Differential
Equations
& Applications
Volume 16, Number 3 (2024), 199–213 doi:10.7153/dea-2024-16-12

EXISTENCE OF SOLUTION FOR HIGHER ORDER NONLINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATION WITH NONLINEAR GROWTH

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(*Communicated by M. Fečkan*)

Abstract. This research paper explores the existence of solution to a higher-order fractional differential equation with a general boundary condition, shedding light on novel extensions beyond existing literature. The equation, characterized by a Caputo fractional derivative exhibits nonlinearity and resonance, making it a compelling subject of study. The investigation employs coincidence degree theory, a robust tool for the examination of differential equations and the identification of solution. Notably, this paper delves into nonlinear growth patterns of function. The main results of the research are accompanied by an illustrative example to clarify the concepts discussed.

1. Introduction

In this paper, we consider the fractional differential equation

$$-\left({}^{\mathrm{C}}D_{a+}^{\alpha}x\right)(t)=f\left(t,x(t),{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(t)\right),\quad a<t<b,\ \ \alpha\in(n-1,n],\ \ n\geqslant 4,\tag{1}$$

together with the boundary condition

$$x^{(i)}(a)=0,\ \ x^{(k)}(b)=\int_a^b x^{(k)}(t)dH(t),\ \ 0\leqslant i\leqslant n-1,\ \ i\neq k+1,\tag{2}$$

where k is any integer between 1 and $n-1$, ${}^{\mathrm{C}}D_{a+}^{\alpha}$ is the Caputo fractional derivative of order α, $f:[a,b]\times\mathbb{R}^2\to\mathbb{R}$, and $\int_a^b x^{(k)}(t)dH(t)$ denotes the Riemann-Stieltjes integral of $x^{(k)}$ with respect to H. We note that the problem (1)–(2) is at resonance in the sense that the corresponding linear homogeneous equation $-\left({}^{\mathrm{C}}D_{a+}^{\alpha}x\right)(t)=0$, $t\in[a,b]$, with the boundary condition (2) has nontrivial solutions. Throughout this article, we assume that the following holds:

(A1) $\int_a^b(t-a)dH(t)=(b-a)$, and $\int_a^b(t-a)^{\alpha-k}dH(t)\neq(b-a)^{\alpha-k}$;

Mathematics subject classification (2020): 26A33, 34B10, 34B15, 34A08, 34B18.
Keywords and phrases: Caputo fractional derivative, existence of solution, nonlinear growth, coincidence degree theory.

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(A2) $f:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies Carathéodory conditions and further let $f:[a,b]\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function and f has the decomposition

$$f(t,x,y)=g(t,x,y)+h(t,x,y)$$

such that $yg(t,x,y)\leqslant 0$ for all $(t,x,y)\in[a,b]\times\mathbb{R}\times\mathbb{R}$. There exist nonnegative constants e_1,e_2,e_3,e_4,e_5 and constants $r,l\in[a,b)$ such that for all $(x,y)\in\mathbb{R}^2$, $t\in[a,b]$

$$|h(t,x,y)|\leqslant e_1|x|+e_2|y|+e_3|x|^r+e_4|y|^l+e_5$$

with

$$\|e_2\|+\frac{\|e_1\|(b-a)^{\alpha-1}}{\Gamma(\alpha)}<\frac{1}{2}, \tag{3}$$

where $\|e_j\|=\|e_j\|_\infty=\max\limits_{a\leqslant t\leqslant b}|e_j(t)|,\ j=1,2,\ldots,5$.

In the analysis of the problem (1)–(2), we employ Mawhin's coincidence theorem. It's noteworthy that Mawhin's coincidence degree theory is formidable approach to investigate the existence of solutions of differential equations. This approach has application in both differential equations and difference equations for the study of existence of solutions. Notably, the application of this technique to fractional differential equations for the study of existence of solution is a relatively recent development, for example [2, 3, 4, 6, 9, 10, 11, 18, 22, 24, 31].

The novelty of our work lies in our consideration of the higher-order fractional derivative with a general boundary condition. We note that we have a second boundary point of order $k\geqslant 1$, which represents a novel extension beyond the earlier boundary conditions discussed in existing literature. For example, when considering $k=0$, prior works such as [8, 16, 17, 18] become relevant. Furthermore, we delve into the analysis of non-linear growth patterns exhibited by a function, as indicated in (A2) above. This exploration holds profound implications across a wide array of applications. Nonlinear growth functions can effectively capture situations where growth is initially rapid but gradually slows down due to factors like limited resources or saturation effects. This ability is crucial for accurately describing scenarios such as population growth and chemical reactions. While a plethora of literature exist regarding linear growth, for example in [2, 3, 9, 10, 18, 22, 31], on the use of nonlinear growth in using coincidence degree theory approach remains relatively unexplored. For instance, in the context of ordinary differential equations, one may refer to [12, 13], and for fractional differential equations, one may refer to [1, 7, 14].

Fractional differential equations are gaining popularity as a modeling tool for complex systems in various fields of science and engineering [15, 21, 28]. To explore the use of boundary value problems with integral boundary conditions, we can refer [30] for phase field models, [26] for heat equations with nonlinear gradient source terms, [20] for applications in biomedical computational fluid dynamics, and [27] for modeling world population growth. When considering applications, it is of significant importance to investigate the existence of solutions for fractional differential equations. Here, we discuss some relevant works that provide us a real motivation for our examination of problem (1)–(2). Ma et al. [18] used coincidence degree theory to study the problem

$$
\begin{gathered}
{}^{\mathrm{C}}D_{0+}^{\alpha}x'(t)=f(t,x(t),x'(t),x''(t)),\quad t\in(0,1),\\
x(0)=x''(0)=0,\quad x(1)=\int_0^1 x(t)dH(t),
\end{gathered}
$$

where $2<\alpha\leqslant 3$, ${}^{\mathrm{C}}D_{0+}^{\alpha}x$ is the standard Caputo fractional derivative. In [5], Bohner et al. applied a Vallée-Poussin theorem and obtained explicit inequality tests for fractional functional differential equations

$$
({}^{C}D_{a+}^{\alpha}x)(t)+\sum_{i=0}^{m}(T_i x^{(i)})(t)=f(t),\quad t\in[a,b],
$$

together with the boundary condition

$$
x^{(i)}(a)=x^{(k)}(b)=0,\ \ 0\leqslant i\leqslant n-1 \text{ and } i\neq k,
$$

where the operator $T: C\to L_{\infty}$ can be an operator with deviation (of delayed or advanced type), an integral operator, or various linear combinations and superpositions. Inspired by the work [5], Srivastava et al. [25] derived three distinct Lyapunov inequalities for the Caputo fractional boundary value problem. Recently, Domonshnitsky et al. [10] investigate the existence of at least one solution to the higher order Riemann–Liouville fractional differential equation

$$
\begin{gathered}
-(D_{0+}^{\alpha}x)(t)=f(t,x(t),D_{0+}^{\alpha-1}x(t)),\quad n-1<\alpha\leqslant n,\quad t\in[0,1],\\
x(0)=x'(0)=\ldots=x^{(n-2)}(0)=0,\quad x^{(k)}(1)=\int_0^1 x^{(k)}(t)dH(t)
\end{gathered}
$$

by using coincidence degree theory. Feckan et al. [11] based on the topological index theory, demonstrated the existence of at least one solution to the problem which consists of Caputo fractional differential equation

$$
({}_{0}^{C}D_{t}^{\alpha}x)(t)=f(t,x(t)),\ \ q\in(m-1,m),\ \ m\in\mathbb{N},
$$

subject to periodic boundary conditions

$$
x^{(k)}(0)=x^{(k)}(T),\ \ k\in\overline{0,m-1},
$$

where $t\in[0,T]$, with $T>1$, $x\in C^{m-1}([0,T],\mathbb{R})$, $f\in C([0,T],\mathbb{R})$ and ${}_{0}^{C}D_{t}^{\alpha}$ represents the generalized Caputo fractional derivative with lower limit at 0.

The rest of this paper is organized as follows. Section 2, contains basic definitions of fractional calculus, essentials of coincidence degree theory and necessary function spaces, norms, and operators are defined to establish our results. Section 3 focuses on establishing the main results of this work. Moving on to Section 4, we provide an example to further illustrate the concepts discussed. Finally, Section 5 provides a discussion, along with potential directions for future research.

2. Preliminaries

DEFINITION 1. (see [15, 21]) The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of an essentially bounded function $f : [a,b] \to \mathbb{R}$ is defined as

$$(I^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \ \ \alpha > 0, \ \ t \in [a,b]. \tag{4}$$

DEFINITION 2. (see [15, 21]) The Caputo fractional derivative of order $\alpha \in (n-1,n]$ for the function $f : [a,b] \to \mathbb{R}$, $n \geqslant 1$, is defined by

$$({}^{C}D^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds. \tag{5}$$

LEMMA 1. (see [15, 21]) *The general solution of the equation* $({}^{C}D^{\alpha}_{a+}x)(t) = 0$ *with* $\alpha > 0$ *and* $n = [\alpha]+1$ *is of the form*

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 - \ldots + c_{n-1}(t-a)^{n-1}, \tag{6}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$.

LEMMA 2. (see [15, 21]) *If* $\alpha > 0$ *and* $n = [\alpha]+1$, *then*

$$(I^{\alpha}_{a+}({}^{C}D^{\alpha}_{a+}x))(t) = x(t) + c_0 + c_1(t-a) + c_2(t-a)^2 - \ldots + c_{n-1}(t-a)^{n-1}, \tag{7}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$.

LEMMA 3. *Assume that* $\alpha \in (n-1,n]$ *and* $f \in L_\infty$. *Then the unique solution of the fractional boundary value problem*

$$\begin{cases} ({}^{C}D^{\alpha}_{a+}x)(t) = f(t), \quad a < t < b, \\ x^{(i)}(a) = 0, \ \ x^{(k)}(b) = 0, \ \ 0 \leqslant i \leqslant n-1 \ \ \text{and} \ \ i \neq k+1, \end{cases} \tag{8}$$

where k is an integer satisfying the inequality $\alpha > k+1$, *where* $k \geqslant 1$, *can be represented by the formula*

$$x(t) = \int_a^b G(t,s)f(s)ds, \tag{9}$$

where Green's function $G_k(t,s)$ *is represented as*

$$G_k(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)}(t-a)^{k+1}(b-s)^{\alpha-k-1} - (t-s)^{\alpha-1}, \\ \qquad\qquad\qquad\qquad a \leqslant s \leqslant t \leqslant b, \\ \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)}(t-a)^{k+1}(b-s)^{\alpha-k-1}, \ a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{10}$$

Proof. Clearly, a solution of

$$({}^C D_{a+}^{\alpha} x)(t) = f(t)$$

can be expressed as

$$x(t) = b_0 + b_1(t-a) + b_2(t-a)^2 + \ldots + b_{n-1}(t-a)^{n-1} - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds.$$

From the boundary condition $x^{(i)}(a) = 0$ for $0 \leqslant i \leqslant n-1$ and $i \neq k+1$, we obtain $b_i = 0$ for $0 \leqslant i \leqslant n-1$, $i \neq k+1$. Since $x^{(k+1)}(a) \neq 0$, we have $b_{k+1} \neq 0$. Consequently,

$$x(t) = b_{k+1}(t-a)^{k+1} - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \tag{11}$$

$$\begin{aligned}
x'(t) &= (k+1)b_{k+1}(t-a)^k - \frac{\alpha-1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} f(s)ds,\\
x''(t) &= (k+1)kb_{k+1}(t-a)^{k-1} - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-3} f(s)ds,\\
&\vdots\\
x^{(k)}(t) &= (k+1)!b_{k+1}(t-a) - \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-k-1} f(s)ds,
\end{aligned}$$

$$x^{(k)}(b) = (k+1)!\, b_{k+1}(b-a) - \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-k-1} f(s)ds.$$

Since, $x^{(k)}(b) = 0$, then

$$b_{k+1} = \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-k-1} f(s)ds.$$

Hence, (11) gives

$$\begin{aligned}
x(t) = \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)\Gamma(\alpha)}(t-a)^{k+1} &\int_a^b (b-s)^{\alpha-k-1} f(s)ds\\
&- \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds
\end{aligned}$$

and

$$\begin{aligned}
x(t) = \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)\Gamma(\alpha)}(t-a)^{k+1} &\left[\int_a^t (b-s)^{\alpha-k-1} f(s)ds\right.\\
&\left. + \int_t^b (b-s)^{\alpha-k-1} f(s)ds\right] - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds.
\end{aligned}$$

This completes the proof. $\square$

We will now present the fundamental aspects of the coincidence degree theory. Let X and Y be the real Banach spaces, and let $M : \mathrm{dom}(M) \subset X \to Y$ be Fredholm operator of index zero, (i.e., $\dim(\mathrm{Ker}(M)) - \mathrm{codim}(\mathrm{Im}(M)) = 0$). If $P : X \to X$ and $Q : Y \to Y$ are two continuous projectors such that $\mathrm{Im}(P) = \mathrm{Ker}(M)$, $\mathrm{Ker}(Q) = \mathrm{Im}(M)$, $X = \mathrm{Ker}(M) \oplus \mathrm{Ker}(P)$ and $Y = \mathrm{Im}(M) \oplus \mathrm{Im}(Q)$, then the inverse operator of $M|_{\mathrm{dom}(M)\cap\mathrm{Ker}(P)} : \mathrm{dom}(M) \cap \mathrm{Ker}(P) \to \mathrm{Im}(M)$ exists and is denoted by K_p (generalized inverse operator of M). If Ω is an open bounded subset of X such that $\mathrm{dom}(M) \cap \Omega \neq 0$, the mapping $N : X \to Y$ will be called M-compact on $\bar{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. That the equation $Mx = Nx$ is solvable can be seen from [19].

THEOREM 1. ([19]) *Let M be a Fredholm operator of index zero and let N be the L-compact on $\overline{\Omega}$. Assume the following conditions are satisfied:*

1) *$Mx \neq \lambda Nx$ for every $(x,\lambda) \in [(\mathrm{dom}(M) \setminus \mathrm{Ker}(M)) \cap \partial\Omega] \times (0,1)$;*

2) *$Nx \notin \mathrm{Im}(M)$ for every $x \in \mathrm{Ker}(M) \cap \partial\Omega$;*

3) *$\deg(QN|_{\mathrm{Ker}(M)}, \mathrm{Ker}(M) \cap \Omega, 0) \neq 0$, where $Q : Y \to Y$ is a projector as above with $\mathrm{Im}(M) = \mathrm{Ker}(Q)$.*

Then, the equation $Mx = Nx$ has at least one solution in $\mathrm{dom}(M) \cap \overline{\Omega}$.

In this article, take the two Banach spacse X and Y as

$$X = \{u : [a,b] \to \mathbb{R} |\ u \in C[a,b],\ {}^{\mathrm{C}}D_{a+}^{\alpha-1}u \in C[a,b]\}, \quad Y = L^1[a,b],$$

with the norms $||u||_X = max\{||u||_\infty, ||{}^{\mathrm{C}}D_{a+}^{\alpha-1}||_\infty\}$, where $||u||_\infty = \max_{t\in[a,b]} |u(t)|$.

Let us define $M : \mathrm{dom}(M) \subset X \to Y$ and $N : X \to Y$ as

$$(Mx)(t) = -({}^{\mathrm{C}}D_{a+}^{\alpha}x)(t),$$

and

$$(Nx)(t) = f(t, x(t), {}^{\mathrm{C}}D_{a+}^{\alpha-1}x(t)), \quad t \in [a,b],$$

where

$$\mathrm{dom}(M) = \Big\{x \in X |\ -{}^{\mathrm{C}}D_{a+}^{\alpha}x \in Y,\ x^{(i)}(a) = 0,$$
$$x^{(k)}(b) = \int_a^b x^{(k)}(t)dH(t),\ 0 \leqslant i \leqslant n-1,\ i \neq k+1\Big\}.$$

Then the boundary value problem (1)–(2) becomes

$$(Mx)(t) = (Nx)(t), \ \ x \in \mathrm{dom}(M).$$

To apply Theorem 1 in the main results of the present article, we define linear continuous projectors $P: X \to X$ and $Q: Y \to Y$ by

$$(Px)(t) = x^{(k)}(b)\frac{(t-a)}{(b-a)}, \tag{12}$$

$$(Qy)(t) = \frac{\alpha-k}{(b-a)^{\alpha-k} - \int_a^b (t-a)^{\alpha-k} dH(t)} \times \left[\int_a^b (b-s)^{\alpha-k-1} y(s)ds - \int_a^b \int_a^t (t-s)^{\alpha-k-1} y(s)dsdH(t)\right] \tag{13}$$

and a generalized inverse operator $K_p : \mathrm{Im}(M) \to \mathrm{dom}(M) \cap \mathrm{Ker}(P)$ of M by

$$\begin{aligned}(K_p y)(t) &= \int_a^b G(t,s)y(s)ds \\ &= \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)}(t-a)^{k+1}\int_a^b (b-s)^{\alpha-k-1}y(s)ds \\ &\quad - \int_a^t (t-s)^{\alpha-1}y(s)ds,\end{aligned} \tag{14}$$

where $G(t,s)$ is given in (10).

3. Main result

In this section, we enhance clarity by presenting several lemmas that establish the prerequisites for the main theorem. We conclude this section with the theorem concerning the existence of a solution.

LEMMA 4. *$M : \mathrm{dom}(M) \subset X \to Y$ is a Fredholm operator of index zero.*

Proof. By Lemma 1 and $Mx = 0$, we have

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_n(t-a)^{n-1},$$

using the boundary conditions (2), gives

$$\mathrm{Ker}(M) = \left\{ b_{k+1}(t-a)^{k+1} :\ b_{k+1} \in \mathbb{R} \right\}.$$

Also,

$$\mathrm{Im}(M) = \left\{ y \in Y : \int_a^b (b-s)^{\alpha-k-1}y(s)ds - \int_a^b \int_a^t (t-s)^{\alpha-k-1}y(s)dsdH(t) = 0 \right\}.$$

Let $x \in \mathrm{dom}(M)$ and $Mx = y$. Then by Lemmas 1,

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_n(t-a)^{n-1} - I_{a+}^{\alpha} y. \tag{15}$$

By boundary conditions, we obtain $c_i = 0$, for $0 \leqslant i \leqslant n-1$, $i \neq k+1$. Therefore $c_{k+1} \neq 0$. Thus, from (15), we have

$$\begin{aligned}
x(t) =& c_{k+1}(t-a)^{k+1} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)ds \\
x'(t) =& c_{k+1}(k+1)(t-a)^{k} - \frac{(\alpha-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}y(s)ds \\
&\vdots \\
x^{(k)}(t) =& c_{k+1}(k+1)k(k+1)\cdots(k-(k-2))(t-a)^{(k-(k-1))} \\
&- \frac{(\alpha-1)\cdots(\alpha-k)}{\Gamma(\alpha)}\int_a^b (t-s)^{\alpha-k-1}y(s)ds \\
x^{(k)}(b) =& c_{k+1}(k+1)!(b-a) - \frac{(\alpha-1)\cdots(\alpha-k)}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-k-1}y(s)ds
\end{aligned}$$

$$\begin{aligned}
\int_a^b x^{(k)}(t)dH(t) = c_{k+1}(k+1)!\int_a^b (t-a)dH(t) \\
- \frac{(\alpha-1)\cdots(\alpha-k)}{\Gamma(\alpha)}\int_a^b\int_a^t (t-s)^{\alpha-k-1}y(s)dsdH(t).
\end{aligned}$$

Given that in Boundary conditions (2), $x^{(k)}(b) = \int_a^b x^{(k)}(t)dH(t)$ and using assumption (A1), $\int_a^b (t-a)dH(t) = (b-a)$, we derive

$$\int_a^b (b-s)^{\alpha-k-1}y(s)ds = \int_a^b\int_a^t (t-s)^{\alpha-k-1}y(s)dsdH(t).$$

On the other hand, if $y \in Y$, then $\int_a^b (b-s)^{\alpha-k-1}y(s)ds = \int_a^b\int_a^t (t-s)^{\alpha-k-1}y(s)dsdH(t)$. If

$$\begin{aligned}
x(t) =& \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)\Gamma(\alpha)}(t-a)^{k+1}\int_a^b (b-s)^{\alpha-k-1}y(s)ds \\
&- \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)ds \\
x'(t) =& \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)\Gamma(\alpha)}(k+1)(t-a)^{k}\int_a^b (b-s)^{\alpha-k-1}y(s)ds \\
&- \frac{(\alpha-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}y(s)ds \\
x''(t) =& \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)\Gamma(\alpha)}(k+1)k(t-a)^{k-1}\int_a^b (b-s)^{\alpha-k-1}y(s)ds \\
&- \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-3}y(s)ds \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
x^{(k)}(t) =& \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)\Gamma(\alpha)}(k+1)!(t-a)\int_a^b (b-s)^{\alpha-k-1}y(s)ds \\
&- \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-k-1}y(s)ds \\
x^{(k)}(b) =& \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(b-a)\Gamma(\alpha)}(b-a)\int_a^b (b-s)^{\alpha-k-1}y(s)ds \\
&- \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-k-1}y(s)ds = 0
\end{aligned}$$

$$\begin{aligned}
\int_a^b x^{(k)}(t)dH(t) =& \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(b-a)\Gamma(\alpha)}\int_a^b (t-a)\int_a^b (b-s)^{\alpha-k-1}y(s)dsdH(t) \\
&- \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{\Gamma(\alpha)}\int_a^b\int_a^t (t-s)^{\alpha-k-1}y(s)dsdH(t) = 0.
\end{aligned}$$

Thus, if $x \in \mathrm{dom}(M)$, then $y \in \mathrm{Im}(M)$ and $Mx = y$, Hence,

$$\mathrm{Im}(M) = \left\{ y \in Y : \int_a^b (b-s)^{\alpha-k-1}y(s)ds - \int_a^b\int_a^t (t-s)^{\alpha-k-1}y(s)dsdH(t) = 0 \right\}.$$

Consequently, $\dim \mathrm{Ker}(M) = 1$ and $\mathrm{Im}(M)$ is closed.

From (12), we see that P is linear and it fulfills idempotence property as $(P^2x)(t) = (Px)(t)$, which means that P is a projection operator. Also, $Ker(P) = \{x \in X | x^{(k)}(b) = 0\}$ and $\mathrm{Im}(P) = \mathrm{Ker}(M)$. For any $x \in X$, together with $x = (x - Px) + Px$, we have $X = \mathrm{Ker}(P) \oplus \mathrm{Ker}(M)$. It is easy to show that $\mathrm{Ker}(M) \cap \mathrm{Ker}(P) = 0$ which implies $X = \mathrm{Ker}(P) \oplus \mathrm{Ker}(M)$. It is not difficult to see that $(Q^2x)(t) = (Qx)(t)$ (see page 12025 in [10] for a similar argument), so Q is a projection operator. Moreover, $\mathrm{Ker}(Q) = \mathrm{Im}(M)$.

Next, for any $y \in Y$, setting $y_1 = y - Qy$, we have $(Qy_1)(t) = Q(y - Q(y))(t) = Qy(t) - Q^2y(t) = 0$. Hence $y_1 \in \mathrm{Im}(M)$ and $Y = \mathrm{Im}(M) + \mathrm{Im}(Q)$. Moreover, it is easy to verify that $\mathrm{Im}(Q) \cap \mathrm{Im}(M) = \{0\}$. Consequently $Y = \mathrm{Im}(M) \oplus \mathrm{Im}(Q)$. Since $\mathrm{Im}(M)$ is a closed subspace of Y and dim $\mathrm{Ker}(M) =$ codim $\mathrm{Im}(M) = 1$, then M is a Fredholm operator of index zero. This proves the lemma. □

LEMMA 5. *K_p is the inverse of $M|_{\mathrm{dom}(M)\cap\mathrm{Ker}(P)}$.*

Proof. If $y \in \mathrm{Im}(M)$, then

$$\begin{aligned}
MK_py =& -{}^{\mathrm{C}}D_{a+}^{\alpha}\Bigg(\frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!(b-a)\Gamma(\alpha)}(t-a)^{(k+1)}\int_a^b (b-s)^{\alpha-k-1} \\
&\qquad\qquad\qquad\qquad - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Bigg) \\
=& \, y.
\end{aligned}$$

For $x \in \mathrm{dom}(M) \cap \mathrm{Ker}(P)$ and $Mx = y$, we have

$$-({}^{\mathrm{C}}D^{\alpha}_{a+}x(t) = y(t), \ \ t \in (a,b)$$

$$x^{(i)}(a) = 0, \ \ x^{(k)}(b) = 0, \ \ 0 \leqslant i \leqslant n-1, \ \ i \neq k+1.$$

Furthermore, for $x \in \mathrm{dom}(M) \cap \mathrm{Ker}(P)$, we have

$$(K_pMx)(t) = \int_a^b G(t,s)({}^{\mathrm{C}}D^{\alpha}_{a+}x(s))ds = \int_a^b G(t,s)y(s)ds = x(t),$$

which implies that $K_p = \left(M|_{\mathrm{dom}(M)\cap\mathrm{Ker}(P)}\right)^{-1}$. The proof is complete. □

LEMMA 6. *$QN : X \to Y$ is continuous and bounded, and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact, where $\Omega \subset X$ is bounded.*

Proof. Since f is continuous, then $QN(\overline{\Omega})$ and $(I-Q)N(\overline{\Omega})$ are bounded. Hence there exists a constant $A > 0$, such that $|(I-Q)Nx(t)| \leqslant A$ for $x \in \overline{\Omega}$ and $t \in [a,b]$. using the Lebesgue dominated convergence theorem, it is clear that $K_p(I-Q)Ny : Y \to Y$ is completely continuous, then by the Arzela-Ascoli theorem, $K_p(I-Q)N(\overline{\Omega})$ is compact. This completes the proof of the theorem. □

Now, we use of the following conditions to prove our results

(A3) There exists a constant $\Phi > 0$ such that if $|{}^{\mathrm{C}}D^{\alpha-1}_{a+}x(t)| > \Phi$ for all $t \in [a,b]$, then $QNx \neq 0$.

(A4) There exists a constant $B > 0$ such that either of the following holds

$$cQN(c(t-a)) < 0 \quad \text{or} \quad cQN(c(t-a)) > 0$$

for $|c| > B$ and $c \in \mathbb{R}$.

LEMMA 7. *If the conditions (A1)–(A4) are satisfied, then set Ω_1, defined by*

$$\Omega_1 = \{x \in \mathrm{dom}(M) \setminus \mathrm{Ker}(M) : Mx = \lambda Nx \textit{ for some } \lambda \in [0,1]\},$$

is bounded.

Proof. For $x \in \Omega_1$, then $x \in \mathrm{dom}(M) \setminus \mathrm{Ker}(M)$ and $Mx = \lambda Nx$, thus $\lambda \neq 0$ and $Nx \in \mathrm{Im}(M) = Ker(Q)$. Thus, $QNx = 0$. For every $x \in \mathrm{dom}(M)$, ${}^{\mathrm{C}}D^{\alpha-1}_{a+}x \in C[a,b]$. By Lemma 2, we have $I^{\alpha-1}_{a+}{}^{\mathrm{C}}D^{\alpha-1}_{a+}x(t) = x(t) + c_0$. Since, $x \in \mathrm{dom}(M)$, $x(a)$ implies that $c = 0$. Therefore,

$$\begin{aligned}
\|x\|_\infty &= \|I^{\alpha-1}_{a+}{}^{\mathrm{C}}D^{\alpha-1}_{a+}x\|_\infty \\
&= \left\| \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2}\,{}^{\mathrm{C}}D^{\alpha-1}_{a+}x(s)ds \right\| \\
&\leqslant \max_{a \leqslant t \leqslant b} \left| \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2}ds \right| \|{}^{\mathrm{C}}D^{\alpha-1}_{a+}x\|_\infty \\
&= \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \|{}^{\mathrm{C}}D^{\alpha-1}_{a+}x\|_\infty. \qquad (16)
\end{aligned}$$

From (A3), there exist $t_0 \in [a,b]$ such that $|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(t_0)| \leqslant \Phi$. Next for $x \in \Omega_1$,

$$ {}^{\mathrm{C}}D_{a+}^{\alpha-1}x(t)\,{}^{\mathrm{C}}D_{a+}^{\alpha}x(t) = \lambda\,{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(t) f(t,x(t),D_{a+}^{\alpha-1}x(t)). $$

Using integration and (16), we proceed

$$ \begin{aligned} \frac{1}{2}\left({}^{\mathrm{C}}D_{a+}^{\alpha-1}x(t)\right)^2 =& \frac{1}{2}\left({}^{\mathrm{C}}D_{a+}^{\alpha-1}x(t_0)\right)^2 + \lambda\int_{t_0}^{t} {}^{\mathrm{C}}D_{a+}^{\alpha-1}x(s) g(s,x(s),{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(s))ds \\ &+ \lambda\int_{t_0}^{t} {}^{\mathrm{C}}D_{a+}^{\alpha-1}x(S) h(s,x(S),{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(S))ds \\ \leqslant& \frac{1}{2}\Phi^2 + \int_a^b |{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(s) h(s,x(s),{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(s))|ds \\ \leqslant& \frac{1}{2}\Phi^2 + \|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(s)\|_\infty \Big(e_1\|x\|_\infty + e_2\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(s)\|_\infty + e_3\|x\|_\infty^r \\ &+ e_4\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x(s)\|_\infty^l + e_5\Big) \\ \leqslant& \frac{1}{2}\Phi^2 + \frac{e_1(b-a)^{\alpha-1}}{\Gamma(\alpha)}\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x\|_\infty^2 + e_2\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x\|_\infty^2 \\ &+ \frac{e_3(b-a)^{r(\alpha-1)}}{(\Gamma(\alpha))^r}\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}\|_\infty^{1+r} + e_4\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x\|_\infty^{1+l} + e_5\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x\|_\infty. \end{aligned} $$

Thus,

$$ \|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x\|_\infty^2 \leqslant \frac{\frac{e_3(b-a)^{r(\alpha-1)}}{(\Gamma(\alpha))^r}\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}\|_\infty^{1+r} + e_4\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x\|_\infty^{1+l} + e_5\|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x\|_\infty + \frac{1}{2}\Phi^2}{\frac{1}{2} - \frac{e_1(b-a)^{\alpha-1}}{\Gamma(\alpha)} - e_2}. $$

Since, $r,l \in [0,1)$ from the previous inequality, there exist $\Phi_1 > 0$ such that

$$ \|{}^{\mathrm{C}}D_{a+}^{\alpha-1}x\|_\infty \leqslant \Phi_1 $$

and (16) shows that

$$ \|x\|_\infty \leqslant \frac{1}{\Gamma(\alpha)}\Phi_1. $$

Therefore Ω_1 is bounded. $\square$

LEMMA 8. *If the assumptions (A1), (A2) and (A4) are satisfied, then the set Ω_2, defined by*

$$ \Omega_2 = \{x : x \in \mathrm{Ker}(M),\ Nx \in \mathrm{Im}(M)\}, $$

is bounded.

Proof. Let $x \in \Omega_2$, $x(t) = c(t-a)$, $c \in \mathbb{R}$, we have $\mathrm{Im}(M) = \mathrm{Ker}(Q)$, and therefore $QNx(c(t-a)) = 0$. By (A4), we have $|c| \leqslant B$, hence Ω_2 is a bounded set. $\square$

For our next result, we define an isomorphism $J : \mathrm{Ker}(M) \to \mathrm{Im}(Q)$ by

$$ J(c(t-a)) = c. $$

LEMMA 9. *If assumptions (A1), (A2), (A4) hold, then the set* Ω_3*, defined by*

$$\Omega_3 = \{x : x \in \mathrm{Ker}(M),\ \lambda Jx + w(1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

with

$$w = \begin{cases} -1, & \textit{if } cQN(c(t-a)) < 0, \\ 1, & \textit{if } cQN(c(t-a)) > 0, \end{cases}$$

is bounded.

Proof. Let $x \in \Omega_3$, we have $x(t) = c(t-a)$, $c \in \mathbb{R}$ and $\lambda c + w(1-\lambda)QN(c(t-a)) = 0$. If $\lambda = 1$, then then $c = 0$. If $\lambda = 0$, by condition (A5), we have $|c| \leqslant B$. Finally, suppose that $\lambda \in (0,1)$. We claim that $|c| \leqslant B$. If $|c| \geqslant B$, then $\lambda c^2 = -w(1-\lambda)cQN(c(t-a)) < 0$, which contradicts $\lambda c^2 > 0$. Thus, our claim remains valid, that is, $|c| \leqslant B$. Thus, Ω_3 is bounded. $\square$

We now prove the main result of this article.

THEOREM 2. *Suppose that the conditions (A1)–(A4) hold. Then problem* (1) *has at least one solution in* X.

Proof. Let Ω any bounded open subset of X such that $\overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3} \subset \Omega$. From Lemma 6, N is M-compact. From Lemmas 7, 8 and 9, it is clear that the assumptions 1) and 2) of Theorem 1 are fulfilled. In order to finalize the proof of the theorem, it is necessary to confirm condition 3) as stated in Theorem 1. We define

$$Z(x,\lambda) = \lambda x + w(1-\lambda)QNx;$$

then it follows from Lemma 9 that $Z(x,\lambda) \neq 0$, $x \in \mathrm{Ker}(M) \cap \partial\Omega$. Thus, by Homotopy property of degree

$$\begin{aligned} \deg(QN|_{\mathrm{Ker}(M)}, \Omega \cap \mathrm{Ker}(M), 0) &= \deg(Z(\cdot,0), \Omega \cap \mathrm{Ker}(M), 0) \\ &= \deg(Z(\cdot,1), \Omega \cap \mathrm{Ker}(M), 0) \\ &= \deg(Z(wJ, \Omega \cap \mathrm{Ker}(M), 0) \neq 0. \end{aligned}$$

Hence, by Theorem 1, the problem (1)–(2) has at least one solution in $\mathrm{dom}(M) \cap \overline{\Omega}$. $\square$

4. Examples

EXAMPLE 1. consider the problem

$$\begin{cases} -({}^{\mathrm{C}}D_{0+}^{\frac{7}{2}}x)(t) = f(t,x(t),{}^{\mathrm{C}}D_{0+}^{\frac{5}{2}}x(t)) \\ x(0) = x'(0) = x'''(0) = 0, \quad x'(1) = \int_0^1 x'(t)dH(t) \end{cases} \tag{17}$$

where $H(t) = 2t$

$$\int_0^1 t d(2t) = 1 \quad \text{and} \quad \int_0^1 t^{\frac{7}{2}-1} d(2t) = \frac{4}{7} \neq 1$$

holds. Thus, (A1) is satisfied. Take $e_1 = \frac{1}{7}$, $e_2 = \frac{1}{5}$, $e_3 = e_4 = r = l = 0$, $e_5 = 3$ and

$$f(t,x,y) = \frac{1}{7(2+\cos y)} + \frac{x}{5(2+\sin x)} + \frac{5}{4}\sin(xy) + \frac{7}{4}\cos(t) - x^4y^5 - y^7.$$

Thus, we have

$$yg(t,x,y) = -x^4y^6 - y^8 \leqslant 0,$$

$$|h(t,x,y)| \leqslant \frac{1}{7} + \frac{|x|}{5} + 3,$$

$$\|e_2\| + \frac{\|e_1\|(b-a)^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{5} + \frac{1}{7 \times 3.32335} = 0.24296 < \frac{1}{2}.$$

Thus, (A2) is satisfied. Moreover, we can choose $\Phi = 20$, we have $f(t,x,y) \neq 0$ if $|y| > 20$. (A3) is also satisfied. Finally, take $B = 10$, when $|c| > 10$,

$$\begin{aligned} cf(t,ct,0) =& c\left(\frac{1}{7(2+1)} + \frac{ct}{5(2+\sin(ct))} + \frac{5}{4}\sin 0 + \frac{7}{4}\cos t\right) \\ =& c\left(\frac{1}{21} + \frac{ct}{10+4\sin(ct)} + \frac{7}{4}\cos t\right) \neq 0, \end{aligned}$$

then we obtain $cQN(c(t-a)) \neq 0$, that is, condition (A4) is satisfied. It follows from Theorem 2 that the problem (17) has at least one solution.

5. Discussion

In this paper, employing coincidence degree theory, we have shown that solutions exist for the higher-order fractional differential equation with general boundary conditions, subject to the specified assumptions. Our work generalizes previous studies by allowing a second boundary point of order $k \geqslant 1$. This novel extension enables the analysis of more complex systems that were not covered by earlier research. A significant aspect of our study is the inclusion of nonlinear growth patterns in the function f. To elucidate the theoretical concepts discussed, we provided an illustrative example.

Several potential avenues for future research based on this paper include applying this technique in conjunction with matrix spectral theory [29], and the method of matrix diagonalization [23]. Additionally, exploring more general fractional derivatives, such as Hilfer or Ψ-Hilfer derivatives, could yield more comprehensive results. It would be intriguing to investigate how the results manifest with these derivatives, as they generalize Caputo fractional derivatives by introducing extra parameters. Moreover, investigating the stability and uniqueness of solutions in this context remains an important area for exploration. It is noteworthy that the boundary condition employed in this study is introduced for the first time in the literature. Consequently, the newly constructed Green's function can be utilized in other methodologies, including different fixed-point theorems, Vallée-Poussin theorem and various numerical methods. Exploring how these techniques yield existence results and the assumptions required on f would be of considerable interest. While this paper focuses on theoretical existence results, implementing numerical methods to approximate the solutions of such fractional differential equations could be a valuable complement to this work.

Acknowledgement. The authors would like to thank the anonymous referee for valuable comments and suggestions, leading to a better presentation of our results.

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(Received December 13, 2023)

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