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ANALYSIS AND SOLUTION OF COMPLEX ORDER DIFFERENTIAL EQUATIONS USING SINGULAR KERNEL

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Abstract. This article employs the two-step Adomian decomposition method (TSADM) to obtain a solution of a fractional differential equation with complex order using the singular kernel operator. Moreover, we have provided conditions for the existence and uniqueness of a solution with the fixed point theorems. Also, we have mentioned examples and solved them with the help of the proposed method and found the analytical solution in one iteration.

1. Introduction

The theory of fractional calculus (FC) has existed for many years. However, it received little attention for a long time because it lacked a corresponding application background. At the beginning of the 21st century, with the development of natural science and increasing engineering demands, people began to explore the potential of fractional calculus. FC has been applied in different research fields. It has been used to study nonlinear fluctuations of solitary gravity waves, gas transportation, lithium-ion battery management, and dissipative acoustic equations. In the control field, the fractional-order controller is a hot research topic. A fractional order controller for a given type of simple model of fractional-order system is given [4, 7, 8, 11, 12, 23].

FC is also used in special cases such as Tempered fractional Brownian motion and can show semi-long range dependency, with correlations falling off like a power law at moderate time scales but becoming short-range dependent at long time scales. The Kolmogorov model for turbulence is now extended to encompass low frequencies. The renowned Davenport wind speed spectrum is used to construct electric power production facilities, and tempered fractional Brownian motion gives a time-domain stochastic process model for it [16, 17]. For various diffusion processes, the mean squared displacement (MSD) varies as a power law of a unique exponent, which can be described by a constant fractional diffusion equation (FDE). For a non-unique scaling exponent, the multi-term fractional or variable fractional order diffusion equations were introduced in the literature. Meanwhile, for some anomalous diffusion without unique diffusion or scaling exponents, various FDE types are based on the distributed-order

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differential equation (DO-DE) using the distributed-order derivative in the model. The retarding sub-diffusion process, in which a plume of particles expands at a logarithmic rate, leading to ultraslow diffusion, is a typical application of this type of FDE.

Another example is the fractional Langevin equation of distributed order, which was first proposed to model the kinetics of retarding sub-diffusion with a decreasing scaling exponent over time and then used to simulate strongly anomalous ultraslow diffusion with the mean square displacement growing as a power of the logarithm of time. Dynamical systems that model real-world features are said to go through two phases. One is from integer-order dynamical systems to fractional-order dynamical systems, and the other is from fractional-order dynamic systems to distributed-order dynamic systems. The DO-DE is intended to model a linear time-invariant input-output (I-O) relationship system based on the frequency domain and response observation. The DO-DE is the time-domain representation of the I-O relationship observed and constructed in the frequency domain [3, 14, 22].

Complex-order fractional derivatives have also attracted attention. A new complex order controller structure and corresponding tuning method based on the numerical method are presented. The iteration methods of complex-order algorithms for the space-time control of linear and nonlinear systems and a complex order force are studied. This article considers a complex-order fractional problem. The complex-order fractional derivative contained in the variational problem is based on the Caputo fractional derivative. The Caputo fractional derivative is a fractional derivative of a function f with a singular kernel. Using the Caputo fractional operators, we have obtained the analytical solution. A new Hermite-Hadamard inequality involving left- and right-sided Caputo fractional integrals via convex functions was investigated. The generalization of Caputo fractional derivatives was raised in 1999 [8–10]. Some properties and important results of the fractional calculus are given in the literature.

The authors of [2] provided a fractional model of phytoplankton species that interact. They research existence, individuality, stability, tenacity, and permanence [1, 15]. Additionally, they have presented a novel technique for demonstrating persistence and permanence, which is helpful for several ecological fractional order models. Ultimately, the authors conducted numerical simulations to verify our analytical results and suggested a discretization technique.

This article has considered the nonlinear complex order differential equations (NL-CODEs) with initial and boundary conditions (IBCs). We solve the proposed problem using the two-step Adomian decomposition method (TSADM) and provide the analytical solution in one iteration. Furthermore, we have given the conditions for the existence and uniqueness of the solution via the fixed point theorem. Also, some examples are discussed in this article to show the method's efficiency.

This paper is organized as follows: Section 2 gives some facts and results related to fractional calculus and related properties. In Section 3, we discuss the TSADM for the NL-CODEs. Section 4 is devoted to the main result of the proposed problema. In section 5, we solved some examples; last, we mentioned the conclusion based on our results in section 6.

2. Preliminaries

In this section, we present some fundamentals definitions and results for which are required for further work in this paper [4–6].

Suppose $C^k([0,\mu],\mathbb{C})$ be the spaces of complex valued functions having smooth derivative up to order $k-1 \in \mathbb{N}\cup\{0\}$ and kth derivatives are continuous (integrable on $[0,\mu]$). The fractional integral of complex order $\theta \in \mathbb{C}, Re\theta > 0$, is defined for a function $\mathscr{M} \in L^1([0,\mu],\mathbb{C})$ (the space of integrable functions on $[0,\mu]$) [4–6].

DEFINITION 1. The Caputo fractional derivative of complex order $\theta \in \mathbb{C}, \{Re(\theta) > 0\}$, of a function $\mathscr{M} : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$ {}^{C}_{0+}D^{\theta}_{\xi}\mathscr{M}(\xi) = {}^{C}_{0+}J^{m-\theta}_{\xi}{}^{C}_{0+}D^{m}_{\xi}\mathscr{M}(\xi) = \frac{1}{\Gamma(m-\theta)}\int_0^{\xi}(\xi-y)^{m-\theta-1}\mathscr{G}^m(y)dy, \quad (2.1) $$

where $m = [Re(\theta)]+1$ and $[Re(\theta)]$ denotes the integral part of the real number θ, and

$$ {}^{C}_{0+}D^{\theta}_{\xi}K = 0. \quad (2.2) $$

The Caputo's fractional operators are linear operators, described as

$$ {}^{C}_{0+}D^{\theta}_{\xi}(Ks(\xi)+Jh(\xi)) = K{}^{C}_{0+}D^{\theta}_{\xi}s(\xi)+J{}^{C}_{0+}D^{\theta}_{\xi}h(\xi), \quad (2.3) $$

where K and J are constants.

DEFINITION 2. The Stirling asymptotic formula of the Gamma function for $\mathscr{H} \in \mathbb{C}$ is

$$ \Gamma(\mathscr{H}) = (2\pi)^{\frac{1}{2}}\mathscr{H}^{\frac{\mathscr{H}-1}{2}}\left[\exp -\mathscr{H}\left(1+\mathscr{O}\left(\frac{1}{\mathscr{H}}\right)\right)\right], \quad (|\arg(\mathscr{H}) < \pi;\ |\mathscr{H}| \to \infty|), \quad (2.4) $$

Further, for $\mathscr{S}, \mathscr{I} \in \mathbb{R}$,

$$ |\Gamma(\mathscr{S}+i\mathscr{I})| = (2\pi)^{\frac{1}{2}}|\mathscr{I}|^{\mathscr{S}-\frac{1}{2}}\exp\left(-\mathscr{S}-\frac{\pi|\mathscr{I}|}{2}\right)\left[1+\mathscr{O}\left(\frac{1}{\mathscr{I}}\right)\right], \quad (\mathscr{I} \to \infty). \quad (2.5) $$

LEMMA 1. *If* $\theta \in \mathbb{C}$, $m \in \mathbb{N}$, *then*

$$ {}^{C}_{0+}D^{\theta}_{\xi}{}^{C}_{0+}J^{\theta}_{\xi}\mathscr{M}(\xi) = \mathscr{M}(\xi). \quad (2.6) $$

$$ {}^{C}_{0+}J^{\theta}_{\xi}{}^{C}_{0+}D^{\theta}_{\xi}\mathscr{M}(\xi) = \mathscr{M}(\xi) - \sum_{\ell=0}^{m-1}\mathscr{M}^{\ell}(0^+)\frac{\xi^{\ell}}{\ell!}, \quad \xi > 0. \quad (2.7) $$

3. The TSADM algorithm

In this section, we discuss the TSADM for the adoped problem.

Consider the nonlinear complex order differential equations (NL-CODEs) [13], described as

$$
\begin{gathered}
{}_0^C D_\beta^\zeta \lambda(\alpha,\beta) = \mathscr{H}(\alpha,\beta,\lambda(\alpha,\beta),\lambda_\beta(\alpha,\beta),\lambda_\alpha(\alpha,\beta),\lambda_{\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta)),\\
(\alpha,\beta)\in[0,1]\times[0,1]=\mathscr{I},\ \ 1<\zeta\leqslant 2,\ \ \zeta=\zeta_1+i\zeta_2,
\end{gathered} \tag{3.1}
$$

with IBCs

$$
\begin{gathered}
\lambda(\alpha,0)=\phi_1(\alpha),\ \lambda_\alpha(\alpha,0)=\phi_2(\alpha)\\
\lambda(0,\beta)=0=\lambda_\alpha(1,\beta)
\end{gathered}
$$

where ${}_0^C D_\beta^\zeta$ is the Caputo fractional derivative of order $\zeta\in\mathbb{C}$.

The algorithm consists of five steps:

Step 1. On applying the inverse operator ${}_0^C I_\beta^\zeta$ of ${}_0^C D_\beta^\zeta$ on both sides of the equation (3.1), we obtain

$$
\begin{aligned}
\lambda(\alpha,\beta) = {} & \phi_1(\alpha)+\alpha\phi_1(\alpha)\\
& + {}_0^C I_\beta^\zeta\big(\mathscr{H}(\alpha,\beta,\lambda(\alpha,\beta),\lambda_\beta(\alpha,\beta),\lambda_\alpha(\alpha,\beta),\lambda_{\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta))\big).
\end{aligned} \tag{3.2}
$$

Step 2. Write the recursion formula for the TSADM from (3.2) in step 1 as

$$
\lambda_0(\alpha,\beta)=\phi_1(\alpha)+\alpha\phi_1(\alpha), \tag{3.3}
$$

and

$$
\lambda_{n+1}(\alpha,\beta)={}_0^C I_\beta^\zeta\Big(\mathscr{H}(\alpha,\beta,\lambda_n(\alpha,\beta),\lambda_{n\beta}(\alpha,\beta),\lambda_{n\alpha}(\alpha,\beta),\lambda_{n\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta))\Big), \tag{3.4}
$$

where $n=1,2,\cdots$.

Step 3. The first iteration (zeroth term) in the equation (3.3) can be splited into several components as

$$
\lambda_0(\alpha,\beta)=\Lambda=\Lambda_1+\Lambda_2+\Lambda_3+\cdots+\Lambda_M, \tag{3.5}
$$

where $\Lambda_1,\Lambda_2,\Lambda_3,\cdots,\Lambda_M$ are the terms obtained by integrating the source term $\lambda(\alpha,\beta)$ and from the associated intial/boundary conditions.

Step 4. Select the first component of λ_0, that is, Λ_1. Then Λ_1 and satisfies (3.1) and that associated the initial conditions, if Λ_1 is the exact solution of (3.1). If the first component do not satisfies either (3.1) or the initial conditions, then we go to the next component and proceed with the same process. If any component in λ_0 satisfies (3.1) and the initial conditions, then that component becomes our exact solution of (3.1) with the initial conditions. If all the components involved in λ_0 do not satisfy (3.1) or the initial conditions, then we go to the next step.

Step 5. Apply the ADM to obtain the solution by choosing $\lambda_0(\alpha,\beta)=\Lambda$ and iterate the solution by using equation (3.4) in Step 2.

4. Convergence analysis and error estimate

In this section, we focus on the convergence of the ADM involved in the TSADM. Here, we present the sufficient conditions for convergence of the method and error estimate for the equation (3.1) and (3.2).

THEOREM 1. *Let $\lambda_n(\alpha,\beta)$ and $\lambda(\alpha,\beta)$ be defined in the Banach space $C(\mathscr{I},\|\cdot\|)$.*
Then the series solution $\{\lambda_n(\alpha,\beta)\}_{n=0}^{\infty}$ defined by

$$\lambda(\alpha,\beta)=\sum_{b=0}^{\infty}\lambda_b(\alpha,\beta)=\lambda_1+\lambda_2+\cdots \tag{4.1}$$

converges to the solution of λ, if $0<\gamma<1$.

Proof. Assume that $(C(\mathscr{I}),\|\cdot\|)$ is the Banach space, the space of all continuous functions on $\mathscr{I}$ with the norm,

$$\|\lambda(\alpha,\beta)\|=\sup_{(\alpha,\beta)\in\mathscr{I}}|\lambda(\alpha,\beta)|. \tag{4.2}$$

Define $\{\mathscr{S}_n\}$ as the sequence of partial sums of the series $\sum_{n=0}^{\infty}\lambda_n(\alpha,\beta)$ by,

$$\begin{aligned}
\mathscr{P}\mathscr{S}_0&=\lambda_0(\alpha,\beta)\\
\mathscr{P}\mathscr{S}_1&=\lambda_0(\alpha,\beta)+\lambda_1(\alpha,\beta)\\
\mathscr{P}\mathscr{S}_2&=\lambda_0(\alpha,\beta)+\lambda_1(\alpha,\beta)+\lambda_2(\alpha,\beta)\\
&\vdots\\
\mathscr{P}\mathscr{S}_n&=\lambda_0(\alpha,\beta)+\lambda_1(\alpha,\beta)+\lambda_2(\alpha,\beta)+\cdots+\lambda_n(\alpha,\beta).
\end{aligned} \tag{4.3}$$

We need to show that $\{\mathscr{P}\mathscr{S}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C(\mathscr{I}),\|\cdot\|)$. For this purpose, we consider,

$$\begin{aligned}
\|\mathscr{P}\mathscr{S}_{n+1}-\mathscr{P}\mathscr{S}_n\|&=\|\lambda_n(\alpha,\beta)\|\leqslant\gamma\|\lambda_n(\alpha,\beta)\|\leqslant\gamma^2\|\lambda_{n-1}(\alpha,\beta)\|\\
&\leqslant\cdots\leqslant\gamma^{n+1}\|\lambda_0(\alpha,\beta)\|.
\end{aligned} \tag{4.4}$$

For every, $n,m\in\mathbb{N},n\geqslant m,$ form (4.4), we have

$$\begin{aligned}
&\|\mathscr{P}\mathscr{S}_n-\mathscr{P}\mathscr{S}_m\|\\
&=\|(\mathscr{P}\mathscr{S}_n-\mathscr{P}\mathscr{S}_{n-1})+(\mathscr{P}\mathscr{S}_{n-1}-\mathscr{P}\mathscr{S}_{n-2})+\cdots+(\mathscr{P}\mathscr{S}_{m+1}-\mathscr{P}\mathscr{S}_m)\|,\\
&\leqslant\|\mathscr{P}\mathscr{S}_n-\mathscr{P}\mathscr{S}_{n-1}\|+\|\mathscr{P}\mathscr{S}_{n-1}-\mathscr{P}\mathscr{S}_{n-2}\|+\cdots+\|\mathscr{P}\mathscr{S}_{m+1}-\mathscr{P}\mathscr{S}_m)\|,\\
&\leqslant\gamma^n\|\lambda_0(\alpha,\beta)\|+\gamma^{n-1}\|\lambda_0(\alpha,\beta)\|+\cdots+\gamma^m\|\lambda_0(\alpha,\beta)\|,\\
&=\frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m+1}\|\lambda_0(\alpha,\beta)\|.
\end{aligned} \tag{4.5}$$

Since $0<\gamma<1$, we have $1-\gamma^{n-m}<1$. Thus

$$\|\mathscr{P}\mathscr{S}_n-\mathscr{P}\mathscr{S}_m\| \leqslant \frac{1-\gamma^{m+1}}{1-\gamma}\max_{(\alpha,\beta)\in\mathscr{I}}\|\lambda_0(\alpha,\beta)\|, \tag{4.6}$$

Again $\lambda_0(\alpha,\beta)$ is bounded gives

$$\lim_{n,m\to 0}\|\mathscr{P}\mathscr{S}_n-\mathscr{P}\mathscr{S}_m\|=0.$$

Therefore, $\{\mathscr{P}\mathscr{S}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $C(\mathscr{I},\|\cdot\|)$, so the series solution of the series $\sum\lambda_b$, converges. □

THEOREM 2. *The maximum absolute truncation error of the series solution in* (4.1) *for* (3.1) *and* (3.2) *is estimated to be*

$$\left|\lambda(\alpha,\beta)-\sum_{b=0}^{m}\lambda_b(\alpha,\beta)\right| \leqslant \frac{1-\gamma^{m+1}}{1-\gamma}\|\lambda_0(\alpha,\beta)\|. \tag{4.7}$$

Proof. From Theorem 1 and (4.5), we have

$$\|\mathscr{P}\mathscr{S}_n-\mathscr{P}\mathscr{S}_m\| \leqslant \frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m+1}\|\lambda_0(\alpha,\beta)\|, \tag{4.8}$$

for $n\geqslant m$.

If $n\to\infty$, then $\mathscr{P}\mathscr{S}_n\to\lambda(\alpha,\beta)$. So,

$$\|\lambda(\alpha,\beta)-\mathscr{P}\mathscr{S}_m\| \leqslant \frac{1-\gamma^{m+1}}{1-\gamma}\|\lambda_0(\alpha,\beta)\|. \tag{4.9}$$

Since $0<\gamma<1$, $1-\gamma^{n-m}<1$, the above inequality becomes

$$\left|\lambda(\alpha,\beta)-\sum_{b=0}^{m}\lambda_b(\alpha,\beta)\right| \leqslant \frac{1-\gamma^{m+1}}{1-\gamma}\|\lambda_0(\alpha,\beta)\|. \tag{4.10}$$

Hence, we have done our task here. □

LEMMA 2. *Let* $\mathscr{H}\in C(\mathscr{I}\times\mathbb{R}^5)$, *then the solution of NL-CODEs* (3.1) *is given by*

$$\begin{aligned}\lambda(\alpha,\beta) &= \phi_1(\alpha)+\alpha\phi_1(\alpha)\\ &\quad +{}_0^C I_\beta^\zeta\big(\mathscr{H}(\alpha,\beta,\lambda(\alpha,\beta),\lambda_\beta(\alpha,\beta),\lambda_\alpha(\alpha,\beta),\lambda_{\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta))\big).\end{aligned} \tag{4.11}$$

To proceed further, we assume that

$[A1]$: There exist $L_i>0$, $i=1,2,3,4$ such that

$$\left|\mathscr{H}(\alpha,\beta,\lambda(\alpha,\beta),\lambda_\beta(\alpha,\beta),\lambda_\alpha(\alpha,\beta),\lambda_{\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta))\right| \leqslant \sum_{i=1}^{4}L_i|\lambda_1-\lambda_2|,$$

for all $\lambda,\lambda_\beta,\lambda_\alpha,\lambda_{\alpha\alpha}\in\mathbb{R}$.

Let $\mathscr{X}=C(\mathscr{K})$ where $\mathscr{K}=\mathscr{I}\times\mathbb{R}^5,\mathbb{R}$ be Banach space with norm $\|\lambda\|_\infty=\sup_{\lambda\in\mathscr{K}}|\lambda(\alpha,\beta)|$.

THEOREM 3. *Under the hypothesis* $[A1]$, *if the condition* $\left[\frac{\sum_{i=1}^{4} L_i}{\Gamma(\zeta+1)}\right] < 1$ *holds, then the proposed problem* (3.1) *has a unique solution.*

Proof. Define the fixed point operator $\Sigma : \mathscr{X} \to \mathscr{X}$ by using (3.1) as

$$\Sigma\lambda(\alpha,\beta) = {}_0^C I_\beta^\zeta \Big(\mathscr{H}(\alpha,\beta,\lambda(\alpha,\beta),\lambda_\beta(\alpha,\beta),\lambda_\alpha(\alpha,\beta),\lambda_{\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta)) \Big). \tag{4.12}$$

Then for any $\lambda_1, \lambda_2 \in \mathscr{X}$, (4.12), we have

$$\begin{aligned}
&\|\Sigma\lambda_1 - \Sigma\lambda_2\|_\infty \\
&= \sup_{\lambda\in\mathscr{K}} |\Sigma\lambda_1(\alpha,\beta) - \Sigma\lambda_2(\alpha,\beta)| \\
&= \sup_{\lambda\in\mathscr{K}} \Big| {}_0^C I_\beta^\zeta \Big(\mathscr{H}(\alpha,\beta,\lambda_1(\alpha,\beta),\lambda_{1\beta}(\alpha,\beta),\lambda_{1\alpha}(\alpha,\beta),\lambda_{1\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta)) \Big) \\
&\quad - {}_0^C I_\beta^\zeta \Big(\mathscr{H}(\alpha,\beta,\lambda_2(\alpha,\beta),\lambda_{2\beta}(\alpha,\beta),\lambda_{2\alpha}(\alpha,\beta),\lambda_{2\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta)) \Big) \Big| \\
&\leqslant {}_0^C I_\beta^\zeta \sum_{i=1}^{4} L_i \Big| \lambda_1 - \lambda_2 \Big| \\
&\leqslant \frac{\sum_{i=1}^{4} L_i \beta^\zeta}{\Gamma(\zeta+1)} \Big| \lambda_1 - \lambda_2 \Big| \\
&\leqslant \frac{\sum_{i=1}^{4} L_i}{\Gamma(\zeta+1)} \|\lambda_1 - \lambda_2\|_\infty.
\end{aligned} \tag{4.13}$$

Finally,

$$\|\Sigma\lambda_1 - \Sigma\lambda_2\|_\infty \leqslant \frac{\sum_{i=1}^{4} L_i}{\Gamma(\zeta+1)} \|\lambda_1 - \lambda_2\|_\infty. \tag{4.14}$$

Hence Σ is a contraction [19, 21], therefore Σ has a unique fixed point. Hence the corresponding problem (3.1) has a unique solution. □

THEOREM 4. *Let* $\eta \subset X$ *be a closed, bounded and convex subset of real Banach space* X *and let* Q_1 *and* Q_2 *be operators on* η *satisfying the following conditions* [20]*:*

(i) $Q_1(\eta') + Q_2(\eta'') \in \eta, \forall \eta', \eta'' \in \eta$,

(ii) Q_1 *is a strict contraction on* η*, that is, there exists a* $r \in [0,1)$ *such that* $|Q_1(x_1) - Q_1(x_2)| \leqslant r|x_1 - x_2|, \forall x_1, x_2 \in \eta$,

(iii) Q_2 *is continuous on* η *and* Q_2 *is a relatively compact subset of* X.

Then there exists at least one solution $\eta' \in \eta$ *such that* $Q_1(\eta') + Q_2(\eta') = \eta'$.

For futher results, let the given hypothesis hold:

$[A2]$: *There exist constant* $A_{\mathscr{H}}, B_{\mathscr{H}}, C_{\mathscr{H}} > 0$ *such that*

$$\begin{aligned}
&|\mathscr{H}(\alpha,\beta,\lambda_2(\alpha,\beta),\lambda_{2\beta}(\alpha,\beta),\lambda_{2\alpha}(\alpha,\beta),\lambda_{2\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta)| \\
&\leqslant A_{\mathscr{H}} + B_{\mathscr{H}}|\lambda| + C_{\mathscr{H}}|\delta|.
\end{aligned}$$

Theorem 3. Under the assumption $[A2]$, *if* $\left[\frac{\Sigma_{i=1}^{4} L_i}{\Gamma(\zeta+1)}\right] < 1$ *holds, then the considered problem* (3.1) *has at leat one solution.*

Proof. Let define two operator from the equation (3.1) as

$$\Sigma_1 \lambda(\alpha,\beta) = \phi_1(\alpha) + \alpha\phi_1(\alpha), \tag{4.15}$$

and

$$\Sigma_2 \lambda(\alpha,\beta) = {}_0^C I_\beta^\zeta \Big(\mathscr{H}(\alpha,\beta,\lambda(\alpha,\beta),\lambda_\beta(\alpha,\beta),\lambda_\alpha(\alpha,\beta),\lambda_{\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta)) \Big). \tag{4.16}$$

Consider $\mathscr{E} = \{\lambda \in \mathscr{X} : \|\lambda\|_\infty \leqslant r\}$. Since $\mathscr{H}$ is continous, so is Σ_1, and letting $\lambda_1, \lambda_2 \in \mathscr{E}$, from the equations (4.15), we have

$$\|\Sigma_1 \lambda_1(\alpha,\beta) - \Sigma_1 \lambda_2(\alpha,\beta)\|_\infty = 0 \tag{4.17}$$

Hence Σ_1 is a contraction. Next task to prove that Σ_2 is compact and continuous, for any $\lambda \in \mathscr{E}$, we have from (4.16).

$$\begin{aligned} \|\Sigma_2 \lambda(\alpha,\beta)\|_\infty &= \sup_{\lambda \in \mathscr{K}} |\Sigma_2 \lambda(\alpha,\beta)| \\ &\leqslant \frac{A_{\mathscr{H}} + B_{\mathscr{H}} r + C_{\mathscr{H}} \|\delta\|_\infty}{\Gamma(\zeta+1)} = \mathscr{P} \end{aligned} \tag{4.18}$$

which implies that $\|\Sigma_2 \lambda(\alpha,\beta)\|_\infty \leqslant \mathscr{P}$. Thus Σ_2 is bounded. Further, let $\alpha_1 < \alpha_2, \beta_1 < \beta_2$ in $\mathscr{K}$, we have

$$\begin{aligned} &|\Sigma_2 \lambda(\alpha_1,\beta_1) - \Sigma_2 \lambda(\alpha_2,\beta_2)| \\ &= \Big| {}_0^C I_{\beta_2}^\zeta \Big(\mathscr{H}(\alpha_2,\beta_2,\lambda(\alpha,\beta),\lambda_\beta(\alpha,\beta),\lambda_\alpha(\alpha,\beta),\lambda_{\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta)) \Big) \\ &\quad - {}_0^C I_{\beta_2}^\zeta \Big(\mathscr{H}(\alpha,\beta,\lambda(\alpha,\beta),\lambda_\beta(\alpha,\beta),\lambda_\alpha(\alpha,\beta),\lambda_{\alpha\alpha}(\alpha,\beta),\delta(\alpha,\beta)) \Big) \Big| \\ &\leqslant \left(\frac{A_{\mathscr{H}} + B_{\mathscr{H}} r + C_{\mathscr{H}} \|\delta\|_\infty}{\Gamma(\zeta+1)} \right) (\beta_2^\zeta - \beta_1^\zeta) \end{aligned} \tag{4.19}$$

which implies that

$$|\Sigma_2 \lambda(\alpha_1,\beta_1) - \Sigma_2 \lambda(\alpha_2,\beta_2)| \leqslant \left(\frac{A_{\mathscr{H}} + B_{\mathscr{H}} r + C_{\mathscr{H}} \|\delta\|_\infty}{\Gamma(\zeta+1)} \right) (\beta_2^\zeta - \beta_1^\zeta). \tag{4.20}$$

From (4.20), we see that if $\beta_1 \to \beta_2$, then the right hand side of (4.20) tends to zero this implies that $|\Sigma_2 \lambda(\alpha_1,\beta_1) - \Sigma_2 \lambda(\alpha_2,\beta_2)| \to 0$ as $\beta_1 \to \beta_2$. Thus the operator defined in (4.15), Σ_2 is continuous. Also, $\Sigma_2(\mathscr{E}) \subset \mathscr{E}$, therefore Σ_2 is compact [18], Σ has at least one fixed point. Hence the problem (3.1) has at least one solution. □

5. Applications

This section solves examples using the TSADM with some remarks.

EXAMPLE 1. Consider the NL-CODEs with IBCs [13] be given by

$$ {}_0^C D_\beta^\zeta \lambda(\alpha,\beta) = \lambda^2(\alpha,\beta) + \lambda_\beta(\alpha,\beta)\lambda_\alpha(\alpha,\beta) + \alpha^3\lambda_{\alpha\beta}(\alpha,\beta) + \delta^2(\alpha,\beta) + \mathscr{F}(\alpha,\beta), \quad \zeta \in (1,2] \tag{5.1} $$

$$ \lambda(\alpha,0) = 0 = \lambda_\beta(\alpha,0) \tag{5.2} $$

$$ \lambda(0,\beta) = 0 = \lambda(1,\beta) \tag{5.3} $$

where $\mathscr{F}(\alpha,\beta) = \frac{\Gamma(3.7)\beta^{2.7-\zeta}}{\Gamma(3.7-\zeta)}\alpha^{3.4} - (\alpha^{3.4}\beta^{2.7})^2 - (9.18\alpha^{5.8}\beta^{4.4}) - (8.16\alpha^{4.4}\beta^{2.7}) - (\alpha^{2.9}\beta^{4.1})^2$. The exact solution is $\beta^{2.7}\alpha^{3.4}$ using TSADM.

On applying inverse operator ${}_0^C I_\beta^\zeta$ of ${}_0^C D_\beta^\zeta$ into (5.1), we obtain

$$ \lambda(\alpha,\beta) = {}_0^C I_\beta^\zeta \Big(\lambda^2(\alpha,\beta) + \lambda_\beta(\alpha,\beta)\lambda_\alpha(\alpha,\beta) + \alpha^3\lambda_{\alpha\beta}(\alpha,\beta) + \delta^2(\alpha,\beta) + \mathscr{F}(\alpha,\beta) \Big). \tag{5.4} $$

The recursion formula for iterations, from the equation (5.4), work as follows

$$ \lambda_0(\alpha,\beta) = {}_0^C I_\beta^\zeta \Big(\mathscr{F}(\alpha,\beta) \Big), \tag{5.5} $$

and

$$ \lambda_{n+1}(\alpha,\beta) = {}_0^C I_\beta^\zeta \Big(\lambda_n^2(\alpha,\beta) + \lambda_{n\beta}(\alpha,\beta)\lambda_{n\alpha}(\alpha,\beta) + \alpha^3\lambda_{n\alpha\beta}(\alpha,\beta) + \delta^2(\alpha,\beta) \Big), \tag{5.6} $$

where $n = 1,2,\cdots,$.

The first iteration of the TSADM can be splitted into five terms as [18]

$$ \lambda_0 = \mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3, \tag{5.7} $$

Now, let us take $\lambda_0 = \mathscr{L}_0$ and whether check this assumption of λ_0 is satisfies (5.1) and also the given initial and boundary conditions. If this choice of λ_0 is approved, then the chosen term is the solution of the problem.

It can be verified that $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$ satisfy the given associated conditions put do not satisfy equation (5.1).

Therefore for the Example 1, we can claim that the proposed method (TSADM) is more efficient to give the exact solution with fewer number iterations as compared to otherexisting nmerical method [18, 19].

EXAMPLE 2. Consider the NL-CODEs with IBCs [13] be given by

$$
\begin{aligned}
{}_0^C D_\beta^\zeta \lambda(\alpha,\beta) &= \cos(\lambda(\alpha,\beta)) + 2\sin(\alpha)\lambda_\alpha(\alpha,\beta) + \lambda_{\alpha\alpha}(\alpha,\beta)\lambda_\beta(\alpha,\beta) \\
&\quad + \delta(\alpha,\beta) + \mathscr{F}(\alpha,\beta), \zeta \in (1,2] \qquad (5.8) \\
\lambda(\alpha,0) &= 0 = \lambda_\beta(\alpha,0) \qquad (5.9) \\
\lambda(0,\beta) &= 0 = \lambda(1,\beta) \qquad (5.10)
\end{aligned}
$$

where

$$
\mathscr{F}(\alpha,\beta) = \frac{\Gamma(4)\beta^{3-\zeta}}{\Gamma(4-\zeta)}\sin(\alpha) - \big(\cos(\beta^3\sin(\alpha) + \beta^3\sin(2\alpha) - 3\beta^5\sin^2(\alpha) + \beta^4\exp(\alpha)\big). \qquad (5.11)
$$

The exact solution is $\beta^3\sin(\alpha)$ using TSADM.

6. Conclusion

The main aim of this work is to find the analytical solution to the considered problems. The proposed method requires no approximation for the solution to obtain high accuracy. The TSADM is easy to apply to the problems and provides an exact solution with one iteration compared to numerical methods. Moreover, we have discussed the new conditions for the existence and uniqueness of the solution using fixed point theorems. We have considered some examples. These examples show the efficiency and applicability of the adopted method. Thus, this study has two main objectives: the high-accuracy solution for the proposed problem and the new results for the existence and uniqueness of the solution.

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