfrak{F}(\mathscr{W}_\tau) \subset \mathscr{W}_\tau$ for some $\tau > 0$.

Step 2. $\mathfrak{F}$ is a contraction mapping on $\mathscr{W}_\tau$.
For all $z_1, z_2 \in \mathscr{W}_\tau$, if $\gamma \in [0, \gamma_1]$, then we obtain

$$\begin{aligned}
&\|(\mathfrak{F}z_1)(\gamma) - (\mathfrak{F}z_2)(\gamma)\| \\
&\leqslant \mathscr{M}\|\mathscr{Q}(z_1) - \mathscr{Q}(z_2)\| \\
&\quad + \left\| \int_0^\gamma (\varphi(\gamma) - \varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma, e)[\Delta(e, z_1(e)) - \Delta(e, z_2(e))]\varphi'(e)de \right\| \\
&\leqslant \mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\|z_1 - z_2\|_{\mathscr{PC}} + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_\Delta(\varphi(\gamma_1) - \varphi(0))^\chi\|z_1 - z_2\|_{\mathscr{PC}} \\
&\leqslant \left(\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}} + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_\Delta(\varphi(\gamma_1) - \varphi(0))^\chi\right)\|z_1 - z_2\|_{\mathscr{PC}}.
\end{aligned} \tag{3.3}$$

If $\gamma \in (\gamma_l, \sigma_l]$, $l = 1, 2, \ldots, \psi$, then we obtain

$$\|(\mathfrak{F}z_1)(\gamma) - (\mathfrak{F}z_2)(\gamma)\| \leqslant \hat{\mathscr{M}}_{\mathscr{P}_l}\|z_1 - z_2\|_{\mathscr{PC}}. \tag{3.4}$$

Similarly, if $\gamma \in (\sigma_l, \gamma_{l+1}]$, $l = 1, 2, \ldots, \psi$, then we obtain

$$\begin{aligned}
&\|(\mathfrak{F}z_1)(\gamma) - (\mathfrak{F}z_2)(\gamma)\| \\
&\leqslant \mathscr{M} \left\| \mathscr{P}_l(\sigma_l, z_1(\gamma_l^-)) - \mathscr{P}_l(\sigma_l, z_2(\gamma_l^-))) \right\| \\
&\quad + \left\| \int_{\sigma_l}^{\gamma} (\varphi(\gamma) - \varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma, e)[\Delta(e, z_1(e)) - \Delta(e, z_2(e))]\varphi'(e)de \right\| \\
&\leqslant \mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l} \|z_1 - z_2\|_{\mathscr{PC}} + \frac{\mathscr{M}}{\Gamma(1+\chi)} \hat{\mathscr{M}}_\Delta (\varphi(\gamma_{l+1}) - \varphi(\sigma_l))^\chi \|z_1 - z_2\|_{\mathscr{PC}} \\
&\leqslant \left(\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l} + \frac{\mathscr{M}}{\Gamma(1+\chi)} \hat{\mathscr{M}}_\Delta (\varphi(\gamma_{l+1}) - \varphi(\sigma_l))^\chi \right) \|z_1 - z_2\|_{\mathscr{PC}}. \qquad (3.5)
\end{aligned}$$

By Eqs. (3.3)–(3.5), we obtain

$$\|(\mathfrak{F}z_1)(\gamma) - (\mathfrak{F}z_2)(\gamma)\| \leqslant \mathscr{O} \|z_1 - z_2\|_{\mathscr{PC}},$$

where

$$\mathscr{O} = \max_{1 \leqslant l \leqslant \psi} \left[\delta_0, \hat{\mathscr{M}}_{\mathscr{P}_l}, \delta_l \right].$$

Hence,

$$\|\mathfrak{F}z_1 - \mathfrak{F}z_2\|_{\mathscr{PC}} \leqslant \mathscr{O} \|z_1 - z_2\|_{\mathscr{PC}}. \qquad (3.6)$$

Thus, $\mathfrak{F}$ is a contraction mapping on $\mathscr{W}_\tau$. Hence, by the fixed point theorem of Banach, there exists a unique mild solution on $\mathscr{J}_1$. □

4. Stability results

DEFINITION 3. [21] A mild solution z of the system (1.1) is said to be stable, if for arbitrary $\varepsilon > 0$, there exists $\kappa > 0$ such that

$$\|z(\gamma) - \hat{z}(\gamma)\| < \varepsilon, \quad \text{whenever} \quad \|z(0) - \hat{z}_0\| < \kappa,$$

where $\hat{z}$ is the mild solution of the system (1.1) with initial conditions $\hat{z}(0) = \hat{z}_0$, and the impulsive conditions $z(\gamma) = \mathscr{P}_l(\gamma, \hat{z}(\gamma_l^-))$, $\gamma \in (\gamma_l, \sigma_l]$, $l = 1, 2, \ldots, \psi$.

THEOREM 2. *If the hypotheses* [A1]–[A4] *are fulfilled. Then the system* (1.1) *has a unique stable mild solution on* $\mathscr{J}_1$, *provided that*

$$\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}} < 1.$$

Proof. By Theorem 1, we obtain that the system (1.1) has a unique mild solution $z(\gamma)$. Let $\hat{z}(\gamma)$ be any mild solution of system (1.1) with conditions $\hat{z}(0) = \hat{z}_0$, and $\hat{z}(\gamma) = \mathscr{P}_l(\gamma, \hat{z}(\gamma_l^-))$, $\gamma \in (\gamma_l, \sigma_l]$, $l = 1, 2, \ldots, \psi$.

Case 1. For $\gamma \in [0,\gamma_1]$, we have

$$\begin{aligned}\|z(\gamma)-\hat{z}(\gamma)\| &\leqslant \mathscr{M}\,\|z_0-\hat{z}_0\|+\mathscr{M}\,\|\mathscr{Q}(z)-\mathscr{Q}(\hat{z})\| \\ &\quad+\left\|\int_0^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma,e)[\Delta(e,z(e))-\Delta(e,\hat{z}(e))]\varphi'(e)de\right\| \\ &\leqslant \mathscr{M}\|z_0-\hat{z}_0\|+\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\|z(\gamma)-\hat{z}(\gamma)\| \\ &\quad+\frac{\mathscr{M}}{\Gamma(\chi)}\hat{\mathscr{M}}_\Delta\int_0^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\|z(e)-\hat{z}(e)\|\varphi'(e)de \\ &\leqslant \mathscr{M}\kappa+\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\|z(\gamma)-\hat{z}(\gamma)\| \\ &\quad+\frac{\mathscr{M}}{\Gamma(\chi)}\hat{\mathscr{M}}_\Delta\int_0^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\|z(e)-\hat{z}(e)\|\varphi'(e)de.\end{aligned}$$

Case 2. For $\gamma \in (\gamma_l,\sigma_l]$, $l=1,2,\ldots,\psi$, we have

$$\begin{aligned}\|z(\gamma)-\hat{z}(\gamma)\| &= \|\mathscr{P}_l(\gamma,z(\gamma_l^-))-\mathscr{P}_l(\gamma,\hat{z}(\gamma_l^-))\| \\ &\leqslant \hat{\mathscr{M}}_{\mathscr{P}_l}\,\|z(\gamma_l^-)-\hat{z}(\gamma_l^-)\|.\end{aligned}$$

Case 3. For $\gamma \in (\sigma_l,\gamma_{l+1}]$, $l=1,2,\ldots,\psi$, we have

$$\begin{aligned}\|z(\gamma)-\hat{z}(\gamma)\| &\leqslant \mathscr{M}\left\|\mathscr{P}_l(\sigma_l,z(\gamma_l^-))-\mathscr{P}_l(\sigma_l,\hat{z}(\gamma_l^-)))\right\| \\ &\quad+\left\|\int_{\sigma_l}^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma,e)[\Delta(e,z(e))-\Delta(e,\hat{z}(e))]\varphi'(e)de\right\| \\ &\leqslant \mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}\|z(\gamma_l^-)-\hat{z}(\gamma_l^-)\| \\ &\quad+\frac{\mathscr{M}}{\Gamma(\chi)}\hat{\mathscr{M}}_\Delta\int_{\sigma_l}^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\|z(e)-\hat{z}(e)\|\varphi'(e)de.\end{aligned}$$

For $\gamma \in \mathscr{J}_1$, we have

$$\begin{aligned}\|z(\gamma)-\hat{z}(\gamma)\| &\leqslant \frac{\mathscr{M}}{1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}}\kappa+\sum_{l=1}^{\psi}\frac{\left(\hat{\mathscr{M}}_{\mathscr{P}_l}+\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}\right)}{1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}}\|z(\gamma_l^-)-\hat{z}(\gamma_l^-)\| \\ &\quad+\frac{\mathscr{M}}{\Gamma(\chi)(1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}})}\hat{\mathscr{M}}_\Delta\int_0^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\|z(e)-\hat{z}(e)\|\varphi'(e)de.\end{aligned}$$

By using impulsive Gronwall's inequality [20], we get

$$\begin{aligned}\|z(\gamma)-\hat{z}(\gamma)\| &\leqslant \frac{\mathscr{M}}{1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}} \\ &\quad\times\kappa\left[\prod_{l=1}^{\psi}\left(1+\frac{(\hat{\mathscr{M}}_{\mathscr{P}_l}+\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l})}{1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}}E_\chi\left(C\Gamma(\chi)(\varphi(\gamma_l)-\varphi(0))^\chi\right)\right)\right]\mathscr{T}_0 \\ &\leqslant \mathscr{T}\kappa,\end{aligned}$$

where

$$\mathscr{T} = \frac{\mathscr{M}}{1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}}\left[\prod_{l=1}^{\psi}\left(1+\frac{(\hat{\mathscr{M}}_{\mathscr{P}_l}+\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l})}{1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}}E_{\chi}\left(C\Gamma(\chi)(\varphi(\gamma_l)-\varphi(0))^{\chi}\right)\right)\right]\mathscr{T}_0,$$

$$C = \frac{\mathscr{M}}{\Gamma(\chi)(1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}})}\hat{\mathscr{M}}_{\Delta}, \quad \mathscr{T}_0 = E_{\chi}\left(C\Gamma(\chi)(\varphi(\gamma)-\varphi(0))^{\chi}\right).$$

Next, we can choose a $\kappa > 0$ such that $\kappa < \dfrac{\varepsilon}{\mathscr{T}}$, then

$$\|z(\gamma)-\hat{z}(\gamma)\| < \varepsilon.$$

Hence, the system (1.1) has a unique stable mild solution on $\mathscr{J}_1$. □

5. Exact controllability

We assume the following hypotheses

[A5]: The linear operators $\Delta_{\sigma_l}^{\gamma_{l+1}} : L^2((\sigma_l,\gamma_{l+1}],\mathscr{S}) \to \mathscr{Z}$, $l = 0,1,\dots,\psi$, defined by

$$\Delta_{\sigma_l}^{\gamma_{l+1}} v = \int_{\sigma_l}^{\gamma_{l+1}} (\varphi(\gamma_{l+1})-\varphi(e))^{\chi-1}\mathscr{T}_{\varphi}^{\chi}(\gamma_{l+1},e)\mathscr{G}v(e)\varphi'(e)de,$$

has bounded invertible operators $(\Delta_{\sigma_l}^{\gamma_{l+1}})^{-1}$ which takes values in $L^2((\sigma_l,\gamma_{l+1}],\mathscr{S})$ $/Ker(\Delta_{\sigma_l}^{\gamma_{l+1}})$ and there exist constants $\Delta_l > 0$ such that

$$\left\|(\Delta_{\sigma_l}^{\gamma_{l+1}})^{-1}\right\| \leqslant \Delta_l.$$

[A6]: The following inequalities hold

$$\mathscr{O}_{\mathfrak{G}} = \max_{1\leqslant l\leqslant\psi}\left[r_0, \ \hat{\mathscr{M}}_{\mathscr{P}_l}, \ r_l\right] < 1,$$

where

$$r_0 = \delta_0\Big(\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}+\frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_{\Delta}(\varphi(\gamma_1)-\varphi(0))^{\chi}\Big),$$

$$r_l = \Big(\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}+\frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_{\Delta}(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^{\chi}\Big), \ l = 1,2,\cdots,\psi,$$

$$\delta_l = \left(1+\frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^{\chi}}{\Gamma(\chi+1)}\right), \ l = 0,1,\cdots,\psi.$$

DEFINITION 4. The system (1.2) is called exactly controllable on $\mathscr{J}_1$ if for every $z_0, z_1 \in \mathscr{Z}$, there exists a suitable control $v \in L^2(\mathscr{J}_1,\mathscr{S})$ such that the mild solution of system (1.2) with respect to v satisfies $z(b) = z_1$.

DEFINITION 5. A function $z : \mathscr{J}_1 \to \mathscr{Z}$ is a mild solution of (1.2) if for every $\gamma \in \mathscr{J}_1$, $z(\gamma)$ fulfills $z(0) = z_0$, and $z(\gamma) = \mathscr{P}_l(\gamma, z(\gamma_l^-))$, $\gamma \in (\gamma_l, \sigma_l]$, $l = 1, 2, \ldots, \psi$, and

$$z(\gamma) = \mathscr{S}_\varphi^\chi(\gamma, 0)[z_0 - \mathscr{Q}(z)] + \int_0^\gamma (\varphi(\gamma) - \varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma, e)[\mathscr{G}v(e) + \Delta(e, z(e))]\varphi'(e)de,$$

for all $\gamma \in [0, \gamma_1]$, $l = 0$ and

$$z(\gamma) = \mathscr{S}_\varphi^\chi(\gamma, \sigma_l)\mathscr{P}_l(\sigma_l, z(\gamma_l^-)) + \int_{\sigma_l}^\gamma (\varphi(\gamma) - \varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma, e)[\mathscr{G}v(e) + \Delta(e, z(e))]\varphi'(e)de, \tag{5.1}$$

for all $\gamma \in (\sigma_l, \gamma_{l+1}]$, $l = 1, 2, \ldots, \psi$.

Next, we define the control function $v(t)$ as follows

$$v_z(\gamma) = \left(\Delta_{\sigma_l}^{\gamma_{l+1}}\right)^{-1} \Bigg[z_{\gamma_{l+1}} - \mathscr{S}_\varphi^\chi(\gamma_{l+1}, \sigma_l)\mathscr{P}_l(\sigma_l, z(\gamma_l^-)) - \int_{\sigma_l}^{\gamma_{l+1}} (\varphi(\gamma_{l+1}) - \varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma_{l+1}, e)\Delta(e, z(e))\varphi'(e)de \Bigg], \quad \forall \gamma \in (\sigma_l, \gamma_{l+1}], \; l = 0, \ldots, \psi, \tag{5.2}$$

where $\mathscr{P}_0(0, \cdot) = z_0 - \mathscr{Q}(z)$.

We put the value of control function $v_z(\gamma)$ from Eq. (5.2) in the Eq. (5.1) and replace γ by γ_{l+1}, $l = 0, 1, \ldots, \psi$, $z(\gamma_{\psi+1}) = z_{\gamma_{\psi+1}} = z_1$, we have

$$\begin{aligned} z(\gamma_{l+1}) &= \mathscr{S}_\varphi^\chi(\gamma_{l+1}, \sigma_l)\mathscr{P}_l(\sigma_l, z(\gamma_l^-)) \\ &\quad + \int_{\sigma_l}^{\gamma_{l+1}} (\varphi(\gamma_{l+1}) - \varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma_{l+1}, e)[\mathscr{G}v_z(e) + \Delta(e, z(e))]\varphi'(e)de \\ &= \mathscr{S}_\varphi^\chi(\gamma_{l+1}, \sigma_l)\mathscr{P}_l(\sigma_l, z(\gamma_l^-)) \\ &\quad + \left(\Delta_{\sigma_l}^{\gamma_{l+1}}\right)\left(\Delta_{\sigma_l}^{\gamma_{l+1}}\right)^{-1} \Bigg(z_{\gamma_{l+1}} - \mathscr{S}_\varphi^\chi(\gamma_{l+1}, \sigma_l)\mathscr{P}_l(\sigma_l, z(\gamma_l^-)) \\ &\quad - \int_{\sigma_l}^{\gamma_{l+1}} (\varphi(\gamma_{l+1}) - \varphi(q))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma_{l+1}, q)\Delta(q, z(q))\varphi'(q)dq \Bigg) \\ &\quad + \int_{\sigma_l}^{\gamma_{l+1}} (\varphi(\gamma_{l+1}) - \varphi(e))^{\chi-1} \mathscr{T}_\varphi^\chi(\gamma_{l+1}, e)\Delta(e, z(e))\varphi'(e)de \\ &= z_{\gamma_{l+1}}. \end{aligned}$$

Hence, control function steers the state from initial state z_0 to target z_1.

THEOREM 3. *If the hypotheses* [A1]–[A3] *and* [A5]–[A6] *are fulfilled. Then the* φ*-Caputo fractional system* (1.2) *exactly controllable on* $\mathscr{J}_1$*, provided that*

$$\max_{1\leqslant l\leqslant\psi}\left[\hat{\mathscr{K}}_{\mathscr{P}_l} + \mathscr{M}\left(1+\frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(b)-\varphi(0))^{\chi}}{\Gamma(\chi+1)}\right)\left(\hat{\mathscr{K}}_{\mathscr{Q}}+\hat{\mathscr{K}}_{\mathscr{P}_l}+\frac{\hat{\mathscr{K}}_{\Delta}(\varphi(b)-\varphi(0))^{\chi}}{\Gamma(1+\chi)}\right)\right]<1. \tag{5.3}$$

Proof. For $\rho>0$, we define

$$\mathscr{R}_\rho=\{z\in\mathscr{PC}(\mathscr{Z}):\|z\|_{\mathscr{PC}}\leqslant\rho\}.$$

Clearly, $\mathscr{R}_\rho$ is a bounded and closed subset of $\mathscr{PC}(\mathscr{Z})$. We define the operator $\mathfrak{G}$ on $\mathscr{R}_\rho$ as follows

$$(\mathfrak{G}z)(\gamma)=\begin{cases}\mathscr{S}_\varphi^{\chi}(\gamma,0)[z_0-\mathscr{Q}(z)]\\ +\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^{\chi}(\gamma,e)\mathscr{G}v_z(e)\varphi'(e)de\\ +\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^{\chi}(\gamma,e)\Delta(e,z(e))\varphi'(e)de & \gamma\in[0,\gamma_1],\ l=0,\\ \mathscr{P}_l(\gamma,z(\gamma_l^-)), & \gamma\in(\gamma_l,\sigma_l],\ l\geqslant1,\\ \mathscr{S}_\varphi^{\chi}(\gamma,\sigma_l)\mathscr{P}_l(\sigma_l,z(\gamma_l^-))\\ +\int_{\sigma_l}^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^{\chi}(\gamma,e)\mathscr{G}v_z(e)\varphi'(e)de\\ +\int_{\sigma_l}^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^{\chi}(\gamma,e)\Delta(e,z(e))\varphi'(e)de & \gamma\in(\sigma_l,\gamma_{l+1}],\ l\geqslant1.\end{cases}$$

Step 1. There exists $\rho>0$ such that $\mathfrak{G}(\mathscr{R}_\rho)\subset\mathscr{R}_\rho$. If we assume that the assertion is not true, then for $\rho>0$, we take $\gamma\in\mathscr{J}_1$ and $z^\rho\in\mathscr{R}_\rho$ such that $\|\mathfrak{G}(z^\rho)(\gamma)\|>\rho$. For $\gamma\in[0,\gamma_1]$, we obtain

$$\begin{aligned}\rho &< \|\mathfrak{G}(z^\rho)(\gamma)\|\\ &\leqslant \|\mathscr{S}_\varphi^{\chi}(\gamma,0)[z_0-\mathscr{Q}(z^\rho)]\|\\ &\quad+\left\|\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^{\chi}(\gamma,e)\Delta(e,z^\rho(e))\varphi'(e)de\right\|\\ &\quad+\left\|\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^{\chi}(\gamma,e)\mathscr{G}v_z(e)\varphi'(e)de\right\|\\ &\leqslant \frac{\Delta_0\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_1)-\varphi(0))^{\chi}}{\Gamma(\chi+1)}\|z_{\gamma_1}\|\\ &\quad+\left(1+\frac{\Delta_0\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_1)-\varphi(0))^{\chi}}{\Gamma(\chi+1)}\right)\\ &\quad\times\left(\mathscr{M}\|z_0\|+\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\rho+\mathscr{M}\|\mathscr{Q}(0)\|+\frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_{\Delta}(1+\rho)(\varphi(\gamma_1)-\varphi(0))^{\chi}\right).\end{aligned}$$

If $\gamma \in (\gamma_l, \sigma_l],\ l = 1,2,\ldots,\psi,$ then we obtain

$$\rho < \|\mathfrak{G}(z^{\rho})(\gamma)\| = \|\mathscr{P}_l(\gamma, z^{\rho}(\gamma_l^-))\|^2 \leqslant \mathscr{K}_{\mathscr{P}_l}(1+\rho). \tag{5.4}$$

Similarly, if $\gamma \in (\sigma_l, \gamma_{l+1}],\ l = 1,2,\ldots,\psi,$ then we obtain

$$\begin{aligned}\rho &< \|\mathfrak{G}(z^{\rho})(\gamma)\| \\ &\leqslant \|\mathscr{S}_{\varphi}^{\chi}(\gamma,\sigma_l)\mathscr{P}_l(\sigma_l, z^{\rho}(\gamma_l^-))\| \\ &\quad + \left\|\int_{\sigma_l}^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_{\varphi}^{\chi}(\gamma,e)\Delta(e,z^{\rho}(e))\varphi'(e)de\right\| \\ &\quad + \left\|\int_{\sigma_l}^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_{\varphi}^{\chi}(\gamma,e)\mathscr{G}v_z(e)\varphi'(e)de\right\| \\ &\leqslant \frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^{\chi}}{\Gamma(\chi+1)}\|z_{\gamma_{l+1}}\| \\ &\quad + \left(1+\frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^{\chi}}{\Gamma(\chi+1)}\right) \\ &\quad \times\left(\mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l}(1+\rho)+\frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_{\Delta}(1+\rho)(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^{\chi}\right).\end{aligned}$$

For every $\gamma \in \mathscr{J}_1,$ we obtain

$$\begin{aligned}\rho &< \|\mathfrak{G}(z^{\rho})(\gamma)\| \\ &\leqslant \mathscr{X}^* + \hat{\mathscr{K}}_{\mathscr{P}_l}\rho + \left(1+\frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(b)-\varphi(0))^{\chi}}{\Gamma(\chi+1)}\right) \\ &\quad \times\left(\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\rho+\mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l}\rho+\frac{\mathscr{M}\hat{\mathscr{K}}_{\Delta}\rho(\varphi(b)-\varphi(0))^{\chi}}{\Gamma(1+\chi)}\right),\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}\mathscr{X}^* = \max_{1\leqslant l\leqslant\psi}&\left\{\frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^{\chi}}{\Gamma(\chi+1)}\|z_{\gamma_{l+1}}\| + \hat{\mathscr{K}}_{\mathscr{P}_l}\right. \\ &+\left(1+\frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^{\chi}}{\Gamma(\chi+1)}\right) \\ &\left.\times\left(\mathscr{M}\|z_0\|+\mathscr{M}\|\mathscr{Q}(0)\|+\mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_l}+\frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_{\Delta}(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^{\chi}\right)\right\}.\end{aligned}$$

Here, $\mathscr{X}^*$ is independent of $\rho,$ both sides of Eq. (5.5) are dividing by ρ and taking $\rho \to \infty$, we obtain

$$1 < \hat{\mathscr{K}}_{\mathscr{P}_l} + \mathscr{M}\left(1+\frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(b)-\varphi(0))^{\chi}}{\Gamma(\chi+1)}\right)\left(\hat{\mathscr{K}}_{\mathscr{Q}}+\hat{\mathscr{K}}_{\mathscr{P}_l}+\frac{\hat{\mathscr{K}}_{\Delta}(\varphi(b)-\varphi(0))^{\chi}}{\Gamma(1+\chi)}\right),$$

which contradicts to Eq. (5.3). Hence, for some $\rho > 0,\ \mathfrak{G}(\mathscr{R}_{\rho}) \subset \mathscr{R}_{\rho}$.

Step 2. $\mathfrak{G}$ is a contraction mapping on $\mathscr{R}_\rho$.
For all $z_1, z_2 \in \mathscr{R}_\rho$, if $\gamma \in [0,\gamma_1]$, then we obtain

$$\begin{aligned}
&\|(\mathfrak{G}z_1)(\gamma)-(\mathfrak{G}z_2)(\gamma)\| \\
\leqslant\ &\mathscr{M}\,\|\mathscr{Q}(z_1)-\mathscr{Q}(z_2)\| \\
&+\left\|\int_0^\gamma (\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma,e)[\Delta(e,z_1(e))-\Delta(e,z_2(e))]\varphi'(e)de\right\| \\
&+\left\|\int_0^\gamma (\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma,e)(v_{z_1}(e)-v_{z_2}(e))\varphi'(e)de\right\| \\
\leqslant\ &\left(1+\frac{\Delta_0\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_1)-\varphi(0))^\chi}{\Gamma(\chi+1)}\right) \\
&\times\left(\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}+\frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_\Delta(\varphi(\gamma_1)-\varphi(0))^\chi\right)\|z_1-z_2\|_{\mathscr{PC}}. \qquad (5.6)
\end{aligned}$$

If $\gamma \in (\gamma_l,\sigma_l]$, $l=1,2,\dots,\psi$, then we obtain

$$\|(\mathfrak{G}z_1)(\gamma)-(\mathfrak{G}z_2)(\gamma)\| \leqslant \hat{\mathscr{M}}_{\mathscr{P}_l}\|z_1-z_2\|_{\mathscr{PC}}. \qquad (5.7)$$

Similarly, if $\gamma \in (\sigma_l,\gamma_{l+1}]$, $l=1,2,\dots,\psi$, then we obtain

$$\begin{aligned}
&\|(\mathfrak{G}z_1)(\gamma)-(\mathfrak{G}z_2)(\gamma)\| \\
\leqslant\ &\mathscr{M}\,\left\|\mathscr{P}_l(\sigma_l,z_1(\gamma_l^-))-\mathscr{P}_l(\sigma_l,z_2(\gamma_l^-)))\right\| \\
&+\left\|\int_{\sigma_l}^\gamma (\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma,e)[\Delta(e,z_1(e))-\Delta(e,z_2(e))]\varphi'(e)de\right\| \\
&+\left\|\int_{\sigma_l}^\gamma (\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma,e)(v_{z_1}(e)-v_{z_2}(e))\varphi'(e)de\right\| \\
\leqslant\ &\left(1+\frac{\Delta_l\mathscr{M}\|\mathscr{G}\|(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^\chi}{\Gamma(\chi+1)}\right) \\
&\times\left(\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}+\frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_\Delta(\varphi(\gamma_{l+1})-\varphi(\sigma_l))^\chi\right)\|z_1-z_2\|_{\mathscr{PC}}. \qquad (5.8)
\end{aligned}$$

By Eqs. (5.6)–(5.8), we obtain

$$\|(\mathfrak{G}z_1)(\gamma)-(\mathfrak{G}z_2)(\gamma)\| \leqslant \mathscr{O}_{\mathfrak{G}}\,\|z_1-z_2\|_{\mathscr{PC}},$$

where

$$\mathscr{O}_{\mathfrak{G}} = \max_{1\leqslant l\leqslant\psi}\left[r_0,\ \hat{\mathscr{M}}_{\mathscr{P}_l},\ r_l\right].$$

Hence,

$$\|\mathfrak{G}z_1-\mathfrak{G}z_2\|_{\mathscr{PC}} \leqslant \mathscr{O}_{\mathfrak{G}}\|z_1-z_2\|_{\mathscr{PC}}. \qquad (5.9)$$

Thus, $\mathfrak{G}$ is a contraction mapping on $\mathscr{R}_\rho$. Hence, by the Banach fixed point theorem, there exists a unique mild solution of system (1.2). Hence, the system (1.2) is controllable on $\mathscr{J}_1$. □

6. Trajectory controllability

Let $\mathscr{E}$ be the set of all functions $z(\cdot)$ defined on $\mathscr{J}_1$ such that $z(0)=z_0$ and $z(b)=z_1$ for all $\gamma \in \mathscr{J}_1$ and the fractional derivative ${}_0^C\mathrm{D}_\varphi^\chi$ exists almost everywhere on $\mathscr{J}_1$. We call $\mathscr{E}$ the set of all feasible trajectories for the system (1.2).

DEFINITION 6. [26] The system (1.2) is said to be trajectory controllable (T-controllable) on $\mathscr{J}_1$, if for $y \in \mathscr{E}$, there exists a control v such that the mild solution of system (1.2) satisfies $z(\gamma)=y(\gamma)$ almost everywhere.

We assume the following hypotheses

[B]: The continuous operator $\mathscr{G}$ is non-zero.

THEOREM 4. *Assume that the hypotheses* [A1]–[A4] *and [B] are hold, then the* φ*-Caputo fractional system* (1.2) *is* T*-controllable on* $\mathscr{J}_1$*, provided that*

$$\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}+\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}<1.$$

Proof. Let $y(\gamma)$ be any given prescribed trajectory on $\mathscr{J}_1$. For, $0<\chi<1$, we define a suitable control function $v(\gamma)$ as follows

$$v(\gamma)=\begin{cases}\dfrac{{}_0^C\mathrm{D}_\varphi^\chi y(\gamma)-\mathscr{H}y(\gamma)-\Delta(\gamma,y(\gamma))}{\mathscr{G}}, & \gamma\in(\sigma_l,\gamma_{l+1}],\ l\geqslant 1\\ 0, & \gamma\in(\gamma_l,\sigma_l],\ l\geqslant 1.\end{cases}$$

Put the value of $v(\gamma)$ in Eq. (1.2) and choose $\kappa(\gamma)=z(\gamma)-y(\gamma)$. We consider IVP defined as follows:

$$\begin{cases}{}_0^C\mathrm{D}_\varphi^\chi\kappa(\gamma)=\mathscr{H}\kappa(\gamma)+[\Delta(\gamma,z(\gamma))-\Delta(\gamma,y(\gamma))], & \gamma\in(\sigma_l,\gamma_{l+1}],\ l\geqslant 1\\ \kappa(\gamma)=\mathscr{P}_l(\gamma,z(\gamma_l^-))-\mathscr{P}_l(\gamma,y(\gamma_l^-)), & \gamma\in(\gamma_l,\sigma_l],\ l\geqslant 1,\\ \kappa(0)+[\mathscr{Q}(z)-\mathscr{Q}(y)]=0.\end{cases} \tag{6.1}$$

Then the mild solution of the IVP (6.1), is given by

$$\kappa(\gamma)=\begin{cases}\mathscr{S}_\varphi^\chi(\gamma,0)[\mathscr{Q}(y)-\mathscr{Q}(z)]\\ +\int_0^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma,e)\\ \times[\Delta(e,z(e))-\Delta(e,y(e))]\varphi'(e)de & \gamma\in[0,\gamma_1],\ l=0,\\ \mathscr{S}_\varphi^\chi(\gamma,\sigma_l)[\mathscr{P}_l(\sigma_l,z(\gamma_l^-))-\mathscr{P}_l(\sigma_l,y(\gamma_l^-))]\\ +\int_{\sigma_l}^\gamma(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_\varphi^\chi(\gamma,e)\\ \times[\Delta(e,z(e))-\Delta(e,y(e))]\varphi'(e)de, & \gamma\in(\sigma_l,\gamma_{l+1}],\ l\geqslant 1.\end{cases} \tag{6.2}$$

For $\gamma \in [0,\gamma_1],$ we obtain

$$\begin{aligned}\|\kappa(\gamma)\| &\leqslant \mathscr{M}\,\|\mathscr{Q}(y)-\mathscr{Q}(z)\| \\ &\quad + \left\|\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_{\varphi}^{\chi}(\gamma,e)[\Delta(e,z(e))-\Delta(e,y(e))]\varphi'(e)de\right\| \\ &\leqslant \mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}\|z-y\| + \frac{\mathscr{M}}{\Gamma(\chi)}\hat{\mathscr{M}}_{\Delta}\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\|\kappa(e)\|\varphi'(e)de.\end{aligned}$$

For each $\gamma \in (\sigma_l,\gamma_{l+1}],\ l=1,2,\ldots,\psi,$ we obtain

$$\begin{aligned}\|\kappa(\gamma)\| &\leqslant \mathscr{M}\,\|\mathscr{P}_l(\sigma_l,z(\gamma_l^-))-\mathscr{P}_l(\sigma_l,y(\gamma_l^-))\| \\ &\quad + \left\|\int_{\sigma_l}^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\mathscr{T}_{\varphi}^{\chi}(\gamma,e)[\Delta(e,z(e))-\Delta(e,y(e))]\varphi'(e)de\right\| \\ &\leqslant \mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}\|z-y\| + \frac{\mathscr{M}}{\Gamma(\chi)}\hat{\mathscr{M}}_{\Delta}\int_{\sigma_l}^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\|\kappa(e)\|\varphi'(e)de.\end{aligned}$$

Hence, for $\gamma \in \mathscr{J}_1,$ we obtain

$$\begin{aligned}&[1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}-\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}]\sup_{\gamma\in\mathscr{J}_1}\|\kappa(\gamma)\| \\ &\leqslant \frac{\mathscr{M}}{\Gamma(\chi)}\hat{\mathscr{M}}_{\Delta}\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\sup_{e\in\mathscr{J}_1}\|\kappa(e)\|\varphi'(e)de.\end{aligned}$$

Then, we get

$$\begin{aligned}\sup_{\gamma\in\mathscr{J}_1}\|\kappa(\gamma)\| &\leqslant \frac{\mathscr{M}\hat{\mathscr{M}}_{\Delta}}{\Gamma(\chi)[1-\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}}-\mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_l}]} \\ &\quad \times\int_0^{\gamma}(\varphi(\gamma)-\varphi(e))^{\chi-1}\sup_{e\in\mathscr{J}_1}\|\kappa(e)\|\varphi'(e)de.\end{aligned}$$

By Gronwall's inequality, we obtain

$$\sup_{\gamma\in\mathscr{J}_1}\|\kappa(\gamma)\| = 0.$$

Hence,

$$\kappa(\gamma) = 0,$$

i.e. $z(\gamma) = y(\gamma)$ almost everywhere. Thus, the φ-Caputo fractional system (1.2) is T-controllable on $\mathscr{J}_1$.

7. Example

Consider the following φ-Caputo fractional system:

$$\begin{cases} {}^{C}_{0}\mathrm{D}^{\chi}_{\varphi} z(\gamma,\alpha) = z_{\alpha\alpha}(\gamma,\alpha) + \dfrac{\gamma e^{-\gamma} z(\gamma,\alpha)}{3(1+|z(\gamma,\alpha)|)}, \ \ \gamma \in (0,0.30] \cup (0.60,1], \ \ \alpha \in [0,\pi], \\ z(\gamma,\alpha) = \dfrac{1}{5}(\sin\gamma) z(0.30^{-},\alpha), \ \ \gamma \in (0.30,0.60], \ \ \alpha \in [0,\pi], \\ z(\gamma,0) = 0 = z(\gamma,\pi), \\ z(0,\alpha) + \dfrac{1}{15} z(\gamma,\alpha) = z_0(\alpha), \end{cases} \tag{7.1}$$

where $\chi = 2/3$ and $0 = \sigma_0 = \gamma_0 < \gamma_1 < \sigma_1 < \gamma_2 = b$, with $\gamma_1 = 0.30$, $\sigma_1 = 0.60$, $\gamma_2 = 1$. Let $\varphi(\gamma) = \gamma$ and $\mathscr{Z} = \mathscr{S} = L^2([0,\pi])$. Define an operator $\mathscr{H} : \mathscr{D}(\mathscr{H}) \subseteq \mathscr{Z} \to \mathscr{Z}$ by $\mathscr{H}\delta = \delta''$ with

$$\mathscr{D}(\mathscr{H}) = \{\delta \in \mathscr{Z} : \delta, \delta' \text{ are absolutely continuous and } \delta'' \in \mathscr{Z}, \ \delta(0) = 0 = \delta(\pi)\}.$$

$\mathscr{H}$ has a discrete spectrum, the normalized eigenvectors $e_n(\alpha) = \sqrt{2/\pi}\sin(n\alpha)$ corresponding to eigenvalue are $-n^2$, $n \in \mathbb{N}$ and $\mathscr{H}$ generates an analytic semigroup $\{\mathscr{T}(\gamma)\}_{\gamma \geqslant 0}$ in $\mathscr{Z}$, which uniformly bounded and defined as

$$\mathscr{T}(\gamma)\alpha = \sum_{n=1}^{\infty} e^{-n^2\gamma} \langle \alpha, e_n \rangle e_n, \ \ \alpha \in \mathscr{Z},$$

with $\|\mathscr{T}(\gamma)\| \leqslant e^{-\gamma} \ \forall \ \gamma \geqslant 0$. Thus, we choose $\mathscr{M} = 1$ that implies that

$$\sup_{\gamma \in [0,\infty)} \|\mathscr{T}(\gamma)\| = 1.$$

Let $z(\gamma)(\alpha) = z(\gamma,\alpha)$ and the functions Δ, $\mathscr{P}_1$ and $\mathscr{Q}$ are defined as

$$\begin{aligned} \Delta(\gamma,z)(\alpha) &= \frac{\gamma e^{-\gamma} z(\gamma,\alpha)}{3(1+|z(\gamma,\alpha)|)}, \\ \mathscr{P}_1(\gamma, z(\gamma_1^{-}))(\alpha) &= \frac{1}{5}(\sin\gamma) z(0.30^{-},\alpha), \\ \mathscr{Q}(z)(\alpha) &= \frac{1}{15} z(\gamma,\alpha). \end{aligned}$$

We obtain $\hat{\mathscr{K}}_{\Delta} = \hat{\mathscr{M}}_{\Delta} = 1/3$, $\hat{\mathscr{K}}_{\mathscr{Q}} = 1/15$, $\hat{\mathscr{K}}_{\mathscr{P}_1} = \hat{\mathscr{M}}_{\mathscr{P}_1} = 1/5$, and

1. $\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}} = 1/15 < 1$.
2. $\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}} + \hat{\mathscr{K}}_{\mathscr{P}_1} + \mathscr{M}\hat{\mathscr{K}}_{\mathscr{P}_1} + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{K}}_{\Delta}(\varphi(b) - \varphi(0))^{\chi} = 0.8359 < 1$.
3. $\max\Big[\mathscr{M}\hat{\mathscr{K}}_{\mathscr{Q}} + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_{\Delta}(\varphi(\gamma_1) - \varphi(0))^{\chi}, \ \hat{\mathscr{M}}_{\mathscr{P}_l}, \ \mathscr{M}\hat{\mathscr{M}}_{\mathscr{P}_1} + \frac{\mathscr{M}}{\Gamma(1+\chi)}\hat{\mathscr{M}}_{\Delta}(\varphi(\gamma_2) - \varphi(\sigma_1))^{\chi}\Big] = \max[0.2321, 1/5, 0.4005] = 0.4005 < 1$.

All hypotheses of Theorem (2) are satisfied. Hence, the system (7.1) has a unique stable mild solution on $\mathscr{J}_1$.

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