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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR $p(x)$–KIRCHHOFF TYPE PROBLEMS WITH NONHOMOGENEOUS NEUMANN CONDITIONS

FARIBA GHAREHGAZLOUEI AND SHAPOUR HEIDARKHANI*

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Abstract. In this paper, we are interested to discuss the existence of multiple solutions for a class of $p(x)$-Kirchhoff type equations with nonhomogeneous Neumann boundary conditions arising in modelling of various phenomena in the study of nonlinear elasticity theory, electro-rheological fluids, and so on. By using a consequence of the local minimum theorem due to Bonanno we look into the existence of one solution under algebraic conditions on the nonlinear term, and two solutions for the problem under algebraic conditions with the classical Ambrosetti-Rabinowitz condition on the nonlinear term. Furthermore, by employing a three-critical-point theorem due to Bonanno and Marano, we guarantee the existence of three solutions for the problem in a special case.

1. Introduction

In this work, we study the existence and multiplicity of solutions for the following perturbed $p(x)$-Kirchhoff-type problem

$$\begin{cases} T(u) = \lambda f(x,u(x)), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \mu g(\gamma(u(x))), & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$T(u) = M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx\Big)(-\Delta_{p(x)}u + \alpha(x)|u|^{p(x)-2}u),$$

$\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$, and let ν be the outward unit normal to $\partial\Omega$. $M : [0,+\infty) \to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leqslant M(t) \leqslant m_1$ for all $t \geqslant 0$. Let $\alpha \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$, $p \in C(\overline{\Omega})$, $\lambda > 0$ and $\mu \geqslant 0$ referred to as control parameters. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function and $\gamma : W^{1,p(x)}(\Omega) \to L^{p(x)}(\partial\Omega)$ is the trace operator.

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* Corresponding author.

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The $p(x)$-Laplacian is a meaningful generalization of the p-Laplacian. In recent years, the study of various mathematical problems with variable exponent problem has received considerable attention. One of the topic fields of partial differential equations that has been continuously noticed is that concerning the Sobolev space with variable exponents, $W^{1,p(.)}(\Omega)$ (where p is a function depending on x); see for example the monograph [15]. The necessary framework for the study of these problems is represented by the function spaces with variable exponent $W^{1,p(x)}(\Omega)$ and $L^{p(x)}(\Omega)$. The basic properties of such spaces can be found in [18, 24, 25].

The problem (1.1) is a generalization of the stationary problem of a model introduced by Kirchhoff [24]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

for $0 < x < L$, $t \geqslant 0$ where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length and ρ_0 the initial axial tension, which extends the classical D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Kirchhoff's model like the problem (1.1) model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density.

Working in the framework of variable exponent spaces, opens the door for multiple applications. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problems of this type is the model of motion of electro-rheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field (see [14, 26, 29] and their references). In addition, the variable exponent spaces are involved in studies hat provide other types of applications, e.g., in image restoration [9] and contact mechanics [6]. Recently, this theory has been expanded by many researchers. For example, Fan and Ji have treated in [17] the problem

$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = f(x,u) + g(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

they proved the existence of infinitely many solutions of the problem under weaker hypotheses by applying a variational principle due to Ricceri and the theory of the variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$. D'Aguì and Sciammetta in [11], investigated the following Neumann problem

$$\begin{cases} -\Delta_{p(x)}u + \alpha(x)|u|^{p(x)-2}u = \lambda f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \mu g(\gamma(u)), & \text{on } \partial\Omega, \end{cases}$$

under an appropriate oscillating behavior of the primitive of the nonlinearity and a suitable growth of the primitive of g at infinity, the existence of infinitely many weak solutions for the problem was obtained. Recently, Kirchhoff type equations involving the $p(x)$-Laplacian have been investigated, but the results are rare. We refer the reader to [7, 8, 10, 12, 13, 19, 21, 22, 27] for an overview of and references on this subject. For example, In [22] the authors studied the existence and multiplicity of solutions for the nonlocal elliptic problem under Neumann boundary condition:

$$\begin{cases} T(u) = \lambda f(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

The authors for the above problem, by using the variational method, under suitable assumptions on f, obtained first one solution, then two weak solutions, when $T(u) = M(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx)(-\Delta_{p(x)}u + \alpha(x)|u|^{p(x)-2}u)$. In [20] using a three-critical-point theorem due to Bonanno and Candito the existence of at least three weak solutions for the problem (1.1) has been discussed, while in [13] the existence of at least one weak solution for the same problem under rather different assumptions on data has been studied using a version of Ricceri's variational principle as given by Bonanno and Molica Bisci.

Here, we deal with the problem (1.1) when the nonlinearity f has the subcritical growth condition, via variational methods, we obtain the existence of at least one, two and three weak solutions for the exact collections of the parameters λ and μ. The main tools are critical point theorems obtained in [1, 2, 5].

This paper is organized as follows: In Section 2, we present some preliminary knowledge on the anisotropic Sobolev spaces with variable exponent. Section 3 contains the main results and the proofs of the main results. We prove the existence of one weak solution in Theorem 4, the existence of two solutions in Theorem 5 and the existence of three weak solutions in Theorem 6 for our Neumann elliptic problem.

2. Preliminaries

In this section, we introduce some definitions and results which will be used in the next section.

In the sequel, we assume that $p \in C(\bar{\Omega})$ verifies the following condition:

$$N < p^- := \inf_{x\in\Omega} p(x) \leqslant p(x) \leqslant p^+ := \sup_{x\in\Omega} p(x) < \infty.$$

We define the variable exponent Lebesgue spaces by

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

$$L^{p(x)}(\partial\Omega) := \left\{ u : \partial\Omega \to \mathbb{R} \text{ measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma < \infty \right\},$$

where $d\sigma$ is the surface measure on $\partial\Omega$. We consider the norms on $L^{p(x)}(\Omega)$ and $L^{p(x)}(\partial\Omega)$, respectively

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{\beta > 0 : \int_\Omega \Big|\frac{u}{\beta}\Big|^{p(x)} dx \leqslant 1\Big\},$$

$$\|u\|_{L^{p(x)}(\partial\Omega)} = \inf\Big\{\eta > 0 : \int_{\partial\Omega} \Big|\frac{u}{\eta}\Big|^{p(x)} d\sigma \leqslant 1\Big\}.$$

Let X be the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ defined by

$$W^{1,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\big\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}. \tag{2.1}$$

The spaces $(L^{p(x)}(\Omega), \|.\|_{L^{p(x)}(\Omega)})$ and $(W^{1,p(x)}(\Omega), \|.\|_{W^{1,p(x)}(\Omega)})$ are separable, reflexive and uniformly convex Banach spaces, (see [18]). Moreover, since $\alpha \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$ is assumed, then the following norm

$$\|u\|_\alpha = \inf\Big\{\sigma > 0 : \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\sigma}\Big|^{p(x)} + \alpha\Big|\frac{u(x)}{\sigma}\Big|^{p(x)}\Big) d\sigma \leqslant 1\Big\},$$

on $W^{1,p(x)}(\Omega)$ is equivalent to that introduce in (2.1). Since $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ (see [18]), and $p^- > N, W^{1,p(x)}(\Omega)$ is continuously embedded in $C^0(\bar{\Omega})$ (the space of continuous functions) and one has

$$\|u\|_{C^0(\bar{\Omega})} \leqslant k_{p^-}\|u\|_{W^{1,p^-}(\Omega)}.$$

When Ω is convex, an explicit upper bound for the constant k_{p^-} is

$$k_{p^-} \leqslant 2^{\frac{p^- -1}{p^-}} \max\Big\{\Big(\frac{1}{\|\alpha\|_1}\Big)^{\frac{1}{p^-}}, \frac{d}{N^{\frac{1}{p^-}}}\Big(\frac{p^- -1}{p^- - N}\operatorname{meas}(\Omega)\Big)^{\frac{p^- -1}{p^-}} \frac{\|\alpha\|_\infty}{\|\alpha\|_1}\Big\},$$

where $\|\alpha\|_1 = \int_\Omega \alpha(x)dx$ and $\|\alpha\|_\infty = \sup_{x\in\Omega}$ $d = \operatorname{diam}(\Omega)$ and $\operatorname{meas}(\Omega)$ is the Lebesgue measure of Ω (see [3, Remark 1]). On the other hand, taking into account that $p^- \leqslant p(x)$, [25, Theorem 2.8] ensures that $L^{p(x)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ and the constant of such embedding does not exceed $1 + \operatorname{meas}(\Omega)$. So, one has

$$\|u\|_{W^{1,p^-}(\Omega)} \leqslant (1 + \operatorname{meas}(\Omega))\|u\|_{W^{1,p(x)}(\Omega)} \leqslant (1 + \operatorname{meas}(\Omega))\|u\|_\alpha.$$

In conclusion, put

$$c = k_{p^-}(1 + \operatorname{meas}(\Omega)),$$

it results

$$\|u\|_{C^0(\bar{\Omega})} \leqslant c\|u\|_\alpha \tag{2.2}$$

for each $u \in W^{1,p(x)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the $\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}dx$ of the space $L^{p(x)}(\Omega)$, we have the following result.

PROPOSITION 1. (See [18]) *If* $u, u_n \in L^{p(x)}(\Omega)$ *and* $p^+ < \infty$, *then*

1. *If* $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \leqslant \rho_{p(x)}(u) \leqslant \|u\|^{p^+}_{L^{p(x)}(\Omega)}$,
2. *If* $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \leqslant \rho_{p(x)}(u) \leqslant \|u\|^{p^-}_{L^{p(x)}(\Omega)}$,
3. $\lim_{n\to\infty} \|u_n - u\|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

PROPOSITION 2. (See [18, 25]) *Let* $\rho_\alpha(u) = \int_\Omega (|\nabla u|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx$ *for* $u \in W^{1,p(x)}(\Omega)$ *we have*

(i) *If* $\|u\|_\alpha \geqslant 1$, *then* $\|u\|_\alpha^{p^-} \leqslant \rho_\alpha(u) \leqslant \|u\|_\alpha^{p^+}$,

(ii) *If* $\|u\|_\alpha \leqslant 1$, *then* $\|u\|_\alpha^{p^+} \leqslant \rho_\alpha(u) \leqslant \|u\|_\alpha^{p^-}$,

(iii) $\lim_{n\to\infty} \|u_n - u\|_\alpha = 0 \Leftrightarrow \lim_{n\to\infty} \rho_\alpha(u_n - u) = 0$.

We introduce the functions $F : \Omega \times \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ and $\hat{M} : [0, +\infty) \to \mathbb{R}$, respectively, as follows

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

$$G(t) = \int_0^t g(\xi)d\xi \quad \text{for all } t \in \mathbb{R},$$

$$\hat{M}(t) = \int_0^t M(\xi)d\xi \quad \text{for all } t \geqslant 0.$$

We define, for any $u \in X = W^{1,p(x)}(\Omega)$, the functionals $\Phi, \Psi_{\lambda,\mu} : X \to \mathbb{R}$ as

$$\Phi(u) := \hat{M}\Big(\int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx \Big), \tag{2.3}$$

and

$$\Psi_{\lambda,\mu}(u) := \int_\Omega F(x,u(x))dx + \frac{\mu}{\lambda} \int_{\partial\Omega} G(\gamma(u(x)))d\sigma. \tag{2.4}$$

We say that a function $u \in X$ is a weak solution of the problem (1.1) if

$$\begin{aligned} &M\Big(\int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx \Big) \\ &\times \int_\Omega (|\nabla u|^{p(x)-2} \nabla u(x) \nabla v(x) + \alpha(x)|u(x)|^{p(x)-2} u(x)v(x))dx \\ &-\lambda \int_\Omega f(x,u(x))v(x)dx - \mu \int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))d\sigma = 0, \end{aligned}$$

holds for all $v \in X$. For our convenience, set

$$G^\tau := a(\partial\Omega) \max_{|\xi| \leqslant \tau} G(\xi), \quad \forall\, \tau > 0,$$

where $a(\partial\Omega) = \int_{\partial\Omega} d\sigma$ and

$$G_\delta := a(\partial\Omega) \inf_{t\in[0,\delta]} G(t), \quad \forall\, \delta \geqslant 1.$$

If g is sign-changing, then clearly $G^\tau \geqslant 0$ and $G_\delta \leqslant 0$.

DEFINITION 1. Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais-Smale condition cut off upper at r (in short $(P.S.)^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

- (e_1) $\{I(u_n)\}$ is bounded,
- (e_2) $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$,
- (e_3) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$

has a convergent subsequence.

The following three theorems are the main tools in the next section to prove result. While the first two results are due to Bonanno, the third one is due to Bonanno and Marano.

THEOREM 1. [2, Theorem 2.3] (see [1, Theorem 5.1]) *Let X be a real Banach space, $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{v} \in X$, with $0 < \Phi(\overline{v}) < r$ such that*

- (E_1) $\dfrac{\sup_{\Phi(u)\leqslant r} \Psi(u)}{r} < \dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$,
- (E_2) *for all* $\lambda \in \Lambda := \left(\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \dfrac{r}{\sup_{\Phi(u)\leqslant r} \Psi(u)} \right)$ *the functional* $I_\lambda := \Phi - \lambda\Psi$ *satisfies* $(P.S.)^{[r]}$ *condition.*

Then, for each $\lambda \in \Lambda$ there is $u_{0,\lambda} \in \Phi^{-1}(0,r)$ such that $I'_\lambda(u_{0,\lambda}) = \vartheta_{X^*}$ and $I_\lambda(u_{0,\lambda}) < I_\lambda(u)$ for all $u \in \Phi^{-1}(0,r)$.

THEOREM 2. [2, Theorem 3.2] *Let X be a real Banach space, $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\right)$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(P.S.)$ condition and it is unbounded from below. Then, for each

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\right)$$

the functional I_λ admits two distinct critical points.

THEOREM 3. [5, Theorem 3.6] *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive and continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that

(E_3) $\frac{\sup_{\Phi(u)\leqslant r}\Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(E_4) *for all* $\lambda \in \Lambda := \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\leqslant r}\Psi(u)}\right)$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

We refer to [4] in which Theorems 1–3 have been successfully applied to obtain the existence of solutions for a class of nonlinear elliptic Dirichlet problems with variable exponent.

3. Main results

We start by giving the existence of one solution for the problem (1.1).

THEOREM 4. *Assume that there exist positive constants $\tau > c$ and $\delta \geqslant 1$, such that*

(H_1) $\quad m_1 p^+ c^{p^-} \|\alpha\|_1 \delta^{p^+} < m_0 p^- \tau^{p^-}$,

(H_2) $\quad \frac{\int_\Omega \sup_{|t|\leqslant\tau} F(x,t)dx}{\tau^{p^-}} < \frac{p^- m_0}{p^+ m_1 c^{p^-}\|\alpha\|_1} \frac{\int_\Omega F(x,\delta)dx}{\delta^{p^+}}$,

(H_3) $\quad F(x,t) \geqslant 0$ *for each* $(x,t) \in \Omega \times \mathbb{R}^+$.

Then, for each

$$\lambda \in \Lambda := \left] \frac{m_1 \delta^{p^+}\|\alpha\|_1}{p^- \int_\Omega F(x,\delta)dx}, \frac{m_0 \tau^{p^-}}{p^+ c^{p^-} \int_\Omega \sup_{|t|\leqslant\tau} F(x,t)dx} \right[, \tag{3.1}$$

and for every nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta_{\lambda,g} > 0$, given by

$$\min\left\{ \frac{m_0\tau^{p^-} - \lambda p^+ c^{p^-} \int_\Omega \sup_{|t|\leqslant\tau} F(x,t)dx}{p^+ c^{p^-} G^\tau}, \frac{m_1 \delta^{p^+}\|\alpha\|_1 - \lambda p^- \int_\Omega F(x,\delta)dx}{p^- G_\delta} \right\},$$

such that for each $\mu \in [0, \delta_{\lambda,g}[$, the problem (1.1) admits at least one nontrivial solution $u_\lambda \in X$ such that $\|u_\lambda\|_\infty \leqslant \tau$.

Proof. Our goal is to apply Theorem 1 to the problem (1.1). To this end, take the real Banach space X with the norm as defined in Section 2, with fix λ and μ as in the conclusion, $\Phi, \Psi_{\lambda,\mu}$ be the functionals defined in (2.3) and (2.4). It is well known that $\Psi_{\lambda,\mu}$ is a differentiable functional whose differential at the point $u \in X$ given by

$$\Psi'_{\lambda,\mu}(u)(v) = \int_\Omega f(x,u(x))v(x)dx + \frac{\mu}{\lambda}\int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))d\sigma$$

for every $v \in X$. Furthermore, $\Psi'_{\lambda,\mu} : X \to X^*$ is a compact operator. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)(v) = \; & M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx\Big) \\ & \times \int_\Omega (|\nabla u|^{p(x)-2}\nabla u(x)\nabla v(x) + \alpha(x)|u(x)|^{p(x)-2}u(x)v(x))dx\end{aligned}$$

for every $v \in X$. Moreover Φ' admits a continuous inverse on X^*. Indeed, According to Theorem 26.A(d) in [28], it is enough to verify that Φ' is coercive, hemicontinuous and uniformly monotone. Assuming $\|u\|_\alpha \geqslant 1$, we have

$$\Phi'(u)(u) \geqslant m_0\|u\|_\alpha^{p^-},$$

since $p^- > 1$ it follows that Φ' is coercive. Since Φ' is the Fréchet derivative of Φ, it follows that Φ' is continuous and bounded. Now, we show that Φ' is uniformly monotone. In fact, for any $\xi, \psi \in \mathbb{R}$, we have the following inequality (see [23]):

$$(|\xi|^{s-2}\xi - |\psi|^{s-2}\psi)(\xi - \psi) \geqslant \begin{cases} 2^{-s}|\xi-\psi|^s, & \text{if } s \geqslant 2, \\ 2^{-s}\frac{|\xi-\psi|^2}{(|\xi|+|\psi|)^{2-s}}, & \text{if } 1 < s < 2. \end{cases} \tag{3.2}$$

At this point, if $p(x) \geqslant 2$, then it follows that, for every $u, v \in X$, we deduce that

$$\begin{aligned} & (\Phi'(u) - \Phi'(v))(u-v) \\ = \; & M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx\Big) \\ & \times \int_\Omega (|\nabla u|^{p(x)-2}\nabla u(\nabla u - \nabla v) + \alpha(x)|u|^{p(x)-2}u(u-v)dx \\ & + M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla v|^{p(x)} + \alpha(x)|v(x)|^{p(x)})dx\Big) \\ & \times\Big(-\int_\Omega (|\nabla v|^{p(x)-2}\nabla v(x))(\nabla u(x) - \nabla v(x)) + \alpha(x)|v(x)|^{p(x)-2}v(x)(u(x)-v(x))dx\Big) \\ \geqslant \; & m_0\int_\Omega ((|\nabla u|^{p(x)-2}\nabla u(x) - |\nabla v|^{p(x)-2}\nabla v(x))(\nabla u(x) - \nabla v(x)) \\ & + \alpha(x)(|u(x)|^{p(x)-2}u(x) - |v(x)|^{p(x)-2}v(x))(u(x)-v(x)))dx \\ \geqslant \; & 2^{-p^-}m_0\|u-v\|_\alpha^{p^-}, \end{aligned}$$

the last inequality is obtained from Proposition 2. On the other hand, if $1 < p(x) < 2$, by Hölder's inequality, we obtain

$$\begin{aligned}
&\int_\Omega \alpha(x)|u(x)-v(x)|^{p(x)}dx \\
&\leqslant \Big(\int_\Omega \frac{\alpha(x)|u(x)-v(x)|^2}{(|u(x)|+|v(x)|)^{2-p(x)}}dx\Big)^{\frac{p(x)}{2}} \Big(\int_\Omega \alpha(x)(|u(x)|+|v(x)|)^{p(x)}dx\Big)^{\frac{2-p(x)}{2}} \\
&\leqslant D_1\Big(\int_\Omega \frac{|u(x)-v(x)|^2}{(|u(x)|+|v(x)|)^{2-p(x)}}dx\Big)^{\frac{p(x)}{2}} (\|u\|_\alpha+\|v\|_\alpha)^{\frac{(2-p(x))p(x)}{2}}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\int_\Omega |\nabla u(x)-\nabla v(x)|^{p(x)}dx \leqslant D_2\Big(\int_\Omega \frac{|\nabla u(x)-\nabla v(x)|^2}{(|\nabla u(x)|+|\nabla v(x)|)^{2-p(x)}}dx\Big)^{\frac{p(x)}{2}} \\
\times\big(\|u\|_\alpha+\|v\|_\alpha\big)^{\frac{(2-p(x))p(x)}{2}}.
\end{aligned}$$

On other side, we have

$$\begin{aligned}
&\langle \Phi'(u)-\Phi'(v), u-v\rangle \\
&\geqslant \frac{D_3}{\big(\|u\|_\alpha+\|v\|_\alpha\big)^{(2-p(x))}}\Big(\int_\Omega |\nabla u(x)-\nabla v(x)|^{p(x)}dx\Big)^{\frac{2}{p(x)}} \\
&\quad+\frac{D_3}{\big(\|u\|_\alpha+\|v\|_\alpha\big)^{(2-p(x))}}\Big(\int_\Omega \alpha(x)|u(x)-v(x)|^{p(x)}dx\Big)^{\frac{2}{p(x)}} \\
&\geqslant \frac{D_4}{\big(\|u\|_\alpha+\|v\|_\alpha\big)^{(2-p(x))}}\Big(\Big(\int_\Omega |\nabla u(x)-\nabla v(x)|^{p(x)}+\alpha(x)|u(x)-v(x)|^{p(x)}dx\Big)^{\frac{2}{p(x)}}\Big) \\
&= \frac{D_4}{\big(\|u\|_\alpha+\|v\|_\alpha\big)^{(2-p(x))}}\|u-v\|^2.
\end{aligned}$$

Thus, by [1, see Proposition 2.1] the functional $I_{\lambda,\mu} = \Phi - \lambda\Psi_{\lambda,\mu}$ verifies $(P.S.)^{[r]}$ condition for each $r > 0$ and so condition (E_2) of Theorem 1 is verified. Therefore, it remains to verify assumption (E_1) of Theorem 1. To this end, we put $r := \frac{m_0}{p^+}(\frac{\tau}{c})^{p^-}$, and pick $w \in X$, defined as

$$w(x) = \begin{cases} \delta, & \text{if } x\in\Omega, \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

one has

$$\frac{m_0\delta^{p^-}}{p^+}\|\alpha\|_1 \leqslant \Phi(w) \leqslant \frac{m_1\delta^{p^+}}{p^-}\|\alpha\|_1. \tag{3.4}$$

Hence, it follows from (H_1) that $0 < \Phi(w) < r$. Now, let $u \in X$ such that $u \in \Phi^{-1}([0,r])$, by Proposition 2, one has

$$\|u\|_\alpha < \max\Big\{\Big(\frac{rp^+}{m_0}\Big)^{\frac{1}{p^+}}, \Big(\frac{rp^+}{m_0}\Big)^{\frac{1}{p^-}}\Big\}.$$

Since $c < \tau$, we obtain

$$|u(x)| \leqslant \|u\|_\infty \leqslant c\|u\|_\alpha < \tau \quad \forall x \in \Omega.$$

Therefore, using (H_2), one has

$$\begin{aligned}
\frac{\sup_{\Phi(u)\leqslant r} \Psi_{\lambda,\mu}(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\left(\int_\Omega F(x,u(x))dx + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(u(x)))d\sigma\right)}{\frac{m_0}{p^+}\left(\frac{\tau}{c}\right)^{p^+}} \\
&\leqslant \frac{\int_\Omega \sup_{|t|\leqslant\tau} F(x,u)dx + \frac{\mu}{\lambda} a(\partial\Omega)\max_{|t|\leqslant\tau} G(t)}{\frac{m_0}{p^+}\left(\frac{\tau}{c}\right)^{p^-}} \\
&= \frac{\int_\Omega \sup_{|t|\leqslant\tau} F(x,u)dx + \frac{\mu}{\lambda} G^\tau}{\frac{m_0}{p^+}\left(\frac{\tau}{c}\right)^{p^-}}.
\end{aligned}$$

On the other hand, we have

$$\Psi_{\lambda,\mu}(w) = \int_\Omega F(x,w(x))dx + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(w(x)))d\sigma \geqslant \int_\Omega F(x,w(x))dx + \frac{\mu}{\lambda} G_\delta.$$

Moreover, thanks to (3.4), one has

$$\frac{\Psi_{\lambda,\mu}(w)}{\Phi(w)} \geqslant \frac{p^- \int_\Omega F(x,\delta)dx}{m_1 \delta^{p^+}\|\alpha\|_1} + \frac{\mu}{\lambda}\frac{p^- G^\tau}{m_1\delta^{p^+}\|\alpha\|_1}.$$

Since $\mu < \delta_{\lambda,\mu}$, we have

$$\mu < \frac{m_0\tau^{p^-} - \lambda p^+ c^{p^-}\int_\Omega \sup_{|t|\leqslant\tau} F(x,t)dx}{p^+ c^{p^-} G^\tau} \tag{3.5}$$

and

$$\mu < \frac{\lambda p^- \int_\Omega F(x,\delta)dx - m_1\delta^{p^+}\|\alpha\|_1}{-p^- G_\delta}. \tag{3.6}$$

From (3.5) and (3.6), we get

$$\frac{p^+ c^{p^-}\int_\Omega \sup_{|t|\leqslant\tau} F(x,t)dx}{m_0\tau^{p^-}} + \frac{\mu}{\lambda}\frac{p^+ c^{p^-} G^\tau}{m_0\tau^{p^-}} < \frac{1}{\lambda},$$

and

$$\frac{p^- \int_\Omega F(x,\delta)dx}{m_1\|\alpha\|_1\delta^{p^+}} + \frac{\mu}{\lambda}\frac{p^- G_\delta}{m_1\|\alpha\|_1\delta^{p^+}} > \frac{1}{\lambda}.$$

Then,

$$\frac{\sup_{\Phi(x)\leqslant r}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Phi(w)}{\Psi(w)},$$

therefor, condition (E_1) of Theorem 1 is verified.

Since $\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\leqslant r}\Psi(u)}\right[$, Theorem 1 with $\bar{v} = w$ guarantees the existence of a local minimum point u_λ for the functional $I_{\lambda,\mu}$ such that $0 < \Phi(u_\lambda) < r$ and so u_λ is a nontrivial weak solution of the problem (1.1) such that $\|u_\lambda\|_\infty < \tau$. $\square$

REMARK 1. Condition (H_2) in Theorem 4 can be replaced by the less general but more easily verifiable condition

$$\int_{\Omega} \sup_{|t|\leqslant\tau} F(x,t)dx < \int_{\Omega} F(x,\delta)dx.$$

As an illustration of Theorem 4, we have the following example.

EXAMPLE 1. Given the domain $\Omega = \{(x,y) \in \mathbb{R}^2, x^2+y^2 \leqslant 1\}$, set

$$M(t) = 2+\cos(t) \ \ \forall\, t \in [0,+\infty[,$$

$$p(x,y) = 2(2+x+y), \ \ (x,y) \in \Omega,$$

then, $m_0 = 1, m_1 = 3$, $p^- = 4$ and $p^+ = 6$. For all $((x,y),t) \in \Omega\times\mathbb{R}$ put $f((x,y),t) = -p(x,y)e^{-\frac{t}{2}}$. By the expression of f we have

$$F((x,y),t) = 2p(x,y)e^{-\frac{t}{2}}, \qquad \forall((x,y),t) \in \Omega\times\mathbb{R}.$$

By choosing $\alpha(x,y) = 1$, $\delta = 1$, $\tau = 48$ and $c^4 = \frac{12^3}{\pi}(1+\pi)^4$, by simple calculations, obviously all assumptions of Theorem 4 are satisfied. Hence, by applying Theorem 4, for every

$$\lambda \in \left(\frac{3\pi e^{\frac{1}{2}}}{8\int_{\Omega} p(x,y)dxdy}, \frac{4^4\pi}{(1+\pi)^4 \int_{\Omega} p(x,y)dxdy} \right),$$

and for each $g:\mathbb{R}\to\mathbb{R}$, there exists $\delta_{\lambda,g} > 0$, such that for each $\mu \in [0,\delta_{\lambda,g}[$, the problem (1.1) admits at least one nontrivial solution $u_\lambda \in X$ such that $\|u_\lambda\|_\infty \leqslant 48$.

In the other case, our goal is to obtain the existence of two distinct solutions for the problem (1.1). The following result is obtained by applying Theorem 2.

THEOREM 5. *Assume that there exists positive constant τ such that $\tau > c$. Moreover, assume that*

(H_4) *there exist $\rho > \frac{p^+ m_1}{m_0}$ and $R > 0$ such that*

$$0 < \rho F(x,t) < t f(x,t)$$

for all $x\in\Omega$ and $|t| \geqslant R$.

Then, for each

$$\lambda \in \Gamma := \left]0, \frac{m_0 \tau^{p^-}}{p^+ c^{p^-} \int_{\Omega} \sup_{|t|\leqslant\tau} F(x,t)dx}\right[,$$

and for every nonnegative continuous function $g:\mathbb{R}\to\mathbb{R}$ satisfying

(H_5) *there exist* $c_1 > 0$ *and function* $r(x) \in \mathscr{C}_+, 0 < r(x) \leqslant r^+ < \underline{p}$ *such that*

$$|g(t)| \leqslant c_1(1 + |t|^{r(x)-1}), \quad \textit{for all} \ \ x \in \partial\Omega, \ t \in \mathbb{R}.$$

There exists $\delta_{\lambda,g} > 0$*, given by in Theorem 4 such that for each* $\mu \in [0, \delta_{\lambda,g}[$*, the problem* (1.1) *admits at least two nontrivial solutions.*

Proof. Let $\Phi, \Psi_{\lambda,\mu}$ be the functionals defined in Theorem 4 which satisfy all regularity assumptions requested in Theorem 2. Arguing as in the proof of Theorem 4, choosing $r := \frac{m_0}{p^+}(\frac{\tau}{c})^{p^-}$, and pick $w \in X$. Now, from condition (H_4), by standard computations, there is a positive constant m such that

$$F(x,t) \geqslant m|t|^{\rho} \quad \text{for all} \quad x \in \Omega. \tag{3.7}$$

Hence, due to the Trace Theorem [16] for the function g, condition (H_5) and (3.7), for every $\lambda \in \Gamma$, $u \in X \setminus \{0\}$ and $t > 1$, we obtain

$$\begin{aligned}
I_{\lambda,\mu}(tu) &= \Phi(tu) - \lambda \int_\Omega F(x, tu(x))dx - \mu \int_{\partial\Omega} G(\gamma(tu(x)))d\sigma \\
&\leqslant \frac{m_1}{p^-} \max\{\|tu\|_\alpha^{p^+}, \|tu\|_\alpha^{p^-}\} - \lambda m t^{\rho} \int_\Omega |u(x)|^{\rho} dx \\
&\quad + c_1 \mu \Big(\|\gamma(tu)\|_{L^1(\partial\Omega)} + \frac{1}{r^-} \|\gamma(tu)\|_{L^{r(x)}(\partial\Omega)}^{r^+} \Big) \\
&\leqslant \frac{m_1 t^{p^+}}{p^-} \max\{\|u\|_\alpha^{p^+}, \|u\|_\alpha^{p^-}\} - \lambda m t^{\rho} \int_\Omega |u(x)|^{\rho} dx \\
&\quad + c_2 t \mu \|u\|_{W^{1,1}(\Omega)} + c_3 t^{r^+} \mu \|u\|_{W^{1,r(x)}(\Omega)}^{r^+}.
\end{aligned}$$

Since $\rho > p^+$, this condition guarantees that $I_{\lambda,\mu}$ is unbounded from below. We recall that $I_{\lambda,\mu}$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $I'_{\lambda,\mu}(u) \in X^*$ given by

$$\begin{aligned}
I'_{\lambda,\mu}(u)(v) = \ & M\Big(\int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx \Big) \\
& \times \int_\Omega (|\nabla u|^{p(x)-2} \nabla u(x) \nabla v(x) + \alpha(x)|u(x)|^{p(x)-2} u(x) v(x))dx \\
& - \lambda \int_\Omega f(x, u(x)) v(x) dx - \mu \int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))d\sigma,
\end{aligned}$$

for every $v \in X$. Finally, we verify that $I_{\lambda,\mu}$ satisfies the (PS)-condition. Indeed, if $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{I_{\lambda,\mu}(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_{\lambda,\mu}(u_n) \to 0$ in X^* as $n \to +\infty$. Then, there exists a positive constant s_0 such that

$$|I_{\lambda,\mu}(u_n)| \leqslant s_0, \quad \|I'_{\lambda,\mu}(u_n)\| \leqslant s_0 \quad \forall n \in \mathbb{N}.$$

Using the conditions (H_4), (H_5) and the definition of $I'_{\lambda,\mu}$, we deduce that, for all $n\in\mathbb{N}$,

$$\begin{aligned}
\rho s_0+s_0\|u_n\|_\alpha &\geqslant \rho I_{\lambda,\mu}(u_n(x))-I'_{\lambda,\mu}(u_n(x))u_n(x)\\
&\geqslant \frac{\rho m_0}{p^+}\int_\Omega(|\nabla u_n|^{p(x)}+\alpha(x)|u_n(x)|^{p(x)})dx-\rho\lambda\int_\Omega F(x,u_n(x))dx\\
&\quad -\rho\mu\int_{\partial\Omega}G(\gamma(u_n(x)))d\sigma-m_1\int_\Omega(|\nabla u_n|^{p(x)}+\alpha(x)|u_n(x)|^{p(x)})dx\\
&\quad +\lambda\int_\Omega f(x,u_n(x))u_n(x)dx+\mu\int_{\partial\Omega}g(\gamma(u_n(x)))\gamma(u_n(x))d\sigma\\
&\geqslant\left(\frac{\rho m_0}{p^+}-m_1\right)\max\{\|u_n\|_\alpha^{p^+},\|u_n\|_\alpha^{p^-}\}\\
&\quad +\lambda\int_\Omega(f(x,u_n(x))u_n(x)-\rho F(x,u_n(x)))dx\\
&\quad -\mu c_2\|u_n\|_{W^{1,1}(\Omega)}-\mu c_3\|u_n\|^{r^+}_{W^{1,r(x)}(\Omega)}\\
&\geqslant\left(\frac{\rho m_0}{p^+}-m_1\right)\max\{\|u_n\|_\alpha^{p^+},\|u_n\|_\alpha^{p^-}\}-s_1\|u_n\|_\alpha-s_2\|u_n\|_\alpha^{r^+}
\end{aligned}$$

for some $s_1,s_2>0$. Since $\rho>\frac{p^+m_1}{m_0}$ it follows $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, since X is a reflexive Banach space we have, up to a subsequence,

$$u_n\rightharpoonup u\ \text{ in } X.$$

By $I'_{\lambda,\mu}(u_n)\to 0$ and $u_n\rightharpoonup u$ in X, we obtain

$$(I'_{\lambda,\mu}(u_n)-I'_{\lambda,\mu}(u))(u_n-u)\to 0.$$

From the continuity of f and g we have

$$\int_\Omega(f(x,u_n)-f(x,u))(u_n-u)dx\to 0,\qquad \text{as}\quad n\to+\infty,$$

and

$$\int_{\partial\Omega}(g(\gamma(u_n))-g(\gamma(u)))(u_n-u)d\sigma\to 0,\qquad \text{as}\quad n\to+\infty.$$

Moreover, an easy computation shows

$$\begin{aligned}
&(I'_{\lambda,\mu}(u_n)-I'_{\lambda,\mu}(u))(u_n-u)\\
&=(\Phi'(u_n)-\Phi'(u))(u_n-u)-\lambda(\Psi'_{\lambda,\mu}(u_n)-\Psi'_{\lambda,\mu}(u))(u_n-u)\\
&\geqslant 2^{-p^-}m_0\|u_n-u\|_\alpha^{p^-}-\lambda\int_\Omega(f(x,u_n(x))-f(x,u(x)))(u_n(x)-u(x))dx\\
&\quad -\mu\int_{\partial\Omega}(g(\gamma(u_n(x)))-g(\gamma(u(x))))(u_n(x)-u(x))d\sigma\\
&\geqslant 2^{-p^-}m_0\|u_n-u\|_\alpha^{p^-}.
\end{aligned}$$

The last of the above inequality is obtained by using (3.2). Combining the last relation with Proposition 1 (iii), we find that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in X. Therefore, $I_{\lambda,\mu}$ satisfies the (PS)-condition and so all hypotheses of Theorem 2 are verified. Hence, applying Theorem 2, for each $\lambda \in \Gamma$ the function $I_{\lambda,\mu}$ admits at least two distinct critical points that are the solutions of the problem (1.1). □

Finally, we discuss the existence of at least three solutions for the problem (1.1).

THEOREM 6. *Assume*

(H_6) *There exist* $s_3 > 0$ *and function* $r(x) \in \mathscr{C}_+, 0 < r(x) \leqslant r^+ < \underline{p}$ *such that*

$$|F(x,t)| \leqslant s_3(1+|t|^{r(x)}), \ \textit{for all} \ (x,t) \in \Omega \times \mathbb{R}.$$

Assume that there exist positive constants $\tau > c$ *and* δ, *such that*

(H_7) $\qquad c^{p^-}\delta^{p^-}\|\alpha\|_1 > \tau^{p^-}$,

and let the assumptions (H_2) *and* (H_3) *in Theorem* 4 *hold. Then for every* $\lambda \in \Lambda$ *as in* (3.1), *and for each* $g : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$|G(t)| \leqslant s_3(1+|t|^{r(x)}), \ \textit{for all}\ x \in \partial\Omega,\ t \in \mathbb{R},$$

there exists $\delta_{\lambda,g} > 0$, *get in Theorem* 4 *such that for each* $\mu \in [0, \delta_{\lambda,g}[$, *the problem* (1.1) *admits at least three distinct solutions.*

Proof. Our aim is to apply Theorem 3. We consider the functionals Φ and $\Psi_{\lambda,\mu}$, as seen before, they satisfy the regularity assumptions requested in Theorem 3. Now, arguing as in the proof of Theorem 4, put $w(x)$ as in (3.3) and $r := \frac{m_0}{p^+}(\frac{\tau}{c})^{p^-}$, bearing in mind (H_7) we obtain

$$\Phi(w) > r > 0.$$

Therefore, according to the proof of Theorem 4, the assumption (E_3) of Theorem 3 holds. Now, we prove that, for each $\lambda \in \Lambda$ the functional $I_{\lambda,\mu}$ is coercive. By using conditions (H_6), and by the Sobolev embedding theorem and the Trace Theorem [16], we easily obtain for all $u \in X$:

$$\begin{aligned}
I_{\lambda,\mu}(u) &\geqslant \frac{m_0}{p^+}\|u\|_\alpha^{p^-} - \lambda\int_\Omega |F(x,t)|dx - \mu\int_{\partial\Omega} G(\gamma(u))d\sigma \\
&\geqslant \frac{m_0}{p^+}\|u\|_\alpha^{p^-} - \lambda s_3\|u\|_{L^{r(x)}(\Omega)}^{r^+} - \mu s_3\|\gamma(u))\|_{L^{r(x)}(\partial\Omega)}^{r^+} - S \\
&\geqslant \frac{m_0}{p^+}\|u\|_\alpha^{p^-} - \lambda s_3\|u\|_{W^{1,r(x)}(\Omega)}^{r^+} - \mu s_3\|u\|_{W^{1,r(x)}(\Omega)}^{r^+} - S \\
&\geqslant \frac{m_0}{p^+}\|u\|_\alpha^{p^-} - \lambda s_3\|u\|_\alpha^{r^+} - \mu s_3\|u\|_\alpha^{r^+} - S
\end{aligned}$$

which $I_{\lambda,\mu} \to +\infty$ as $\|u\| \to +\infty$. Hence the functional $I_{\lambda,\mu}$ is coercive, also condition (E_4) holds. So, for each $\lambda \in \Lambda$, Theorem 3 implies that the functional $I_{\lambda,\mu}$ admits at least three critical points in X that are solutions of the problem (1.1). □

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Fariba Gharehgazlouei
Department of Mathematics
Faculty of Sciences, Razi University
67149 Kermanshah, Iran
e-mail: f.gharehgazloo@yahoo.com

Shapour Heidarkhani
Department of Mathematics
Faculty of Sciences, Razi University
67149 Kermanshah, Iran
e-mail: sh.heidarkhani@razi.ac.ir