Differential
Equations
& Applications
Volume 16, Number 3 (2024), 183–198
doi:10.7153/dea-2024-16-11

MASS EFFECT ON AN ELLIPTIC PDE INVOLVING TWO HARDY–SOBOLEV CRITICAL EXPONENTS

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(*Communicated by P. Souplet*)

Abstract. We let Ω be a bounded domain of $\mathbb{R}^3$ and Γ be a smooth closed curve contained in Ω. We study existence of positive solutions $u \in H_0^1(\Omega)$ to the equation

$$-\Delta u + hu = \lambda \rho_\Gamma^{-s_1} u^{5-2s_1} + \rho_\Gamma^{-s_2} u^{5-2s_2} \qquad \text{in } \Omega$$

where $h : \Omega \to \mathbb{R}$ is a function and ρ_Γ is the distance function to Γ. We prove existence of solutions depending on the regular part of the Green function of linear operator. We prove the existence of positive mountain pass solutions for this Euler-Lagrange equation depending on the mass which is the regular part of the Green function of the linear operator $-\Delta + h$.

1. Introduction

In this paper, we are concerned with the mass effect on the existence of mountain pass solutions of the following nonlinear partial differential equation involving two Hardy-Sobolev critical exponents in $\mathbb{R}^3$. More precisely, letting Ω to be a bounded domain of $\mathbb{R}^3$, Γ a smooth closed curve contained, $h : \Omega \to \mathbb{R}$ a function and λ a real parameter, we consider

$$\begin{cases} -\Delta u(x) + hu(x) = \lambda \dfrac{u^{5-2s_1}(x)}{\rho_\Gamma^{s_1}(x)} + \dfrac{u^{5-2s_2}(x)}{\rho_\Gamma^{s_2}(x)} & \text{in } \Omega \\ u(x) > 0 \quad \text{and} \quad u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\rho_\Gamma(x) := \inf_{y \in \Gamma} |y - x|$$

is the distance function to the curve Γ and for $0 < s_2 < s_1 < 2$, $2^*_{s_1} := 6 - 2s_1$ and $2^*_{s_2} := 6 - 2s_2$ are two critical Hardy-Sobolev exponents. To study equation (1.1), we consider the following non-linear functional $\Psi : H_0^1(\Omega) \to \mathbb{R}$ defined by:

$$\Psi(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega h(x) u^2 dx - \frac{\lambda}{2^*_{s_1}} \int_\Omega \rho_\Gamma^{-s_1}(x) |u|^{2^*_{s_1}} dx$$
$$- \frac{1}{2^*_{s_2}} \int_\Omega \rho_\Gamma^{-s_2}(x) |u|^{2^*_{s_2}} dx, \tag{1.2}$$

Mathematics subject classification (2020): 35J08, 35A23, 35J60.
Keywords and phrases: Two Hardy-Sobolev critical exponents, Green function, positive mass, mountain pass solution, curve singularity.

Paper DEA-16-11

where $h:\Omega\to\mathbb{R}$ is a function and $H^1_0(\Omega)$ is the completion of $\mathscr{C}^\infty_c(\Omega)$ in

$$H^1(\Omega):=\{u\in L^2(\Omega):\ \nabla u\in (L^2(\Omega))^3\}.$$

Thanks to the continuous embedding of $H^1_0(\Omega)$ into the weighted Lebesgue spaces

$$L^{2^*_s}(\Omega,\rho_\Gamma^{-s}):=\left\{u:\Omega\to\mathbb{R} \text{ such that } \|u\|_{L^{2^*_s}(\Omega,\rho_\Gamma^{-s})}:=\left(\int_\Omega |u|^{2^*_s}\rho_\Gamma^{-s}dx\right)^{1/2^*_s}<\infty\right\},$$

the functional Ψ is well defined. Then there exists a positive constant $r>0$ and $u_0\in H^1_0(\Omega)$ such that $\|u_0\|_{H^1_0(\Omega)}>r$ and

$$\inf_{\|u\|_{H^1_0(\Omega)}=r}\Psi(u)>\Psi(0)\geqslant\Psi(u_0),$$

see for instance Lemma 4.5 in the paper of the author and Diatta [1]. Then the point $(0,\Psi(0))$ is separated from the point $(u_0,\Psi(u_0))$ by a ring of mountains. Set

$$c^*:=\inf_{P\in\mathscr{P}}\max_{v\in P}\Psi(v), \tag{1.3}$$

where $\mathscr{P}$ is the class of continuous paths in $H^1_0(\Omega)$ connecting 0 to u_0. Since $2^*_{s_2}>2^*_{s_1}$, the function $t\longmapsto\Psi(tv)$ has the unique maximum for $t\geqslant 0$. Furthermore, we have

$$c^*:=\inf_{u\in H^1_0(\Omega),u\geqslant 0,u\neq 0}\max_{t\geqslant 0}\Psi(tu).$$

Due to the fact that the embedding of $H^1_0(\Omega)$ into the weighted Lebesgue spaces $L^{2^*_{si}}(\rho_\Gamma^{-si})$ is not compact, the functional Ψ does not satisfy the Palais-Smale condition. Therefore, in general c^* might not be a critical value for Ψ.

To recover compactness, we study the following non-linear problem: let $x=(y,z)\in\mathbb{R}\times\mathbb{R}^2$ and consider

$$\begin{cases}-\Delta u=\lambda\dfrac{u^{2^*_{s_1}-1}(x)}{|z|^{s_1}}+\dfrac{u^{2^*_{s_2}-1}}{|z|^{s_2}} & \text{in } \mathbb{R}^3\\ u(x)>0 & \text{in } \mathbb{R}^3.\end{cases} \tag{1.4}$$

To obtain solutions of (1.4), we consider the functional $\Phi:\mathscr{D}^{1,2}(\mathbb{R}^3)\to\mathbb{R}$ defined by

$$\Phi(u):=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx-\frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^3}|z|^{-s_1}|u|^{2^*_{s_1}}dx-\frac{1}{2^*_{s_2}}\int_{\mathbb{R}^3}|z|^{-s_2}|u|^{2^*_{s_2}}dx, \tag{1.5}$$

$\mathscr{D}^{1,2}(\mathbb{R}^3)$ is the completion of $\mathscr{C}^\infty_c(\mathbb{R}^3)$ with respect to the norm

$$u\longmapsto\sqrt{\int_{\mathbb{R}^3}|\nabla u|^2dx}.$$

Next, we define

$$\beta^*:=\inf_{u\in D^{1,2}(\mathbb{R}^3),u\geqslant 0,u\neq 0}\max_{t\geqslant 0}\Phi(tu).$$

Then we get compactness provided

$$c^* < \beta^*,$$

see Proposition 4.1 in [1]. Therefore the existence, symmetry and decay estimates of non-trivial solution $w \in \mathscr{D}^{1,2}(\mathbb{R}^3)$ of (1.4) play an important role in problem (1.1). Then we have the following results.

PROPOSITION 1. *Let $0 \leqslant s_2 < s_1 < 2$, $\lambda \in \mathbb{R}$. Then equation*

$$\begin{cases} -\Delta u = \lambda \dfrac{u^{2^*_{s_1}-1}(x)}{|z|^{s_1}} + \dfrac{u^{2^*_{s_2}-1}}{|z|^{s_2}} & \text{in } \mathbb{R}^3 \\ u(x) > 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.6}$$

has a positive ground state solution $w \in \mathscr{D}^{1,2}(\mathbb{R}^3)$ depending only on $|y|$ and $|z|$. Moreover there exists two positive constants C_1 and C_2 such that

$$\frac{C_1}{1+|x|} \leqslant w(x) \leqslant \frac{C_2}{1+|x|} \qquad \text{in } \mathbb{R}^3 \tag{1.7}$$

and for $|x| = |(t,z)| \leqslant 1$, we have

$$|\nabla w(x)| + |x||D^2 w(x)| \leqslant C_2 |z|^{1-s_1} \tag{1.8}$$

and if $|x| = |(t,z)| \geqslant 1$, we have

$$|\nabla w(x)| + |x||D^2 w(x)| \leqslant C_2 \max(1, |z|^{-s_1})|x|^{1-N}. \tag{1.9}$$

For a complete proof, we refer to the paper of Fabbri-Mancini-Sandeep and Proposition 2.4 in [1]. Next, we let $G(x,y)$ be the Dirichlet Green function of the operator $-\Delta + h$, with Dirichlet boundary conditions. It satisfies

$$\begin{cases} -\Delta_x G(x,y) + h(x)G(x,y) = 0 & \text{for every } x \in \Omega \setminus \{y\} \\ G(x,y) = 0 & \text{for every } x \in \partial\Omega. \end{cases} \tag{1.10}$$

In addition there exists a continuous function $\mathbf{m} : \Omega \to \mathbb{R}$ and a positive constant $c > 0$ such that

$$G(x,y) = \frac{c}{|x-y|} + c\,\mathbf{m}(y) + o(1) \qquad \text{as } x \to y, \tag{1.11}$$

see for instance [3]. We call the function $\mathbf{m} : \Omega \to \mathbb{R}$ the *mass* of $-\Delta + h$ in Ω. We note that $-\mathbf{m}$ is occasionally called the *Robin function* of $-\Delta + h$ in the literature. Then our main result is the following:

THEOREM 2. *Let $0 \leqslant s_2 < s_1 < 2$ and Ω be a bounded domain of $\mathbb{R}^3$. Consider Γ a smooth closed curve contained in Ω. Let $h : \Omega \to \mathbb{R}$ be a given function such that the linear operator $-\Delta + h$ is coercive. We assume that there exists $y_0 \in \Gamma$ such that*

$$m(y_0) > 0. \tag{1.12}$$

Then there exists $u \in H_0^1(\Omega) \setminus \{0\}$ *non-negative solution of*

$$-\Delta u(x) + hu(x) = \lambda \frac{u^{5-2s_1}(x)}{\rho_\Gamma^{s_1}(x)} + \frac{u^{5-2s_2}(x)}{\rho_\Gamma^{s_2}(x)} \qquad \text{in } \Omega.$$

In contrast to the case $N \geqslant 4$ (see [1] for more details), the existence of solution does not depend on the local geometry of the singularity but on the location of the curve Γ. Besides in the study of Hardy-Sobolev equations in domains with interior singularity for the three dimensional case, the effect of the mass plays an important role in the existence of positive solutions. For Hardy-Sobolev inequality on Riemannian manifolds with singularity a point, Jaber [7] proved the existence of positive solutions when the mass is positive. We refer also to [8] for existence of mountain pass solution to a Hardy-Sobolev equation with an additional perturbation term. For the Hardy-Sobolev equations on domains with singularity a curve, we refer to the papers of the author and Fall [3] and the author and Ijaodoro [6]. We also suggest to the interested readers the nice work of Druet [2], Schoen-Yau [9] and [10] for more details related to the positive mass theorem. We also mention that this paper is the 3-dimensional version of the work of thye author [1].

The proof of Theorem 2 relies on test function methods. Namely we build appropriate test functions allowing to compare c^* and β^*. Near the concentration point $y_0 \in \Gamma$, the test function is similar to the test function in the case $N \geqslant 4$ but away from it is replaced with the regular part of the Green function which makes apear the mass, see Section 3.

2. Tool box

We consider the function

$$\mathscr{R} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \qquad x \mapsto \mathscr{R}(x) = \frac{1}{|x|}$$

which satisfies

$$-\Delta \mathscr{R} = 0 \qquad \text{in } \mathbb{R}^3 \setminus \{0\}. \tag{2.1}$$

We denote by G the solution to the equation

$$\begin{cases} -\Delta_x G(y,\cdot) + hG(y,\cdot) = 0 & \text{in } \Omega \setminus \{y\}. \\ G(y,\cdot) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

and satisfying

$$G(x,y) = \mathscr{R}(x-y) + O(1) \qquad \text{for } x,y \in \Omega \text{ and } x \neq y. \tag{2.3}$$

We note that G is proportional to the Green function of $-\Delta + h$ with Dirichlet boundary conditions.

We let $\chi \in C_c^\infty(-2,2)$ with $\chi \equiv 1$ on $(-1,1)$ and $0 \leqslant \chi < 1$. For $r > 0$, we consider the cylindrical symmetric cut-off function

$$\eta_r(t,z) = \chi\left(\frac{|t|+|z|}{r}\right) \qquad \text{for every } (t,z) \in \mathbb{R}\times\mathbb{R}^2. \tag{2.4}$$

Then there exists a positive contant C such that:

$$\eta_r \equiv 1 \ \text{ in } \ Q_r, \qquad \eta_r \in H_0^1(Q_{2r}), \qquad |\nabla \eta_r| \leqslant \frac{C}{r} \ \text{ in } \ \mathbb{R}^3,$$

where $Q_r := (-r,r)\times B_{\mathbb{R}^2}(0,r)$ and $B_{\mathbb{R}^2}(0,r)$ is the 2-dimensional euclidean ball centered at the origin and of radius r. For $y_0 \in \Omega$, we let $r_0 \in (0,1)$ such that

$$y_0 + Q_{2r_0} \subset \Omega. \tag{2.5}$$

We define the function $M_{y_0} : Q_{2r_0} \to \mathbb{R}$ given by

$$M_{y_0}(x) := G(y_0, x+y_0) - \eta_r(x)\frac{1}{|x|} \qquad \text{for every } x \in Q_{2r_0}. \tag{2.6}$$

It follows from (2.3) that $M_{y_0} \in L^\infty(Q_{r_0})$. By (2.2) and (2.1),

$$|-\Delta M_{y_0}(x) + h(x)M_{y_0}(x)| \leqslant \frac{C}{|x|} = C\mathscr{R}(x) \qquad \text{for every } x \in Q_{r_0},$$

where as $\mathscr{R} \in L^p(Q_{r_0})$ for every $p \in (1,3)$. Hence by elliptic regularity theory, see [4], $M_{y_0} \in W^{2,p}(Q_{r_0/2})$ for every $p \in (1,3)$. Therefore by Morrey's embdding theorem, we deduce that

$$\|M_{y_0}\|_{C^{1,\rho}(Q_{r_0/2})} \leqslant C \qquad \text{for every } \rho \in (0,1). \tag{2.7}$$

In view of (1.11), the mass of the operator $-\Delta + h$ in Ω at the point $y_0 \in \Omega$ is given by

$$\mathbf{m}(y_0) = M_{y_0}(0). \tag{2.8}$$

Next, we have the following result which will be important in the sequel.

LEMMA 1. *Let $w \in \mathscr{D}^{1,2}(\mathbb{R}^3)$ given by Proposition 1 and consider the function $v_\varepsilon : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ given by*

$$v_\varepsilon(x) = \varepsilon^{-1} w\left(\frac{x}{\varepsilon}\right).$$

Then there exists a constant $\Lambda > 0$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ (still denoted by ε) such that

$$v_\varepsilon(x) \to \frac{\Lambda}{|x|} \qquad \text{and} \qquad \nabla v_\varepsilon(x) \to -\Lambda\frac{x}{|x|^3} \qquad \text{for all most every } x \in \mathbb{R}^3$$

and

$$v_\varepsilon(x) \to \frac{\Lambda}{|x|} \qquad \text{and} \qquad \nabla v_\varepsilon(x) \to -\Lambda\frac{x}{|x|^3} \qquad \text{for every } x \in \mathbb{R}^3 \setminus \{z=0\}. \tag{2.9}$$

Proof. By Proposition 1, we have that (v_ε) is bounded in $C^2_{loc}(\mathbb{R}^3 \setminus \{z=0\})$. Therefore by Arzelá-Ascolli's theorem v_ε converges to v in $C^1_{loc}(\mathbb{R}^3 \setminus \{z=0\})$. In particular,

$$v_\varepsilon \longrightarrow v \qquad \text{and} \qquad \nabla v_\varepsilon \longrightarrow \nabla v \qquad \text{almost every where on } \mathbb{R}^3.$$

It is plain, from (1.7), that

$$0 < \frac{C_1}{\varepsilon + |x|} \leqslant v_\varepsilon(x) \leqslant \frac{C_2}{\varepsilon + |x|} \qquad \text{for almost every } x \in \mathbb{R}^3. \tag{2.10}$$

By (1.4), we have

$$-\Delta v_\varepsilon(x) = \lambda \varepsilon^{2-s_1} \frac{v_\varepsilon^{5-2s_1}(x)}{|z|^{s_1}} + \varepsilon^{2-s_2} \frac{v_\varepsilon^{5-2s_2}(x)}{|z|^{s_2}} \qquad \text{in } \mathbb{R}^3. \tag{2.11}$$

Newt, we let $\varphi \in C_c^\infty\left(\mathbb{R}^3 \setminus \{0\}\right)$. We multiply (2.11) by φ and integrate by parts to get

$$-\int_{\mathbb{R}^3} v_\varepsilon \Delta\varphi dx = \lambda \varepsilon^{2-s_1} \int_{\mathbb{R}^3} \frac{v_\varepsilon^{5-2s_1}(x)}{|z|^{s_1}} \varphi(x) dx + \varepsilon^{2-s_2} \int_{\mathbb{R}^3} \frac{v_\varepsilon^{5-2s_2}(x)}{|z|^{s_2}} \varphi(x) dx.$$

By (2.10) and the dominated convergence theorem, we can pass to the limit in the above identity and deduce that

$$\Delta v = 0 \qquad \text{in } \mathscr{D}'\left(\mathbb{R}^3 \setminus \{0\}\right).$$

In particular v is equivalent to a non-negative function to a function of class $C^\infty\left(\mathbb{R}^3 \setminus \{0\}\right)$ which is still denoted by v. Thanks to (2.10), by Bôcher's theorem, see [5], there exists a constant $\Lambda > 0$ such that $v(x) = \frac{\Lambda}{|x|}$. The proof of the lemma is thus finished. □

We finish this section by the following estimates. Thanks to the decay estimates in Proposition 1, we have

LEMMA 2. *Let $r_0 \in (0,1)$ such that (2.5) holds. Then there exists a constant $C > 0$ such that for every $\varepsilon, r \in (0, r_0/2)$ and for $s \in (0,2)$, we have*

$$\int_{Q_{r/\varepsilon}} |\nabla w|^2 dx \leqslant C \max\left(1, \frac{\varepsilon}{r}\right), \qquad \int_{Q_{r/\varepsilon}} |w|^2 dx \leqslant C \max\left(1, \frac{r}{\varepsilon}\right), \tag{2.12}$$

$$\int_{Q_{r/\varepsilon}} w|\nabla w| dx \leqslant C \max\left(1, \log\frac{r}{\varepsilon}\right), \tag{2.13}$$

$$\int_{Q_{r/\varepsilon}} |\nabla w| dx \leqslant C \max\left(1, \frac{r}{\varepsilon}\right), \qquad \int_{Q_{r/\varepsilon}} |w| dx \leqslant C \max\left(1, \frac{r^2}{\varepsilon^2}\right) \tag{2.14}$$

and

$$\varepsilon^2 \int_{Q_{r/\varepsilon}} |z|^{-s}|x|^2 w^{2^*_s} dx + \varepsilon \int_{Q_{4r/\varepsilon} \setminus Q_{r/\varepsilon}} |z|^{-s} w^{2^*_s - 1} dx + \int_{\mathbb{R}^3 \setminus Q_{r/\varepsilon}} |z|^{-s} w^{2^*_s} dx = o(\varepsilon), \tag{2.15}$$

where $Q_{r/\varepsilon} := (-r/\varepsilon, r/\varepsilon) \times B_{\mathbb{R}^2}(0, r/\varepsilon)$.

Proof. The inequalities in (2.12), (2.13) and (2.14) are an immediate consequence of (1.7) and (1.8) and we get estimation (2.15) thanks to (1.7) and (1.9). □

3. Proof of the main result

Given $y_0 \in \Gamma \subset \Omega \subset \mathbb{R}^3$, we let r_0 as defined in (2.5). For $r \in (0, r_0/2)$, we consider $F_{y_0} : Q_r \to \Omega$ parameterizing a neighborhood of y_0 in Ω, with the property that $F_{y_0}(0) = y_0$,

$$\rho_\Gamma(F_{y_0}(x)) = |z|, \qquad \text{for all } x = (t,z) \in Q_r. \tag{3.1}$$

Moreover in these local coordinates, we have

$$g_{ij}(x) = \delta_{ij} + O(|x|) \tag{3.2}$$

and

$$\sqrt{|g|}(x) = 1 + \langle A, z\rangle + O\left(|x|^2\right), \tag{3.3}$$

where $A \in \mathbb{R}^2$ is the vector curvature of Γ and $|g|$ stands for the determinant of g, see [3] for more details related to this parametrization.

Next,we let $w \in \mathscr{D}^{1,2}(\mathbb{R}^3)$ given by Proposition 1. For $\varepsilon > 0$, we consider $u_\varepsilon : \Omega \to \mathbb{R}$ given by

$$u_\varepsilon(y) := \varepsilon^{-1/2}\eta_r(F_{y_0}^{-1}(y))w\left(\frac{F_{y_0}^{-1}(y)}{\varepsilon}\right).$$

We can now define the test function $\Psi_\varepsilon : \Omega \to \mathbb{R}$ by

$$\Psi_\varepsilon(y) = u_\varepsilon(y) + \varepsilon^{1/2}\Lambda\,\eta_{2r}(F_{y_0}^{-1}(y))M_{y_0}(F_{y_0}^{-1}(y)) \tag{3.4}$$

It is plain that $\Psi_\varepsilon \in H_0^1(\Omega)$ and

$$\Psi_\varepsilon\left(F_{y_0}(x)\right) = \varepsilon^{-1/2}\eta_r(x)w\left(\frac{x}{\varepsilon}\right) + \varepsilon^{1/2}\Lambda\,\eta_{2r}(x)M_{y_0}(x) \qquad \text{for every } x \in \mathbb{R}^N.$$

To alleviate the notations, we will write ε instead of ε_n and we will remove the subscript y_0, by writing M and F in the place of M_{y_0} and F_{y_0} respectively. We define

$$\widetilde{\eta}_r(y) := \eta_r(F^{-1}(y)), \quad V_\varepsilon(y) := v_\varepsilon(F^{-1}(y))$$

and

$$\widetilde{M}_{2r}(y) := \eta_{2r}(F^{-1}(y))M(F^{-1}(y)),$$

where $v_\varepsilon(x) = \varepsilon^{-1}w\left(\frac{x}{\varepsilon}\right)$. With these notations, (3.4) becomes

$$\Psi_\varepsilon(y) = u_\varepsilon(y) + \varepsilon^{\frac{1}{2}}\Lambda\widetilde{M}_{2r}(y) = \varepsilon^{\frac{1}{2}}\widetilde{\eta}_r(y)V_\varepsilon(y) + \varepsilon^{\frac{1}{2}}\Lambda\widetilde{M}_{2r}(y). \tag{3.5}$$

In the sequel we define $\mathscr{O}_r(\varepsilon)$ as

$$\lim_{r\to 0}\frac{\mathscr{O}_r(\varepsilon)}{\varepsilon} = 0.$$

Then we have the following.

LEMMA 3. *We have*

$$\int_\Omega |\nabla\Psi_\varepsilon|^2 dy + \int_\Omega h|\Psi_\varepsilon|^2 dy = \int_{\mathbb{R}^3} |\nabla w|^2 dx + \pi\varepsilon \boldsymbol{m}(y_0)\Lambda^2 + \mathscr{O}_r(\varepsilon), \tag{3.6}$$

as $\varepsilon \to 0$.

Proof. Recalling (3.5), direct computations give

$$\begin{aligned}\int_{F(Q_{2r})\setminus F(Q_r)} |\nabla\Psi_\varepsilon|^2 dy &= \int_{F(Q_{2r})\setminus F(Q_r)} |\nabla u_\varepsilon|^2 dy + \varepsilon\Lambda^2 \int_{F(Q_{2r})\setminus F(Q_r)} |\nabla \widetilde{M}_{2r}|^2 dy \\ &\quad + 2\varepsilon^{1/2}\Lambda \int_{F(Q_{2r})\setminus F(Q_r)} \nabla(u_\varepsilon)\cdot\nabla\widetilde{M}_{2r} dy \\ &= \varepsilon \int_{F(Q_{2r})\setminus F(Q_r)} |\nabla(\widetilde{\eta}_r V_\varepsilon)|^2 dy + \varepsilon\Lambda^2 \int_{F(Q_{2r})\setminus F(Q_r)} |\nabla\widetilde{M}_{2r}|^2 dy \\ &\quad + 2\varepsilon\Lambda \int_{F(Q_{2r})\setminus F(Q_r)} \nabla(\widetilde{\eta}_r V_\varepsilon)\cdot\nabla\widetilde{M}_{2r} dy. \end{aligned}\tag{3.7}$$

By (2.4), $\eta_r v_\varepsilon = \eta_r \varepsilon^{-1} w(\cdot/\varepsilon)$ is cylindrically symmetric. Therefore by the change variable $y = F(x)$ and using (3.2), we get

$$\begin{aligned}&\varepsilon \int_{F(Q_{2r})\setminus F(Q_r)} |\nabla(\widetilde{\eta}_r V_\varepsilon)|^2 dy \\ &= \varepsilon \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_r v_\varepsilon)|_g^2 \sqrt{g} dx \\ &= \varepsilon \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_r v_\varepsilon)|^2 dx + O\left(\varepsilon r^2 \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_r v_\varepsilon)|^2 dx\right). \end{aligned}\tag{3.8}$$

By computing, we find that

$$\begin{aligned}&\varepsilon \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_r v_\varepsilon)|^2 dx \\ &\leqslant \varepsilon \int_{Q_{2r}\setminus Q_r} |\nabla v_\varepsilon|^2 dx + \varepsilon \int_{Q_{2r}\setminus Q_r} v_\varepsilon^2 |\nabla\eta_r|^2 dx + 2\varepsilon \int_{Q_{2r}\setminus Q_r} v_\varepsilon |\nabla v_\varepsilon||\nabla\eta_r| dx \\ &\leqslant \varepsilon \int_{Q_{2r}\setminus Q_r} |\nabla v_\varepsilon|^2 dx + \frac{C}{r^2}\varepsilon \int_{Q_{2r}\setminus Q_r} v_\varepsilon^2 dx + \frac{C}{r}\varepsilon \int_{Q_{2r}\setminus Q_r} v_\varepsilon |\nabla v_\varepsilon| dx \\ &= \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} |\nabla w|^2 dx + C\frac{\varepsilon}{r^2} \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} w^2 dx + \frac{C}{r}\varepsilon \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} w|\nabla w| dx. \end{aligned}$$

From this and (2.12) and (2.13), we get

$$O\left(\varepsilon r^2 \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_r v_\varepsilon)|^2 dx\right) = \mathscr{O}_r(\varepsilon).$$

We replace this in (3.8) to have

$$\varepsilon \int_{F(Q_{2r})\setminus F(Q_r)} |\nabla(\widetilde{\eta}_r V_\varepsilon)|^2 dy = \varepsilon \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_r v_\varepsilon)|^2 dx + \mathscr{O}_r(\varepsilon). \tag{3.9}$$

We have the following estimates

$$0 \leqslant v_\varepsilon \leqslant C|x|^{-1} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\} \qquad \text{and} \qquad |\nabla v_\varepsilon(x)| \leqslant C|x|^{-2} \quad \text{for } |x| \geqslant \varepsilon, \tag{3.10}$$

which easily follows from (1.7), (3.2) and (2.1). By these estimates, (3.2), (3.3) and (2.7) together with the change of variable $y = F(x)$, we have

$$\begin{aligned}\varepsilon \int_{F(Q_{2r})\setminus F(Q_r)} \nabla(\widetilde{\eta}_r V_\varepsilon)\cdot\nabla\widetilde{M}_{2r}dy &= \varepsilon\int_{Q_{2r}\setminus Q_r}\nabla(\eta_r v_\varepsilon)\cdot\nabla M dx \\ &\quad + O\left(\varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla v_\varepsilon|dx + \frac{\varepsilon}{r}\int_{Q_{2r}\setminus Q_r} v_\varepsilon dx\right) \\ &= \varepsilon\int_{Q_{2r}\setminus Q_r}\nabla(\eta_r v_\varepsilon)\cdot\nabla M dx + \mathscr{O}_r(\varepsilon).\end{aligned}$$

This with (3.9), (2.7) and (3.7) give

$$\begin{aligned}\int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\Psi_\varepsilon|^2 dy &= \varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla(\eta_r v_\varepsilon)|^2 dx + \varepsilon\Lambda^2\int_{Q_{2r}\setminus Q_r}|\nabla(\eta_{2r}M)|^2 dx \\ &\quad + 2\varepsilon\Lambda\int_{Q_{2r}\setminus Q_r}\nabla(\eta_r v_\varepsilon)\cdot\nabla M dx + \mathscr{O}_r(\varepsilon).\end{aligned}$$

Thanks to Lemma 1 and (3.10), we can thus use the dominated convergence theorem to deduce that, as $\varepsilon \to 0$,

$$\int_{Q_{2r}\setminus Q_r}|\nabla(\eta_r v_\varepsilon)|^2 dx = \Lambda^2\int_{Q_{2r}\setminus Q_r}|\nabla(\eta_r\mathscr{R})|^2 dx + o(1). \tag{3.11}$$

Similarly, we easily see that

$$\int_{Q_{2r}\setminus Q_r}\nabla(\eta_r v_\varepsilon)\cdot\nabla M dx = \Lambda\int_{Q_{2r}\setminus Q_r}\nabla(\eta_r\mathscr{R})\cdot\nabla M dx + o(1) \qquad \text{as } \varepsilon \to 0.$$

This and (3.11), then give

$$\begin{aligned}\int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\Psi_\varepsilon|^2 dy &= \varepsilon\Lambda^2\int_{Q_{2r}\setminus Q_r}|\nabla(\eta_r\mathscr{R})|^2 dx + \varepsilon\Lambda^2\int_{Q_{2r}\setminus Q_r}|\nabla M|^2 dx \\ &\quad + 2\varepsilon\Lambda^2\int_{Q_{2r}\setminus Q_r}\nabla(\eta_r\mathscr{R})\cdot\nabla M dx + \mathscr{O}_r(\varepsilon) \\ &= \varepsilon\Lambda^2\int_{Q_{2r}\setminus Q_r}|\nabla(\eta_r\mathscr{R} + M)|^2 dx + \mathscr{O}_r(\varepsilon).\end{aligned} \tag{3.12}$$

Since the support of Ψ_ε is contained in Q_{4r} while the one of η_r is in Q_{2r}, it is easy to deduce from (2.7) that

$$\int_{\Omega\setminus F(Q_{2r})}|\nabla\Psi_\varepsilon|^2 dy = \varepsilon\Lambda^2\int_{F(Q_{4r})\setminus F(Q_{2r})}|\nabla\widetilde{M}_{2r}|^2 dy = \mathscr{O}_r(\varepsilon)$$

and from Lemma 2, that

$$\int_{\Omega\setminus F(Q_r)} h|\Psi_\varepsilon|^2 dy = \varepsilon\Lambda^2\int_{F(Q_{4r})\setminus F(Q_r)} h|\eta_r V_\varepsilon + \widetilde{M}_{2r}|^2 dy = \mathscr{O}_r(\varepsilon).$$

Therefore by (3.12), we conclude that

$$\begin{aligned}&\int_{\Omega\setminus F(Q_r)}|\nabla\Psi_\varepsilon|^2dy+\int_{\Omega\setminus F(Q_r)}h|\Psi_\varepsilon|^2dy\\&=\varepsilon\Lambda^2\int_{Q_{2r}\setminus Q_r}|\nabla(\eta_r\mathscr{R}+M)|^2dx+\varepsilon\Lambda^2\int_{Q_{2r}\setminus Q_r}h(\cdot+y_0)|\eta_r\mathscr{R}+M|^2dx+\mathscr{O}_r(\varepsilon).\end{aligned}$$

Recall that $G(x+y_0,y_0)=\eta_r(x)\mathscr{R}(x)+M(x)$ for ever $x\in Q_{2r}$ and that by (2.2),

$$-\Delta_x G(x+y_0,y_0)+h(x+y_0)G(x+y_0,y_0)=0 \qquad \text{for every } x\in Q_{2r}\setminus Q_r.$$

Therefore, by integration by parts, we find that

$$\begin{aligned}&\int_{\Omega\setminus F(Q_r)}|\nabla\Psi_\varepsilon|^2dy+\int_{\Omega\setminus F(Q_r)}h|\Psi_\varepsilon|^2dy\\&=\Lambda^2\varepsilon\int_{\partial(Q_{2r}\setminus Q_r)}(\eta_r\mathscr{R}+M)\frac{\partial(\eta_r\mathscr{R}+M)}{\partial\overline{\nu}}d\sigma(x)+\mathscr{O}_r(\varepsilon),\end{aligned}$$

where $\overline{\nu}$ is the exterior normal vectorfield to $Q_{2r}\setminus Q_r$. Thanks to (2.7), we finally get

$$\begin{aligned}&\int_{\Omega\setminus F(Q_r)}|\nabla\Psi_\varepsilon|^2dy+\int_{\Omega\setminus F(Q_r)}h|\Psi_\varepsilon|^2dy\\&=-\varepsilon\Lambda^2\int_{\partial Q_r}\mathscr{R}\frac{\partial\mathscr{R}}{\partial\nu}d\sigma(x)-\varepsilon\Lambda^2\int_{\partial Q_r}M\frac{\partial\mathscr{R}}{\partial\nu}d\sigma(x)+\mathscr{O}_r(\varepsilon),\end{aligned}\tag{3.13}$$

where ν is the exterior normal vectorfield to Q_r.

Next we make the expansion of $\int_{F(Q_r)}|\nabla\Psi_\varepsilon|^2dy$ for r and ε small. First, we observe that, by Lemma 2 and (2.7), we have

$$\begin{aligned}&\int_{F(Q_r)}|\nabla\Psi_\varepsilon|^2dy\\&=\int_{F(Q_r)}|\nabla u_\varepsilon|^2dy+\varepsilon\Lambda^2\int_{F(Q_r)}|\nabla M|^2dy+2\varepsilon^{1/2}\Lambda\int_{F(Q_r)}\nabla u_\varepsilon\cdot\nabla\widetilde{M}_{2r}dy\\&=\int_{Q_{r/\varepsilon}}|\nabla w|^2dx+O\left(\varepsilon^2\int_{Q_{r/\varepsilon}}|x|^2|\nabla w|^2dx+\varepsilon^2\int_{Q_{r/\varepsilon}}|\nabla w|dx\right)+\mathscr{O}_r(\varepsilon)\\&=\int_{Q_{r/\varepsilon}}|\nabla w|^2dx+\mathscr{O}_r(\varepsilon).\end{aligned}$$

By integration by parts and using (2.15), we deduce that

$$\begin{aligned}\int_{F(Q_r)}|\nabla\Psi_\varepsilon|^2dy&=\int_{\mathbb{R}^3}|\nabla w|^2dx+\int_{\partial Q_{r/\varepsilon}}w\frac{\partial w}{\partial\nu}d\sigma(x)+\mathscr{O}_r(\varepsilon)\\&=\int_{\mathbb{R}^3}|\nabla w|^2dx+\varepsilon\int_{\partial Q_r}v_\varepsilon\frac{\partial v_\varepsilon}{\partial\nu}d\sigma(x)+\mathscr{O}_r(\varepsilon).\end{aligned}\tag{3.14}$$

Now (3.10), (2.9) and the dominated convergence theorem yield, for fixed $r>0$ and $\varepsilon \to 0$,

$$
\begin{aligned}
&\int_{\partial Q_r} v_\varepsilon \frac{\partial v_\varepsilon}{\partial \nu} d\sigma(x) \\
&= \int_{\partial B_{\mathbb{R}^2}(0,r)} \int_{-r}^{r} v_\varepsilon(t,z) \nabla v_\varepsilon(t,z) \cdot \frac{z}{|z|} d\sigma(z) dt + 2\int_{B_{\mathbb{R}^2}} v_\varepsilon(r,z) \partial_t v_\varepsilon(r,z) dz \\
&= \Lambda^2 \int_{\partial B_{\mathbb{R}^2}(0,r)} \int_{-r}^{r} \mathscr{R}(t,z) \nabla \mathscr{R}(t,z) \cdot \frac{z}{|z|} d\sigma(z) dt + 2\Lambda^2 \int_{B_{\mathbb{R}^2}} \mathscr{R}(r,z) \partial_t \mathscr{R}(r,z) dz + o(1) \\
&= \Lambda^2 \int_{\partial Q_r} \mathscr{R} \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) + o(1), \qquad (3.15)
\end{aligned}
$$

where $B_{\mathbb{R}^2}$ is the unit ball of the euclidean space $\mathbb{R}^2$. Moreover (2.14) implies that

$$\int_{F(Q_r)} h \Psi_\varepsilon^2 dy = \mathscr{O}_r(\varepsilon).$$

From this together with (3.14) and (3.15), we obtain

$$\int_{F(Q_r)} |\nabla \Psi_\varepsilon|^2 dy + \int_{F(Q_r)} h \Psi_\varepsilon^2 dy = \int_{\mathbb{R}^3} |\nabla w|^2 dx + \Lambda^2 \varepsilon \int_{\partial Q_r} \mathscr{R} \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) + \mathscr{O}_r(\varepsilon).$$

Combining this with (3.13), we then have

$$\int_{\Omega} |\nabla \Psi_\varepsilon|^2 dy + \int_{\Omega} h \Psi_\varepsilon^2 dy = \int_{\mathbb{R}^3} |\nabla w|^2 dx - \varepsilon \Lambda^2 \int_{\partial Q_r} M \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) + \mathscr{O}_r(\varepsilon) + o(\varepsilon). \qquad (3.16)$$

Recalling that $\mathscr{R}(x) = \frac{1}{|x|}$, we have

$$
\begin{aligned}
\int_{\partial Q_r} \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) &= -\int_{\partial Q_r} \frac{x \cdot \nu(x)}{|x|^3} d\sigma(x) \\
&= \int_{B_{\mathbb{R}^2}(0,r)} \frac{-2r}{r^2+|z|^2} dz - 2\pi \int_{-r}^{r} \frac{r^3}{r^2+t^2} dt \\
&= -\pi^2(1+r^2). \qquad (3.17)
\end{aligned}
$$

Since (recalling (2.8)) $M(y) = M(0) + O(r) = \mathbf{m}(y_0) + O(r)$ in Q_{2r}, we get (3.6). This then ends the proof. □

We finish by the following expansion

LEMMA 4.

$$
\begin{aligned}
&\frac{\lambda}{2^*_{s_1}} \int_{\Omega} \rho_\Gamma^{-s_1} |\Psi_\varepsilon|^{2^*_{s_1}} dy + \frac{1}{2^*_{s_2}} \int_{\Omega} \rho_\Gamma^{-s_2} |\Psi_\varepsilon|^{2^*_{s_2}} dy \\
&= \frac{\lambda}{2^*_{s_1}} \int_{\mathbb{R}^3} |z|^{-s_1} |w|^{2^*_{s_1}} dx + \frac{1}{2^*_{s_2}} \int_{\mathbb{R}^3} |z|^{-s_2} |w|^{2^*_{s_2}} dx + \varepsilon \pi^2 \Lambda^2 \boldsymbol{m}(y_0) + \mathscr{O}_r(\varepsilon).
\end{aligned}
$$

Proof. Let $p>2$. Then there exists a positive constant $C(p)$ such that

$$||a+b|^p-|a|^p-pab|a|^{p-2}|\leqslant C(p)\left(|a|^{p-2}b^2+|b|^p\right)\qquad \text{for all } a,b\in\mathbb{R}.$$

As a consequence, we obtain, for $s\in(0,2)$, that

$$\begin{aligned}
&\int_\Omega \rho_\Gamma^{-s}|\Psi_\varepsilon|^{2^*_s}dy\\
&=\int_{F(Q_r)}\rho_\Gamma^{-s}|u_\varepsilon+\varepsilon^{\frac12}\Lambda\widetilde{M}_{2r}|^{2^*_s}dy+\int_{F(Q_{4r})\setminus F(Q_r)}\rho_\Gamma^{-s}|u_\varepsilon+\varepsilon^{\frac12}\Lambda\widetilde{M}_{2r}|^{2^*_s}dy\\
&=\int_{F(Q_r)}\rho_\Gamma^{-s}|u_\varepsilon|^{2^*_s}dy+2^*_s\Lambda\varepsilon^{1/2}\int_{F(Q_r)}\rho_\Gamma^{-s}|u_\varepsilon|^{2^*_s-1}\widetilde{M}_{2r}dy\\
&\quad+O\left(\int_{F(Q_{4r})}\rho_\Gamma^{-s}|u_\varepsilon|^{2^*_s-2}\left(\varepsilon^{1/2}\widetilde{M}_{2r}\right)^2dy+\int_{F(Q_{4r})}\rho_\Gamma^{-s}|\varepsilon^{1/2}\widetilde{M}_{2r}|^{2^*_s}dy\right)\\
&\quad+O\left(\int_{F(Q_{4r})\setminus F(Q_r)}\rho_\Gamma^{-s}|u_\varepsilon|^{2^*_s}dy+2^*_s\Lambda\varepsilon^{1/2}\int_{F(Q_{4r})\setminus F(Q_r)}\rho_\Gamma^{-s}|u_\varepsilon|^{2^*_s-1}\widetilde{M}_{2r}dy\right).
\end{aligned}\tag{3.18}$$

By Hölder's inequality and (3.3), we have

$$\begin{aligned}
&\int_{F(Q_{4r})}\rho_\Gamma^{-s}|u_\varepsilon|^{2^*_s-2}\left(\varepsilon^{1/2}\widetilde{M}_r\right)^2dy\\
&\leqslant\varepsilon\|u_\varepsilon\|^{2^*_s-2}_{L^{2^*_s}(F(Q_{4r});\rho_\Gamma^{-s})}\|\widetilde{M}_{2r}\|^2_{L^{2^*_s}(F(Q_{4r});\rho_\Gamma^{-s})}\\
&=\varepsilon\|w\|^{2^*_s-2}_{L^{2^*_s}(Q_{4r};|z|^{-s}\sqrt{|g|})}\|\widetilde{M}_{2r}\|^2_{L^{2^*_s}(F(Q_{4r});\rho_\Gamma^{-s})}\\
&\leqslant\varepsilon(1+Cr)\|\widetilde{M}_{2r}\|^2_{L^{2^*_s}(F(Q_{4r});\rho_\Gamma^{-s})}\\
&=\mathscr{O}_r(\varepsilon).
\end{aligned}\tag{3.19}$$

where, we recall that

$$L^{2^*_s}(\Omega,\rho_\Gamma^{-s}):=\{u:\Omega\to\mathbb{R}\text{ such that }\int_\Omega|u|^{2^*_s}\rho_\Gamma^{-s}dx<\infty\}.$$

Furthermore, since $2^*_s>2$, by (2.7), we easily get

$$\int_{F(Q_{4r})}\rho_\Gamma^{-s}|\varepsilon^{1/2}\widetilde{M}_{2r}|^{2^*_s}dy=o(\varepsilon).\tag{3.20}$$

Moreover by change of variables and (2.15), we also have

$$\begin{aligned}
&\int_{F(Q_{4r})\setminus F(Q_r)}\rho_\Gamma^{-s}|u_\varepsilon|^{2^*_s}dy+2^*_s\Lambda\varepsilon^{1/2}\int_{F(Q_{4r})\setminus F(Q_r)}\rho_\Gamma^{-s}|u_\varepsilon|^{2^*_s-1}\widetilde{M}_{2r}dy\\
&\leqslant C\int_{Q_{4r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^{-s}|w|^{2^*_s}dx+C\varepsilon\int_{Q_{4r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^{-s}|w|^{2^*_s-1}dx=o(\varepsilon).
\end{aligned}$$

By this, (3.18), (3.20) and (3.19), it results

$$\int_\Omega \rho_\Gamma^{-s}|\Psi_\varepsilon|^{2_s^*}dy = \int_{F(Q_r)} \rho_\Gamma^{-s}|u_\varepsilon|^{2_s^*}dy + 2_s^*\Lambda\varepsilon^{1/2}\int_{F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2_s^*-1}\widetilde{M}_{2r}dy + \mathscr{O}_r(\varepsilon).$$

We define $B_\varepsilon(x) := M(\varepsilon x)\sqrt{|g_\varepsilon|}(x) = M(\varepsilon x)\sqrt{|g|}(\varepsilon x)$. Then by the change of variable $y = \frac{F(x)}{\varepsilon}$ in the above identity and recalling (3.3), then by oddness, we have

$$\begin{aligned}
\int_\Omega \rho_\Gamma^{-s}|\Psi_\varepsilon|^{2_s^*}dy &= \int_{Q_{r/\varepsilon}} |z|^{-s}w^{2_s^*}\sqrt{|g_\varepsilon|}dx + 2_s^*\varepsilon\Lambda\int_{Q_{r/\varepsilon}} |z|^{-s}|w|^{2_s^*-1}B_\varepsilon dx + \mathscr{O}_r(\varepsilon)\\
&= \int_{Q_{r/\varepsilon}} |z|^{-s}w^{2_s^*}dx + 2_s^*\varepsilon\Lambda\int_{Q_{r/\varepsilon}} |z|^{-s}|w|^{2_s^*-1}B_\varepsilon dx + \mathscr{O}_r(\varepsilon)\\
&\quad + O\left(\varepsilon^2\int_{Q_{r/\varepsilon}} |z|^{-s}|x|^2 w^{2_s^*}dx\right)\\
&= \int_{\mathbb{R}^3} |z|^{-s}|w|^{2_s^*}dx + 2_s^*\varepsilon\Lambda\int_{Q_{r/\varepsilon}} |z|^{-s}|w|^{2_s^*-1}B_\varepsilon dx\\
&\quad + O\left(\int_{\mathbb{R}^3\setminus Q_{r/\varepsilon}} |z|^{-s}w^{2_s^*}dx + \varepsilon^2\int_{Q_{r/\varepsilon}} |z|^{-s}|x|^2 w^{2_s^*}dx\right) + \mathscr{O}_r(\varepsilon).
\end{aligned}$$

By (2.15) we then have

$$\int_\Omega \rho_\Gamma^{-s}|\Psi_\varepsilon|^{2_s^*}dy = \int_{\mathbb{R}^3} |z|^{-s}|w|^{2_s^*}dx + 2_s^*\varepsilon\Lambda\int_{Q_{r/\varepsilon}} |z|^{-s}|w|^{2_s^*-1}B_\varepsilon dx + \mathscr{O}_r(\varepsilon). \tag{3.21}$$

Therefore for $0 < s_2 < s_1 < 2$, we have

$$\begin{aligned}
&\frac{\lambda}{2_{s_1}^*}\int_\Omega \rho_\Gamma^{-s_1}|\Psi_\varepsilon|^{2_{s_1}^*}dy + \frac{1}{2_{s_2}^*}\int_\Omega \rho_\Gamma^{-s_2}|\Psi_\varepsilon|^{2_{s_2}^*}dy\\
&= \frac{\lambda}{2_{s_1}^*}\int_{\mathbb{R}^3} |z|^{-s_1}|w|^{2_{s_1}^*}dx + \frac{1}{2_{s_2}^*}\int_{\mathbb{R}^3} |z|^{-s_2}|w|^{2_{s_2}^*}dx\\
&\quad + \varepsilon\Lambda\lambda\int_{Q_{r/\varepsilon}} |z|^{-s_1}|w|^{2_{s_1}^*-1}B_\varepsilon dx + \varepsilon\Lambda\int_{Q_{r/\varepsilon}} |z|^{-s_2}|w|^{2_{s_2}^*-1}B_\varepsilon dx + \mathscr{O}_r(\varepsilon).
\end{aligned}$$

We multiply (1.4) by $B_\varepsilon \in \mathscr{C}^1(\overline{Q_r})$ and we integrate by parts to get

$$\begin{aligned}
&\lambda\int_{Q_{r/\varepsilon}} |z|^{-s_1}|w|^{2_{s_1}^*-1}B_\varepsilon dx + \int_{Q_{r/\varepsilon}} |z|^{-s_2}|w|^{2_{s_2}^*-1}B_\varepsilon dx\\
&= \int_{Q_{r/\varepsilon}} \nabla w\cdot\nabla B_\varepsilon dx - \int_{\partial Q_{r/\varepsilon}} B_\varepsilon\frac{\partial w}{\partial \nu}d\sigma(x)\\
&= \int_{Q_{r/\varepsilon}} \nabla w\cdot\nabla B_\varepsilon dx - \int_{\partial Q_r} B_1\frac{\partial v_\varepsilon}{\partial \nu}d\sigma(x).
\end{aligned}$$

Since $|\nabla B_\varepsilon| \leqslant C\varepsilon$, by Lemma 1 and (2.7), we then have

$$\varepsilon \int_{Q_{r/\varepsilon}} \nabla w \cdot \nabla B_\varepsilon dx = O\left(\varepsilon^2 \int_{Q_{r/\varepsilon}} |\nabla w| dx\right) = \mathscr{O}_r(\varepsilon).$$

Consequently, on the one hand,

$$\begin{aligned}&\lambda \varepsilon \int_{Q_{r/\varepsilon}} |z|^{-s_1} |w|^{2^*_{s_1}-1} B_\varepsilon dx + \varepsilon \int_{Q_{r/\varepsilon}} |z|^{-s_2} |w|^{2^*_{s_2}-1} B_\varepsilon dx \\ &= -\varepsilon \int_{\partial Q_r} B_1 \frac{\partial v_\varepsilon}{\partial \nu} d\sigma(x) + \mathscr{O}_r(\varepsilon).\end{aligned}$$

On the other hand by Lemma 1, (2.7) and the dominated convergence theorem, we get

$$\begin{aligned}\int_{\partial Q_r} B_1 \frac{\partial v_\varepsilon}{\partial \nu} d\sigma(x) &= \Lambda \int_{\partial Q_r} B_1 \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) + o(1) \\ &= \Lambda M(0) \int_{\partial Q_r} \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) + O(r) + o(1),\end{aligned}$$

so that

$$\begin{aligned}&\lambda \varepsilon c \int_{Q_{r/\varepsilon}} |z|^{-s_1} |w|^{2^*_{s_1}-1} B_\varepsilon dx + \varepsilon c \int_{Q_{r/\varepsilon}} |z|^{-s_2} |w|^{2^*_{s_2}-1} B_\varepsilon dx \\ &= -\varepsilon \Lambda^2 M(0) \int_{\partial Q_r} \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) + \mathscr{O}_r(\varepsilon).\end{aligned}$$

It then follows from (3.21) that

$$\begin{aligned}&\frac{\lambda}{2^*_{s_1}} \int_\Omega \rho_\Gamma^{-s_1} |\Psi_\varepsilon|^{2^*_{s_1}} dy + \frac{1}{2^*_{s_2}} \int_\Omega \rho_\Gamma^{-s_2} |\Psi_\varepsilon|^{2^*_{s_2}} dy \\ &= \frac{\lambda}{2^*_{s_1}} \int_{\mathbb{R}^3} |z|^{-s_1} |w|^{2^*_{s_1}} dx + \frac{1}{2^*_{s_2}} \int_{\mathbb{R}^3} |z|^{-s_2} |w|^{2^*_{s_2}} dx - \varepsilon \Lambda^2 M(0) \int_{\partial Q_r} \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) + \mathscr{O}_r(\varepsilon).\end{aligned}$$

Thanks to (3.17), we have

$$\int_{\partial Q_r} \frac{\partial \mathscr{R}}{\partial \nu} d\sigma(x) = -\pi^2(1+r^2).$$

Since $M(0) = \mathbf{m}(y_0)$, see (2.8), the proof of the lemma is thus finished. □

Now we are in position to complete the proof of our main result.

Proof of Theorem 2. The proof is mainly based on the mountain pass lemma: let Ω be a bounded domain of $\mathbb{R}^3$, Γ be a smooth closed curve contained in Ω, $0 \leqslant s_2 < s_1 < 2$ and $h : \Omega \to \mathbb{R}$ be a function such that the linear operator $-\Delta + h$ is coercive. assume that

$$c^* := \sup_{t \geqslant 0} \Psi(tw) < \Psi(w) =: \beta^*,$$

where $w \in \mathscr{D}^{1,2}(\mathbb{R}^3)$ is given by Proposition 1. Then there exists $u \in H^1_0(\Omega)$ solution of the Euler-Lagrange equation:

$$-\Delta u + hu = \lambda \rho_\Gamma^{-s_1} u^{5-2s_1} + \rho_\Gamma^{-s_2} u^{5-2s_2} \qquad \text{in } \Omega,$$

see for instance Propostion 4.1. in [1]. Next, combining Lemma 3 and Lemma 4 and recalling (1.2) and (1.5), we have

$$J(tu_\varepsilon) = \Psi(tw) + \mathscr{M}_{r,\varepsilon}(tw), \tag{3.22}$$

for some function $\mathscr{M}_{r,\varepsilon} : \mathscr{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ satisfying

$$\mathscr{M}_{r,\varepsilon}(w) = -\frac{\varepsilon}{2} c^2 \pi^2 m(y_0) + \mathscr{O}_r(\varepsilon).$$

Since $2^*_{s_2} > 2^*_{s_1}$, $\Psi(tu_\varepsilon)$ has a unique maximum, we have

$$\max_{t \geqslant 0} \Psi(tw) = \Psi(w) = \beta^*.$$

Therefore, the maximum of $J(tu_\varepsilon)$ occurs at $t_\varepsilon := 1 + o_\varepsilon(1)$. Thanks to assumption (1.12), we have

$$\mathscr{M}_{r,\varepsilon}(w) < 0.$$

Therefore

$$\max_{t \geqslant 0} J(tu_\varepsilon) := J(t_\varepsilon u_\varepsilon) \leqslant \Psi(t_\varepsilon w) + \mathscr{M}_{r,\varepsilon}(t_\varepsilon w) \leqslant \Psi(t_\varepsilon w) < \Psi(w) = \beta^*.$$

We thus get the desired result. □

Acknowledgements. The authors would like to express his gratitude to the referees for several useful remarks and suggestions.

REFERENCES

[1] A. DIATTA AND E. H. A. THIAM, *Influence of the curvature in the existence of solutions for a two Hardy-Sobolev critical exponents*, `arxiv.org/pdf/2309.04768`.

[2] O. DRUET, *Elliptic equations with critical Sobolev exponents in dimension 3*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire, vol. 19, no. 2, no longer published by Elsevier, 2002.

[3] M. M. FALL AND E. H. A. THIAM, *Hardy-Sobolev inequality with singularity a curve*, (2018): 151–181.

[4] D. GILBARG, DAVID AND N. S. TRUDINGER, *Elliptic partial differential equations of second order*, vol. 224, no. 2, Berlin: Springer, 1977.

[5] W. K. HAYMANN AND P. B. KENNEDY, *Subharmonic functions*, Academic Press, London, 1976.

[6] I. E. IJAODORO AND E. H. A. THIAM, *Influence of an lp-perturbation on Hardy-Sobolev inequality with singularity a curve*, Opuscula Mathematica 41.2 (2021): 187–204.

[7] H. JABER, *Hardy-Sobolev equations on compact Riemannian manifolds*, Nonlinear Anal. **103** (2014), 39–54.

[8] H. JABER, *Mountain pass solutions for perturbed Hardy-Sobolev equations on compact manifolds* Analysis, 2016, vol. 36, no. 4, 287–296.
[9] R. SCHOEN AND S. YAU, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.
[10] R. SCHOEN AND S. YAU, *Proof of the positive action-conjecture in quantum relativity*, Phys. Rev. Lett. **42** (1979), no. 9, 547–548.

(Received December 1, 2023)

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