Differential
Equations
& Applications
Volume 16, Number 3 (2024), 215–225 doi:10.7153/dea-2024-16-13

AN IMPROVED GRADIENT ESTIMATE FOR SOLUTIONS TO VERY SINGULAR QUASILINEAR ELLIPTIC EQUATIONS IN WEIGHTED LORENTZ SPACES

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(*Communicated by M. Feng*)

Abstract. The gradient estimates in the weighted Lorentz spaces for solutions to a class of quasilinear elliptic equations with a measure on the right-hand side are established using the idea of level-set inequality. Regularity results obtained in this paper are concerned with the quasilinear elliptic equations driven by p-Laplacian, under certain smoothness assumptions on the boundary of domain Ω and the data of the problem. Especially, this paper studies the "very singular" case for the growth exponent p, i.e. when $1 < p \leqslant \frac{3n-2}{2n-1}$. As far as we know, the presence of measure source term μ (being a bounded Radon measure) makes the study of regularity theory more challenging due to the notion of solutions and their reasonable existence. The contribution of this paper is the extension of previous results in weighted Lebesgue spaces in [9, 15].

1. Introduction

This paper deals with the gradient estimates for solutions to a class of quasilinear elliptic equations with measure data

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) & = \mu \quad \text{in } \Omega, \\ \qquad u & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in the setting of weighted Lorentz spaces. The operator on the left-hand side is driven by p-Laplacian and the given data μ on the right-hand side is a signed Radon measure defined on Ω with finite total mass. It also allows us to define μ on the whole space $\mathbb{R}^n$ by letting $\mu(\mathbb{R}^n \setminus \Omega) = 0$, $n \geqslant 2$. More precisely, the coefficient A is a Carathéodory vector field satisfying two following conditions: there is a constant $\Lambda > 0$ such that

$$|A(x,\xi)| + |\langle \nabla_\xi A(x,\xi), \xi\rangle| \leqslant \Lambda |\xi|^{p-1},$$
$$\langle A(x,\xi_1) - A(x,\xi_2), \xi_1 - \xi_2\rangle \geqslant \Lambda^{-1}(|\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}}|\xi_1 - \xi_2|^2$$

hold for almost every $x \in \Omega$ and every pair $(\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$.

Mathematics subject classification (2020): 26D15, 26A51, 32F99, 41A17.
Keywords and phrases: Quasilinear equation, measure data, p-capacity thickness, very singular case, Lorentz estimates, Muckenhoupt weights.
This research is supported by Dr. Minh-Phuong Tran.
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In the past years, special attention has been devoted to the study of elliptic equations/systems with measure data μ. Further, to the question of explaining the existence of the very weak solutions, there has been a substantial amount of work in the specific case when p is singular. The existence and regularity of solutions to measure data problems have been investigated by many authors in the past years. Here, we may mention a few for instance by Boccardo et al. [1, 2, 3] for the existence, and by Mingione [7], Hung-Phuc [8, 9], Tran-Nguyen [10, 11, 12, 14] for regularity results, together with a huge literature concerning the problems with measure data by different authors.

To provide detailed results regarding the regularity results of equation (1.1), let us briefly summarize these works as follows. For the case when $\frac{3n-2}{2n-1} < p \leqslant 2 - \frac{1}{n}$, p is said "very singular", under various assumptions on the data and domain Ω, Nguyen-Phuc in [8, 16], Tran-Nguyen in [10, 11, 12] proved the global gradient estimates for (renormalized) solutions to (1.1) in weighted (or non-weighted) Lebesgue and Lorentz also Lorentz-Morrey spaces. With two different points of view for the data of problem (1.1), [9] and [14] have established the global regularity estimates for (renormalized) solutions in the case when p is very singular, $1 < p \leqslant \frac{3n-2}{2n-1}$.

Inspired by these above results, in this paper, we are devoted to extending results to the weighted Lorentz spaces, for this very singular growth exponent. Our work is inherited from what has been done with the good-λ or level-set inequality in previous studies [9, 15]. So, here we show how to extend these above results.

2. Preliminaries

2.1. Notation

In order to make the paper more clearly, from now on we always consider $\Omega \subset R^n$, $n \geqslant 2$ an open bounded region. As usual, we write a ball with center $x_0 \in \mathbb{R}^n$ and radius $\rho > 0$ as $B_\rho(x_0)$. Throughout the estimates in proofs, we shall denote by C the symbol of a general positive constant. The actual value of C is not important, it may change and we continue to denote it by C in the chain of estimates without loss of generality. Parentheses are used to clarify the dependence of the constant C on the certain parameters. If necessary, we also use specific constants such as C_1, C_2, etc.

Moreover, for any measurable subset $\mathscr{D} \subset \mathbb{R}^n$, let $h \in L^1_{loc}(\mathbb{R}^n)$ be a measurable mapping, we shall denote the average integral of h on $\mathscr{D}$ by

$$\fint_{\mathscr{D}} h(x)dx = \frac{1}{\mathscr{L}^n(\mathscr{D})} \int_{\mathscr{D}} h(x)dx,$$

where $\mathscr{L}^n(\mathscr{D})$ or $|D|$ is the notation of Lebesgue measure of a measurable set $\mathscr{D}$ in $\mathbb{R}^n$.

2.2. Assumption on domain: p-capacity thickness complement

As aforementioned in previous sections, there have been many suitable assumptions on Ω considered to obtain the global estimates. In the limit of our work, we consider a weak assumption on Ω: the domain whose complement satisfies the p-capacity.

We say that a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-capacity thick complement if the complement $\mathbb{R}^n \setminus \Omega$ satisfies the p-capacity uniform thickness condition. To be more precise, there exist two constants $c_0, r_0 > 0$ such that

$$\mathrm{cap}_p((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_\rho(\xi), B_{2\rho}(\xi)) \geqslant c_0 \mathrm{cap}_p(\overline{B}_\rho(\xi), B_{2\rho}(\xi)) \tag{2.1}$$

for every $\xi \in \mathbb{R}^n \setminus \Omega$ and $0 < \rho \leqslant r_0$. Here, the notation $\mathrm{cap}_p(\mathscr{U}, \Omega)$ stands for the p-capacity of the compact set $\mathscr{U} \subseteq \Omega$ that is defined by

$$\mathrm{cap}_p(\mathscr{U}, \Omega) = \inf \left\{ \int_\Omega |\psi|^p dx : \psi \in C_0^\infty(\Omega), \psi \geqslant 1 \text{ on } \mathscr{U} \right\}. \tag{2.2}$$

We refer the reader to [5, Chapter 2] and [13, 10] for further details of the definition and properties of variational capacity, also the domain satisfying this type of condition.

It is worth noticing that this assumption is very mild and essential for higher integrability results up to the boundary. To better understand, domains which their complements satisfies p-capacity uniform thickness contain domains with Lipschitz continuous boundaries or satisfy a uniform exterior corkscrew condition.

2.3. Assumption on measure μ

In this paper, we define $\mathfrak{M}_b(\Omega)$ as the space of all Radon measures on Ω with bounded total variation. With respect to the p-capacity, for any measure $\mu \in \mathfrak{M}_b(\Omega)$, we denote μ^+ and μ^- the positive and negative part of μ.

It was known from [6] that any $\mu \in \mathfrak{M}_b(\Omega)$ can be decomposed uniquely in the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. Furthermore, any $\mu_0 \in \mathfrak{M}_0(\Omega)$ is able to write $\mu_0 = g - \mathrm{div}(F)$ for $g \in L^1(\Omega)$ and $F \in L^{\frac{p}{p-1}}(\Omega;\mathbb{R}^n)$. For the focusing on our proofs in this paper, we will not discuss much about the Radon measure. Although, due to the importance of measure itself, the reader is referred to the [7, 9, 10, 3, 17] for additional details related to the measure data and their properties.

In what follows, when we mention the problem(1.1) with measure data μ, the case p is very singular $\left(1 < p \leqslant \frac{3n-2}{2n-1}\right)$ and the domain $\Omega \subset \mathbb{R}^n$ satisfies the p-capacity thick complement assumption on Ω, we shall shortly write assumption (H).

2.4. Renormalized solution

The solutions to measure data problem (1.1) considered in this paper is understood in the sense of "*renormalized solutions*". Note that when estimate the problem with measure data and p is very sinular $\left(1 < p \leqslant 2 - \frac{1}{n}\right)$, the distributional solutions do not belong to $W^{1,1}_{\mathrm{loc}}(\Omega)$. This is the important point motivating to give a sense of derivative ∇u generalizing the usual concept of weak derivative in $W^{1,1}_{\mathrm{loc}}(\Omega)$.

First, for every $l > 0$, let us define the so-called two-sided truncation operator $T_l : \mathbb{R} \to \mathbb{R}$ as follows. For any $t \in \mathbb{R}$,

$$T_l(t) = \max\{-l, \min\{l, t\}\},$$

and this value belongs to $W_0^{1,p}(\Omega)$. In the distributional sense, for a finite measure $\mu_l \in \Omega$, it satisfies that

$$-\mathrm{div}(A(x, \nabla T_l(u))) = \mu_l.$$

We remark here that from [1, Lemma 2.1], for a measurable function f defined in Ω satisfying $T_l(f) \in W_{\mathrm{loc}}^{1,p}(\Omega)$, for any $l > 0$, there exists a measurable function $V : \Omega \to \mathbb{R}^n$ such that $\nabla T_l(f) = \chi_{\{|f| \leqslant l\}} V$ almost everywhere in Ω. The function V is called *distributional gradient* of f, and when no confusion arises, we shall still write V as ∇u throughout this paper.

We reproduce the definition of renormalized solution to (1.1) here and we suggest the readers have a look in [1, 3] for a detailed explanation.

DEFINITION 1. Let a measure $\mu \in \mathfrak{M}_b(\Omega)$ be decomposed as $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. Then, a measurable function u is called a renormalized solution of (1.1) if $T_l(u) \in W_0^{1,p}(\Omega)$ for any $l > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{n}{n-1}$. Moreover, for any $l > 0$, there exist two non-negative Radon measures $\lambda_l^+, \lambda_l^- \in \mathfrak{M}_0(\Omega)$ concentrated on the sets $u = l$ and $u = -l$, respectively such that $\mu_l^+ \to \mu_s^+$, $\mu_l^- \to \mu_s^-$ in the narrow topology of measures and that

$$\int_{\{|u|<l\}} \langle A(x, \nabla u), \nabla \varphi \rangle dx = \int_{\{|u|<l\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda_l^+ - \int_\Omega \varphi d\lambda_l^-,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

2.5. Maximal operators

Let $0 \leqslant \alpha \leqslant \infty$. The fractional maximal function $\mathbf{M}_\alpha$ of a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ is delineated by

$$\mathbf{M}_\alpha f(x) = \sup_{\rho>0} \rho^\alpha \fint_{B_\rho(x)} |f(y)| dy, \quad x \in \mathbb{R}^n. \tag{2.3}$$

REMARK 1. When $\alpha = 0$, it coincides with the Hardy-Littlewood maximal function, $\mathbf{M}_0 f = \mathbf{M} f$, given by

$$\mathbf{M} f(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |f(y)| dy, \quad x \in \mathbb{R}^n \tag{2.4}$$

for a locally integrable function f in $\mathbb{R}^n$. Moreover, for a prescribed measure μ defined in $\mathbb{R}^n$, the fractional maximal function of μ, denoted by $\mathbf{M}_1(\mu)$, is defined as following

$$\mathbf{M}_1(\mu)(x) := \sup_{\rho>0} \frac{|\mu|(B_\rho(x))}{\rho^{n-1}}, \quad \forall x \in \mathbb{R}^n.$$

LEMMA 1. (Boundedness of maximal operator $\mathbf{M}$) *The operator* $\mathbf{M}$ *is bounded from* $L^s(\mathbb{R}^n)$ *to* $L^{s,\infty}$ *for* $s \geqslant 1$ *and* $\in \left[0, \frac{n}{s}\right)$. *This means, ther is a positive constant* $C = C(n,s)$ *such that:*

$$\mathscr{L}^n(\{x \in \mathbb{R}^n : \mathbf{M}(f)(x) > \lambda\}) \leqslant C \frac{1}{\lambda^s} \int_{\mathbb{R}^n} |f(y)|^s dy \tag{2.5}$$

for all $h \in L^s(\mathbb{R}^n)$ *and* $\lambda > 0$.

LEMMA 2. *There exists a posituve constant* C *such that*

$$\begin{aligned} &\left[q \int_0^\infty \lambda^s (|\{y \in \Omega : \mathbf{M}(h)(y) > \lambda\}|)^{\frac{s}{q}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{s}} \\ &\leqslant C \left[q \int_0^\infty \lambda^s (|\{y \in \Omega : h(y) > \lambda\}|)^{\frac{s}{q}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{s}}, \end{aligned} \tag{2.6}$$

where $q > 1$, $0 < s \leqslant \infty$ *and for all* $h \in L^{q,s}(\mathbb{R}^n)$.

2.6. Muckenhoupt weights

We call that a weight $\omega \in L^1_{loc}(\mathbb{R}^n)$ belongs to Muckenhoupt class $\mathbf{A}_q$ for $1 \leqslant q \leqslant \infty$ if it satisfies

- For $1 < q < \infty$:

$$[\omega]_{\mathbf{A}_q} = \sup_{B \subset \mathbb{R}^n} \left(\fint_B \omega(y) dy\right) \left(\fint_B \omega(y)^{\frac{-1}{q-1}} dy\right)^{q-1}$$

- For $q = 1$:

$$[\omega]_{\mathbf{A}_1} = \sup_{B \subset \mathbb{R}^n} \left(\fint_B \omega(y) dy\right) \sup_{y \in B} \frac{1}{\omega(y)} < \infty,$$

- For $q = \infty$, there are two positive constants C and ν such that

$$\omega(E) \leqslant C \left(\frac{\mathscr{L}^n(E)}{\mathscr{L}^n(B)}\right)^\nu \omega(B)$$

for all balls B in $\mathbb{R}^n$ and all measurable subsets E of B. In this case, (C,ν) stands for $[\omega]_{\mathbf{A}_\infty}$ and we shall simply write $[\omega] = (C,\nu)$.

2.7. Weighted Lorentz spaces

The weighted Lorentz space $L^{q,s}_{\omega}(\Omega)$ is defined for $0 < q < \infty$, $0 < s < \infty$ and the Muckenhoupt weight $\omega \in \mathbf{A}_{\infty}$ by the side of all Lebesgue measure function h on Ω such that $\|h\|_{L^{q,s}_{\omega}(\Omega)} < +\infty$, where

$$\|h\|_{L^{q,s}_{\omega}(\Omega)} = \begin{cases} \left[q\int_0^{\infty} \lambda^s \omega\left(\{x \in \Omega : |h(x)| > \lambda\}\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{s}}, & \text{if } s < \infty, \\ \sup_{\lambda>0} \lambda\, \omega\left(\{x \in \Omega : |h(x)| > \lambda\}\right)^{\frac{1}{q}}, & \text{if } s = \infty. \end{cases} \tag{2.7}$$

If $\omega = 1$, the weighted Lorentz space $L^{q,s}_{\omega}(\Omega)$ becomes the un-weighted (classical) Lorentz space $L^{q,s}(\Omega)$. Furthermore, in the case of weighted Lorentz spaces, when $q = s$, $L^{q,s}_{\omega}(\Omega)$ coincides the weighted Lebesgue space $L^{q}_{\omega}(\Omega)$ which is defined by the set of all measure functions h such that

$$\|h\|_{L^{q}_{\omega}(\Omega)} := \left(\int_{\Omega} |h(x)|^q \omega(x) dx\right)^{\frac{1}{q}} < +\infty.$$

REMARK 2. We can aplly lemma 2.6 for weighted Lorentz space:

$$\begin{aligned} &\left[q\int_0^{\infty} \lambda^s (\omega(\{y \in \Omega : \mathbf{M}(h)(y) > \lambda\}))^{\frac{s}{q}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{s}} \\ &\leqslant C\left[q\int_0^{\infty} \lambda^s (\omega(\{y \in \Omega : h(y) > \lambda\}))^{\frac{s}{q}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{s}}. \end{aligned}$$

3. Main regularity results

We have two main results for global regularity of renormalized solutions u of (1.1). In order to prove two main results, it is necessary to apply the good-λ inequality (or level-set inequality) that has been shown in [15, Lemma 4.1]. However, due to the presence of Muckenhoupt weight, we shall state and prove a weighted version of this result.

For every $\xi \in \Omega$ and $0 < \rho \leqslant r_0/10$, let v be a solution to the equation

$$\begin{cases} -\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) &= 0 \quad \text{in } \Omega_{2\rho}(\xi), \\ \qquad v &= u \quad \text{on } \partial\Omega_{2\rho}(\xi) \end{cases} \tag{3.1}$$

where $\Omega_{\rho}(\xi) = B_{\rho}(\xi) \cap \Omega$.

Here, let us recall the "*reverse Hölder* inequality" that related to the homogeneous problem (3.1). We could refer to previous papers [4, 15] and especially to references therein for the proof. It is well-known that there exists $\Theta > p$ such that

$$\left(\fint_{\Omega_{\rho}(\xi)} |\nabla v|^{\Theta} dx\right)^{\frac{1}{\Theta}} \leqslant C\left(\fint_{\Omega_{2\rho}(\xi)} |\nabla v|^{t} dx\right)^{\frac{1}{t}} \tag{3.2}$$

for every $t \in (0, p]$.

THEOREM 1. *Let $\omega \in \mathbf{A}_\infty$ with $[\omega] = (c_0, \nu)$. Assume that u is a renormalized solution to equation (1.1) satisfying $|\nabla u| \in L^{2-p}(\Omega)$ under assumption (H). Then for any $0 < \gamma < \frac{n(p-1)}{n-1}$, there exists a constant $\varepsilon_0 \in (0,1)$ such that the following inequality*

$$\begin{aligned}
&\omega\left(\left\{x \in \Omega: \left[\mathbf{M}(|\nabla u|^\gamma)\right]^{\frac{1}{\gamma}} > \varepsilon^{-\frac{1}{\Theta\nu}}\lambda\right\}\right) \\
&\leqslant C\varepsilon\omega\left(\left\{x \in \Omega: \left[\mathbf{M}(|\nabla u|^\gamma)(x)\right]^{\frac{1}{\gamma}} > \lambda\right\}\right) \\
&\qquad + C\omega\left(\left\{x \in \Omega: \left[\mathbf{M}_1(\mu)(x)\right]^{\frac{1}{p-1}} > \varepsilon^{\frac{1}{\gamma\nu}-\frac{1}{\Theta\nu}}\lambda\right\}\right) \\
&\qquad\qquad + C\omega\left(\left\{x \in \Omega: \left[\mathbf{M}(|\nabla u|^{2-p})(x)\right]^{\frac{1}{2-p}} > \varepsilon^{-\frac{1}{\nu}}\lambda\right\}\right)
\end{aligned} \tag{3.3}$$

holds for any $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$.

Proof. For simplicity of notation, for each $\gamma > 0$ and $\lambda > 0$, let us first introduce several level sets as follows

$$\begin{aligned}
&\mathbb{P}(\gamma,\lambda) := \left\{x \in \Omega: \left[\mathbf{M}(|\nabla u|^\gamma)(x)\right]^{\frac{1}{\gamma}} > \lambda\right\}, \\
&\mathbb{Q}(\lambda) := \left\{x \in \Omega: \left[\mathbf{M}_1(\mu)(x)\right]^{\frac{1}{p-1}} > \lambda\right\}, \\
&\mathbb{P}^c(\gamma,\lambda) = \Omega \setminus \mathbb{P}(\gamma,\lambda),\ \mathbb{Q}^c(\lambda) = \Omega \setminus \mathbb{Q}(\lambda).
\end{aligned}$$

The goal inequality (3.3) can be rewritten as

$$\omega\left(\mathbb{P}(\gamma, \varepsilon^{-a}\lambda)\right) \leqslant C\left[\varepsilon\omega\left(\mathbb{P}(\gamma,\lambda)\right) + \omega\left(\mathbb{Q}(\varepsilon^b\lambda)\right) + \omega\left(\mathbb{P}(2-p, \varepsilon^{-1}\lambda)\right)\right]. \tag{3.4}$$

Let us recall the following level set inequality which was proved in [15]. The authors proved that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $0 < \gamma \leqslant \frac{n(p-1)}{n-1}$, we have

$$\begin{aligned}
&\mathscr{L}^n\left(\left\{x \in \Omega: \left[\mathbf{M}(|\nabla u|^\gamma)(x)\right]^{\frac{1}{\gamma}} > \varepsilon^{-\frac{1}{\Theta}}\lambda; \left[\mathbf{M}_1(\mu)(x)\right]^{\frac{1}{p-1}} \leqslant \varepsilon^{\frac{1}{\gamma}-\frac{1}{\Theta}}\lambda;\right.\right. \\
&\left.\left.\left[\mathbf{M}(|\nabla u|^{2-p})(x)\right]^{\frac{1}{2-p}} \leqslant \varepsilon^{-1}\lambda\right\}\right) \\
&\leqslant C\varepsilon\mathscr{L}^n\left(\left\{x \in \Omega: \left[\mathbf{M}(|\nabla u|^\gamma)(x)\right]^{\frac{1}{\gamma}} > \lambda\right\}\right).
\end{aligned}$$

With our notation, this inequality is equivalent to

$$\mathscr{L}^n\left(\mathbb{P}(\gamma, \varepsilon^{-\frac{1}{\Theta}}\lambda) \cap \mathbb{P}^c(2-p, \varepsilon^{-1}\lambda) \cap \mathbb{Q}^c(\varepsilon^{\frac{1}{\gamma}-\frac{1}{\Theta}}\lambda)\right) \leqslant C\varepsilon\mathscr{L}^n\left(\mathbb{P}(\gamma,\lambda)\right). \tag{3.5}$$

One remarks that if $D_1 \subset D_2$ and $\omega \in \mathbf{A}_\infty$ with $[\omega] = (c_0, \nu)$, we have

$$\omega(D_1) \leqslant c_0 \left[\frac{\mathscr{L}^n(D_1)}{\mathscr{L}^n(D_2)}\right]^\nu \omega(D_2).$$

Applying this inequality with the following fact

$$\mathbb{P}(\gamma, \varepsilon^{-\frac{1}{\Theta}}\lambda) \subset \mathbb{P}(\gamma, \lambda),$$

one obtains from (3.5) that

$$\omega\left(\mathbb{P}(\gamma, \varepsilon^{-\frac{1}{\Theta}}\lambda) \cap \mathbb{P}^c(2-p, \varepsilon^{-1}\lambda) \cap \mathbb{Q}^c(\varepsilon^{\frac{1}{\gamma}-\frac{1}{\Theta}}\lambda)\right) \leqslant C\varepsilon^\nu \omega\left(\mathbb{P}(\gamma, \lambda)\right). \tag{3.6}$$

Let us take $\varepsilon_0 = \varepsilon_0^\nu$ and $\varepsilon = \varepsilon^\nu$ in (3.6). That means for every $\varepsilon \in (0, \varepsilon_0)$, there holds

$$\omega\left(\mathbb{P}(\gamma, \varepsilon^{-\frac{1}{\Theta\nu}}\lambda) \cap \mathbb{P}^c(2-p, \varepsilon^{-\frac{1}{\nu}}\lambda) \cap \mathbb{Q}^c(\varepsilon^{\frac{1}{\gamma\nu}-\frac{1}{\Theta\nu}}\lambda)\right) \leqslant C\varepsilon \omega\left(\mathbb{P}(\gamma, \lambda)\right). \tag{3.7}$$

We now apply the following inequality

$$\omega(D_1 \cup D_2) \leqslant C\left[\omega(D_1) + \omega(D_2)\right],$$

to get that

$$\begin{aligned}
\omega\left(\mathbb{P}(\gamma, \varepsilon^{-\frac{1}{\Theta\nu}}\lambda)\right) &\leqslant C\omega\left(\mathbb{P}(\gamma, \varepsilon^{-\frac{1}{\Theta\nu}}\lambda) \cap \mathbb{P}^c(2-p, \varepsilon^{-\frac{1}{\nu}}\lambda) \cap \mathbb{Q}^c(\varepsilon^{\frac{1}{\gamma\nu}-\frac{1}{\Theta\nu}}\lambda)\right) \\
&\quad + C\omega\left(\mathbb{P}(2-p, \varepsilon^{-\frac{1}{\nu}}\lambda) \cap \mathbb{Q}(\varepsilon^{\frac{1}{\gamma\nu}-\frac{1}{\Theta\nu}}\lambda)\right) \\
&\leqslant C\omega\left(\mathbb{P}(\gamma, \varepsilon^{-\frac{1}{\Theta\nu}}\lambda) \cap \mathbb{P}^c(2-p, \varepsilon^{-\frac{1}{\nu}}\lambda) \cap \mathbb{Q}^c(\varepsilon^{\frac{1}{\gamma\nu}-\frac{1}{\Theta\nu}}\lambda)\right) \\
&\quad + C\omega\left(\mathbb{P}(2-p, \varepsilon^{-\frac{1}{\nu}}\lambda)\right) + C\omega\left(\mathbb{Q}(\varepsilon^{\frac{1}{\gamma\nu}-\frac{1}{\Theta\nu}}\lambda)\right).
\end{aligned} \tag{3.8}$$

We may conclude (3.4) by combining two estimates in (3.7) and (3.8). The proof is complete. □

3.1. In weighted Lorentz spaces

We are in the position of the first global regularity result of solutions in weighted Lorentz spaces. It will be stated in the following theorem.

THEOREM 2. *Let $\omega \in \mathbf{A}_\infty$ with $[\omega] = (c_0, \nu)$. Assume that u is a renormalized solution to equation (1.1) satisfying $|\nabla u| \in L^{2-p}(\Omega)$ under assumption (H). Then for any $q \in (2-p, \Theta)$ and $0 < s \leqslant \infty$, the renormalized solution u of equation (1.1) satisfies*

$$\|\nabla u\|_{L^{q,s}_\omega(\Omega)} \leqslant C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{q,s}_\omega(\Omega)}. \tag{3.9}$$

Proof. Following Theorem 1, there exists $\theta > p$, $\lambda > 0$, $C > 0$, $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 \in (0,1)$ such that

$$\begin{aligned}
&\omega\left(\left\{x \in \Omega:\ \left[\mathbf{M}(|\nabla u|^\gamma)\right]^{\frac{1}{\gamma}} > \varepsilon^{-\frac{1}{\Theta\nu}}\lambda\right\}\right)\\
&\quad\leqslant C\varepsilon\omega\left(\left\{x \in \Omega:\ \left[\mathbf{M}(|\nabla u|^\gamma)(x)\right]^{\frac{1}{\gamma}} > \lambda\right\}\right)\\
&\qquad + C\omega\left(\left\{x \in \Omega:\ \left[\mathbf{M}_1(\mu)(x)\right]^{\frac{1}{p-1}} > \varepsilon^{\frac{1}{\gamma\nu}-\frac{1}{\Theta\nu}}\lambda\right\}\right)\\
&\qquad\qquad + C\omega\left(\left\{x \in \Omega:\ \left[\mathbf{M}(|\nabla u|^{2-p})(x)\right]^{\frac{1}{2-p}} > \varepsilon^{-\frac{1}{\nu}}\lambda\right\}\right).
\end{aligned}$$

We replace $a = \frac{1}{\theta\nu}$, $b = \frac{1}{\gamma\nu} - \frac{1}{\theta\nu}$, then the inequality is rewritten:

$$\begin{aligned}
&\omega\left(\left\{x \in \Omega:\ \left[\mathbf{M}(|\nabla u|^\gamma)\right]^{\frac{1}{\gamma}} > \varepsilon^{-a}\lambda\right\}\right)\\
&\quad\leqslant C\varepsilon\omega\left(\left\{x \in \Omega:\ \left[\mathbf{M}(|\nabla u|^\gamma)(x)\right]^{\frac{1}{\gamma}} > \lambda\right\}\right)\\
&\qquad + C\omega\left(\left\{x \in \Omega:\ \left[\mathbf{M}_1(\mu)(x)\right]^{\frac{1}{p-1}} > \varepsilon^{b}\lambda\right\}\right)\\
&\qquad\qquad + C\omega\left(\left\{x \in \Omega:\ \left[\mathbf{M}(|\nabla u|^{2-p})(x)\right]^{\frac{1}{2-p}} > \varepsilon^{-\frac{1}{\nu}}\lambda\right\}\right).
\end{aligned}$$

Next, we firstly investigate global gradient estimates of solution u:

$$\begin{aligned}
&\|(\mathbf{M}(|\nabla u|^\gamma))^{\frac{1}{\gamma}}\|^s_{L^{q,s}_\omega}\\
&= q\varepsilon^{-as}\int_0^\infty \lambda^s \omega\left(\{(\mathbf{M}(|\nabla u|^\gamma))^{\frac{1}{\gamma}} > \varepsilon^{-a}\lambda\}\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}\\
&\leqslant C\varepsilon^{\frac{s}{q}-as}\int_0^\infty \lambda^s \omega\left(\{x \in \Omega : [\mathbf{M}(|\nabla u|^\gamma)(x)]^{\frac{1}{\gamma}} > \lambda\}\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}\\
&\quad + C\varepsilon^{-as}\int_0^\infty \lambda^s \omega\left(x \in \Omega : [\mathbf{M}_1(\mu)(x)]^{\frac{1}{p-1}} > \varepsilon^b\lambda\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}\\
&\quad + C\varepsilon^{-as}\int_0^\infty \lambda^s \omega\left(\{x \in \Omega : \left[\mathbf{M}(|\nabla u|^{2-p})(x)\right]^{\frac{1}{2-p}} > \varepsilon^{-\frac{1}{\nu}}\lambda\}\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}\\
&\leqslant C\varepsilon^{\frac{s}{q}-as}\|(\mathbf{M}(|\nabla u|^\gamma))^{\frac{1}{\gamma}}\|^s_{L^{q,s}_\omega(\Omega)} + C\varepsilon^{-as-bs}\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|^s_{L^{q,s}_\omega(\Omega)}\\
&\quad + C\varepsilon^{-as+\frac{1}{\nu}}\|(\mathbf{M}(|\nabla u|)^{2-p})^{\frac{1}{2-p}}\|^s_{L^{q,s}_\omega(\Omega)}.
\end{aligned}$$

We have an estiamte $\|(\mathbf{M}(|\nabla u|)^{2-p})^{\frac{1}{2-p}}\|^s_{L^{q,s}_\omega(\Omega)} = \|(\mathbf{M}(|\nabla u|)^{2-p})\|^{\frac{s}{2-p}}_{L^{\frac{q}{2-p},\frac{s}{2-p}}_\omega(\Omega)}$.

It implies to:

$$\begin{aligned}\|(\mathbf{M}(|\nabla u|^{\gamma}))^{\frac{1}{\gamma}}\|^{s}_{L^{q,s}_{\omega}(\Omega)} \leqslant{}& C\varepsilon^{\frac{s}{q}-as}\|(\mathbf{M}(|\nabla u|^{\gamma}))^{\frac{1}{\gamma}}\|^{s}_{L^{q,s}_{\omega}(\Omega)} + C\varepsilon^{-as-bs}\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|^{s}_{L^{q,s}_{\omega}(\Omega)}\\ &+C\varepsilon^{-as+\frac{1}{v}}\|(\mathbf{M}(|\nabla u|)^{2-p})\|^{\frac{s}{2-p}}_{L^{\frac{q}{2-p},\frac{s}{2-p}}_{\omega}(\Omega)}.\end{aligned}$$

Using the Lemma 2.6, it leads to:

$$\begin{aligned}\|(\mathbf{M}(|\nabla u|^{\gamma}))^{\frac{1}{\gamma}}\|^{s}_{L^{q,s}_{\omega}(\Omega)} \leqslant{}& C\varepsilon^{\frac{s}{q}-as}\|(\mathbf{M}(|\nabla u|^{\gamma}))^{\frac{1}{\gamma}}\|^{s}_{L^{q,s}_{\omega}(\Omega)} + C\varepsilon^{-as-bs}\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|^{s}_{L^{q,s}_{\omega}(\Omega)}\\ &+C\varepsilon^{-as+\frac{1}{v}}\|(|\nabla u|^{2-p})\|^{\frac{s}{2-p}}_{L^{\frac{q}{2-p},\frac{s}{2-p}}_{\omega}(\Omega)}.\end{aligned}$$

This gives:

$$\begin{aligned}\|(\mathbf{M}(|\nabla u|^{\gamma}))^{\frac{1}{\gamma}}\|^{s}_{L^{q,s}_{\omega}(\Omega)} \leqslant{}& C\varepsilon^{\frac{s}{q}-as}\|(\mathbf{M}(|\nabla u|^{\gamma}))^{\frac{1}{\gamma}}\|^{s}_{L^{q,s}_{\omega}(\Omega)} + C\varepsilon^{-as-bs}\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|^{s}_{L^{q,s}_{\omega}(\Omega)}\\ &+C\varepsilon^{-as+\frac{1}{v}}\|(|\nabla u|)\|^{s}_{L^{q,s}_{\omega}(\Omega)}.\end{aligned}$$

We choose $\varepsilon \in (0,\varepsilon_0)$ such that: $\begin{cases} C\varepsilon^{\frac{s}{q}-as} \leqslant \frac{1}{2} \\ C\varepsilon^{-as+\frac{1}{v}} \leqslant \frac{1}{4} \end{cases}$, re-examine into our estimate, we conclude that:

$$\|(\mathbf{M}(|\nabla u|^{\gamma}))^{\frac{1}{\gamma}}\|^{s}_{L^{q,s}_{\omega}(\Omega)} \leqslant \frac{1}{2}\|(\mathbf{M}(|\nabla u|^{\gamma}))^{\frac{1}{\gamma}}\|^{s}_{L^{q,s}_{\omega}(\Omega)} + C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|^{s}_{L^{q,s}_{\omega}(\Omega)}$$

and obtain:

$$\|(|\nabla u|)\|_{L^{q,s}_{\omega}(\Omega)} \leqslant C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|^{s}_{L^{q,s}_{\omega}(\Omega)}.$$

By this final estimate, Theorem 2 is complete. □

4. Conclusion

This paper gives global regularity of renormalized solution for the very singular quasilinear elliptic equations in weighted Lorentz spaces with measure data. The results were based on the -capacity thickness complement, renormalized solution, the Carathéodory vector field. This study also was inspired and extended results from previous papers which we read before. The results also extend the range of the study for space of solutions of the equation

$$\begin{cases} -\mathrm{div}(x,\nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The authors would like to thank Minh-Phuong Tran for the helpful inspiration, the interesting discussions, and her constant encouragement in this research. Also, we would like to thank the Editors and Referees for investigating our manuscript, critical remarks, as well as for their helpful suggestions.

The next studies can be viewed as extending the problems as a wider spaces of solutions, the solutions are equal to 0 on the boundary, or changing conditions in domain Ω.

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(Received May 25, 2024)

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