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REGULARITY FOR NON–UNIFORMLY ELLIPTIC DOUBLE OBSTACLE PROBLEMS WITH FRACTIONAL MAXIMAL OPERATORS

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Abstract. This paper aims to establish a global estimate for solutions to non-uniformly elliptic double obstacle problems in Lorentz and Orlicz-Sobolev spaces. In this study, we build upon the technique introduced $\mathbf{M}_\alpha$ by Tran and Nguyen in their paper [28]. This technique relies on the concept of the good $-\lambda$ inequality proposed by Mingione and the definition of the distribution function by Grafakos. We make use of certain familiar assumptions about non-smooth domains. Additionally, we employ function spaces, inequalities, and several lemmas to support our proof.

1. Introduction

Our aim in this paper is to establish a global estimate for solutions to non-uniformly elliptic obstacle problems. We consider the following equation

$$-\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u\Big)=-\operatorname{div}\Big(|\mathbf{F}|^{p-2}\mathbf{F}+|\mathbf{F}|^{q-2}\mathbf{F}\Big)\ \text{ in } \Omega, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is an open bounded domain with $n\geqslant 2$; $\mathbf{F}:\Omega\to\mathbb{R}^n$ is a vector field, and the growth exponents p,q satisfy $1<p\leqslant q<n$. In the remainder of this paper, we shall define function $\mathscr{H}$ as the following

$$\mathscr{H}(\xi):=|\xi|^p+|\xi|^q.$$

Problem (1.1) is closely related to the problem of finding the minimum of the energy function

$$\mathbb{K}\ni u\mapsto\int_\Omega\Big(|\nabla u|^p+|\nabla u|^q\Big)-\Big\langle|\mathbf{F}|^{p-2}\mathbf{F}+|\mathbf{F}|^{q-2}\mathbf{F},\nabla u\Big\rangle dx, \tag{1.2}$$

with

$$\mathbb{K}:=\Big\{u\in W_0^{1,\mathscr{H}}(\Omega):\psi_1\leqslant u\leqslant\psi_2\ \text{ a.e }\Omega\Big\},$$

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where $\psi_1, \psi_2 \in W^{1,\mathscr{H}}(\Omega)$ are two constraint functions satisfying $\psi_1 \leqslant \psi_2$ a.e. Ω and $\psi_1 \leqslant 0 \leqslant \psi_2$ on $\partial\Omega$.

This research paper is based on the two-phase (p,q)-Laplacian problem, which has been studied and achieved many results [10, 16, 29, 32]. In addition, the (p,q)-Laplace equation problem has many applications in life. In physics, the equation is applied to composite materials-synthetic materials formed by mixing two or more different compounds that often have superior properties compared to the original materials. Fiber-reinforced plastic is widely used in our daily lives. Its common applications include making rocket engine casings, aircraft casings, auto components, construction materials, water pipes, etc. [2, 3, 22].

The obstacle problems are interesting and useful in a variety of scientific and engineering domains, including biology, computer science, mechanics, economics, engineering, etc. Referring back to Stefan's seminal work [23], this problem can be framed as an obstruction problem. Stefan researched heat transmission in a homogeneous medium experiencing phase transition, usually ice melting in water or water passing to ice. Numerous applications of this problem can be found in areas including interacting particle systems, mathematical finance, Hele-Shaw flow, fluid filtration (Dam problem), phase transitions (Stefan problem), elasticity, optimal stopping, etc. For further details and applications of obstacle-type problems, we refer the reader to [8,17,21]. There have been several notable works in the derivation of regularity theory for both equations and obstacle problems involving (p,q)-Laplacian. We refer the reader to [10, 16, 29, 32] and the references therein for more contributions to these problems.

In this study, we continue to exploit the idea of using so-called "good-λ" with fractional maximal operators $\mathbf{M}_\alpha$ by Tran and Nguyen in [24–26, 31, 32] to obtain a global regularity estimate for solutions to the double-obstacle problem corresponding to (1.1) in Lorentz spaces. Our proof in this paper not only deals with the problems involving (p,q)-Laplacian but also with a more general class of non-uniformly elliptic problems. In the results of this paper, we not only solve the (p,q)-Laplace problem but also extend the solution to a class of problems with a more general form as follows

$$-\mathrm{div}\mathscr{A}\big(x,\nabla u\big) = -\mathrm{div}\mathscr{B}\big(x,\mathbf{F}\big) \text{ in } \Omega. \tag{1.3}$$

We can summarize the above problem with two constraints ψ_1, ψ_2 as follows
Find a solution u that satisfies

$$-\mathrm{div}\mathscr{A}\big(x,\nabla u\big) \leqslant -\mathrm{div}\mathscr{B}\big(x,\mathbf{F}\big) \text{ a.e. in } \mathbb{K}. \tag{1.4}$$

The nonlinear operators $\mathscr{A}, \mathscr{B} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are Carathéodory vector functions that satisfy the existence condition $0 < \nu \leqslant L$ is such that

$$\begin{cases} |\mathscr{A}(x,\xi)| + |\mathscr{B}(x,\xi)| + |\partial_\xi \mathscr{A}(x,\xi)||\xi| \leqslant L\Big(|\xi|^{p-1} + |\xi|^{q-1}\Big), \\ \nu\Big(|\xi|^{p-2} + |\xi|^{q-2}\Big)|\eta|^2 \leqslant \Big\langle \partial_\xi \mathscr{A}(x,\xi)\eta, \eta \Big\rangle, \end{cases} \tag{1.5}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\eta \in \mathbb{R}^n$. Here, ∂_ξ represents the partial derivative with respect to the variable ξ and $\langle \cdot, \cdot \rangle$ is the usual inner product. It is easy to see that the divergent operator on the left side of equation (1.1) is a specific case of the nonlinear operator $\mathscr{A}$ when $\mathscr{A}(\xi) = |\xi|^{p-2}\xi + |\xi|^{q-2}\xi$. Furthermore, from the condition $(1.5)_2$, for $1 < p < q$, we get

$$\left\langle \mathscr{A}(x,\xi) - \mathscr{A}(x,\eta), \xi - \eta \right\rangle \geqslant \tilde{\nu}\left(\left(|\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} + \left(|\xi|^2 + |\eta|^2\right)^{\frac{q-2}{2}} \right)|\xi - \eta|^2,$$

where $\tilde{\nu}$ is a positive constant depends on $n, p, q,$ and ν. And for the case $2 \leqslant p < q$, it is reduced as follows:

$$\left\langle \mathscr{A}(x,\xi) - \mathscr{A}(x,\eta), \xi - \eta \right\rangle \geqslant \tilde{\nu}\left(|\xi - \eta|^p + |\xi - \eta|^q\right).$$

The operator $\mathscr{B} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is the Carathéodory vector function defined

$$|\mathscr{B}(x,\xi)| \leqslant L\left(|\xi|^{p-1} + |\xi|^{q-1}\right), \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n. \tag{1.6}$$

Let us now turn to describing the two-obstacle functions ψ_1 and $\psi_2 \in W^{1,\mathscr{H}}(\Omega)$ such that $\psi_1 \leqslant \psi_2$ and $\psi_1 \leqslant 0 \leqslant \psi_2$ on $\partial\Omega$ (we shall discuss the details of Orlicz spaces $L^{\mathscr{H}}(\Omega)$ and Orlicz-Sobolev spaces $W^{1,\mathscr{H}}(\Omega)$ in Section 2.1 below).

We say that $u \in \mathbb{K}$ is a weak solution to the double-obstacle problem (1.4) if the following variational inequality

$$\int_\Omega \mathscr{A}\left(x, \nabla u\right) \cdot \left(\nabla v - \nabla u\right) dx \geqslant \int_\Omega \mathscr{B}\left(x, \mathbf{F}\right) \cdot \left(\nabla v - \nabla u\right) dx, \tag{1.7}$$

holds for all $v \in \mathbb{K}$, where $\mathbf{F} : \Omega \to \mathbb{R}^n$ is a vector-valued measurable function such that $\mathscr{H}(\mathbf{F}) \in L^1(\Omega)$.

For shortness of notation, we denote the word **input** to represent the given data of the problem that the generic constant C may depend on. More precisely, we shall briefly use that word for the set of

$$\mathbf{input} = \left\{ n, p, q, \alpha, \nu, L, \|\mathscr{H}(\mathbf{F})\|_{L^1(\Omega)}, \|\mathscr{H}(\nabla\psi_1)\|_{L^1(\Omega)}, \|\mathscr{H}(\nabla\psi_2)\|_{L^1(\Omega)} \right\}.$$

There have been a lot of regularity results concerning the (p,q)-Laplace equations or more general problems. We address the interested reader to [6, 11–13, 27, 32] for gradient estimates of solutions to non-uniformly quasilinear elliptic equations governed by two or multi-phase. In this work, we simply study the regularity of double-phase problems under a two-obstacle constraint. Specifically, our results are preserved in terms of fractional maximal operators $\mathbf{M}_\alpha$. As far as we know, regularity results via fractional maximal operators were first mentioned in [9, 18, 19], and later established by Tran and Nguyen in [26–28, 30], etc. Due to the meaning of regularity theory via the use of $\mathbf{M}_\alpha$ discussed in their studies, our main results also follow the same lines as them.

This is how the current paper is organized. The formulation of non-uniformly elliptic problems with two obstacles is given in the current Section 1 and summary.

The introduction to this problem, such as a historical review of the development of the relative problem, presents the primary findings, some annotations, and hypotheses on the obstacle problem. In Section 2, we provide a list of notation and some function spaces, especially necessary background information on fractional maximal distribution for use in main proofs. Section 3 is dedicated to some technical lemmas to prove our main theorems. The statements of Theorem 1 and Theorem 2 are devoted in Section 4, and these are two important theorems and also possible future research topics of this paper. In the last section, we discussed proofs of main theorems.

2. Preliminaries

In this section, we are going to collect some notation, properties and initial definitions for later use, and some necessary background on function spaces that will be mentioned in the rest of this paper.

2.1. Notation

Firstly, domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is assumed to be an open-bounded domain. Throughout the paper, we use the symbol C to denote a generic positive constant. The value of C will not necessarily be the same at each occurrence, depending only on the dimension and some constants appearing in the statements. Moreover, the dependence of C on some given parameters will be emphasized between parentheses. Next, we let $\mathscr{L}^n(E)$ stand for the Lebesgue measure of a measurable set E in $\mathbb{R}^n$. In addition, we will write $\mathrm{diam}(\Omega)$ for the diameter of Ω, that is defined as

$$\mathrm{diam}(\Omega) = \sup_{\xi_1, \xi_2 \in \Omega} |\xi_1 - \xi_2|.$$

In what follows, we shall denote the integral average of a function $h \in L^1_{loc}(\mathbb{R}^n)$ over the measurable subset E of $\mathbb{R}^n$ as

$$\fint_E h(x)dx = \frac{1}{\mathscr{L}^n(E)} \int_E h(x)dx.$$

Finally, the open n-dimensional Euclidean ball in $\mathbb{R}^n$ of radius $\rho > 0$ and center ξ will be denoted by $B_\rho(\xi)$, that is the set $\{z \in \mathbb{R}^n : |z - \xi| < \rho\}$. We also denote here $\Omega_\rho(\xi) = B_\rho(\xi) \cap \Omega$, which is considered as the “surface ball” when the center ξ lies on $\partial\Omega$.

2.2. Some function spaces

Although the emphasis of the paper is not on the existence of solutions but rather on their regularity, we must determine the spaces where the solutions exist. Orlicz-Sobolev (O-S) spaces that we shall denote $W^{1,\mathscr{H}}(\Omega)$ from now on. For more details, the interesting reader may see Harjulehto and Hästö [15].

DEFINITION 1. (Orlicz spaces) The Orlicz space, denoted $L^{\mathscr{H}}(\Omega)$, is the set of all measurable functions $f:\Omega\to\mathbb{R}$ such that

$$\int_{\Omega}\mathscr{H}(|f|)dx<+\infty, \tag{2.1}$$

where, $L^{\mathscr{H}}(\Omega)$ is the Banach space corresponding to the following norm

$$\|f\|_{L^{\mathscr{H}}(\Omega)}=\inf\left\{\lambda>0:\int_{\Omega}\mathscr{H}\left(\frac{|f|}{\lambda}\right)dx\leqslant 1\right\}. \tag{2.2}$$

DEFINITION 2. (Orlicz-Sobolev spaces) Orlicz-Sobolev spaces, often denoted by $W^{1,\mathscr{H}}(\Omega)$, is the set of all measurable functions $f\in L^{\mathscr{H}}(\Omega)$ such that its distributional gradient vector ∇f belongs to $L^{\mathscr{H}}(\Omega,\mathbb{R}^n)$. On $W^{1,\mathscr{H}}(\Omega)$, the corresponding norm is defined by

$$\|f\|_{W^{1,\mathscr{H}}(\Omega)}=\|f\|_{L^{\mathscr{H}}(\Omega)}+\|\nabla f\|_{L^{\mathscr{H}}(\Omega,\mathbb{R}^n)}. \tag{2.3}$$

The space $W_0^{1,\mathscr{H}}(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\mathscr{H}}(\Omega)$.

We will recall the definition of Lorentz spaces. The definition of Lorentz spaces has been mentioned in [12, 14]. Lorentz spaces play a important role in our study.

DEFINITION 3. (Lorentz spaces) The Lorentz space, often denoted by $L^{s,t}(\Omega)$ with $0<s<\infty$ and $0<t\leqslant\infty$, is defined as the set of all measurable function f on Ω such that the following quantity for $0<t<\infty$

$$\|f\|_{L^{s,t}(\Omega)}:=\left[\int_0^{\infty}\lambda^t\mathscr{L}^n(\{y\in\Omega:|f(y)|>\lambda\})^{\frac{t}{s}}\frac{d\lambda}{\lambda}\right]^{\frac{1}{t}}<\infty,$$

and when $t=\infty$ it satisfies

$$\|f\|_{L^{s,t}(\Omega)}:=\sup_{\lambda>0}\lambda\mathscr{L}^n\left(\{y\in\Omega:|f(y)|>\lambda\}\right)^{\frac{1}{s}}<\infty.$$

It is worthy to remark that when $t=\infty$, the Lorentz spaces are then known as the Marcinkiewicz spaces. Moreover, it is worth remarking that when $s=t$, the Lorentz spaces $L^{s,s}(\Omega)$ are nothing but Lebesgue spaces $L^s(\Omega)$. In particular, it is well known that for some $0<r\leqslant s\leqslant t\leqslant\infty$, it holds that

$$L^t(\Omega)\subset L^{s,r}(\Omega)\subset L^s(\Omega)\subset L^{s,t}(\Omega)\subset L^r(\Omega).$$

Now we turn to the definition of the fractional maximum function in the spirit of [12]. This is a very important tool to prove our result in the next section.

DEFINITION 4. (Fractional maximal functions and their cut-off versions) Let $0 \leqslant \alpha \leqslant n$. The fractional maximal function $\mathbf{M}_\alpha$ of a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\mathbf{M}_\alpha f(x) = \sup_{\rho>0} \rho^\alpha \fint_{B_\rho(x)} |f(y)| dy, \quad x \in \mathbb{R}^n, \tag{2.4}$$

when $\alpha = 0$, it coincides with the Hardy-Littlewood maximal function, $\mathbf{M}_0 f = \mathbf{M} f$, defined by

$$\mathbf{M} f(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |f(y)| dy, \quad x \in \mathbb{R}^n,$$

for any given locally integrable function f in $\mathbb{R}^n$. On the other hand, in order to prove the main results in this paper, It is necessary to define the cut-off versions of these maximal operators. Let $\gamma > 0$ and α should be belong to $[0,n]$. We define two additional cut-off versions of $\mathbf{M}_\alpha f$ in (2.4) as follows:

$$\mathbf{M}_\alpha^\gamma f(x) = \sup_{0<\rho<\gamma} \rho^\alpha \fint_{B_\rho(x)} f(y) dy,$$

and

$$\mathbf{T}_\alpha^\gamma f(x) = \sup_{\rho \geqslant \gamma} \rho^\alpha \fint_{B_\rho(x)} f(y) dy.$$

We remark that if $\alpha = 0$ then $\mathbf{M}_\alpha^\gamma f = \mathbf{M}^r f$ and $\mathbf{T}_\alpha^\gamma f = \mathbf{T}^r f$ for all $f \in L^1_{loc}(\mathbb{R}^n)$.

Based on definition 3, we will state the significant lemma for our proof in this paper.

LEMMA 1. (The boundedness property of $\mathbf{M}_\alpha$) *Let $f \in L^s(\mathbb{R}^n)$ with $s \geqslant 1$ and $\alpha \in [0, n/s)$. There exists $C = C(n, \alpha, s) > 0$ such that for any $\lambda > 0$ there holds*

$$\mathscr{L}^n\left(\{x \in \mathbb{R}^n : \mathbf{M}_\alpha f(x) > \lambda\}\right) \leqslant C\left(\frac{1}{\lambda^s} \int_{\mathbb{R}^n} |f(y)|^s dy\right)^{\frac{n}{n-\alpha s}}.$$

This Lemma has been proved in the paper [27, Lemma 2.8] by Tran, Nguyen.

3. Preparatory lemmas

Next, let us state some preliminary technical lemmas that will be applied to our main proofs in Section 5.

LEMMA 2. (Covering Lemma) *Let Ω be a bounded domain of $\mathbb{R}^n$ such that $\partial\Omega \in C^{1,\sigma^+}$. Let $0 < \varepsilon < 1$, $r > 0$.*

(i) *Suppose that two measurable subsets of* $V \subset W$ *of* Ω *satisfy*

$$\mathscr{L}^n(V) \leqslant \varepsilon \mathscr{L}^n(B_r).$$

(ii) *Assume moreover that for every* $x \in \Omega$*, if* $\mathscr{L}^n(V \cap B_r(x)) > \varepsilon \mathscr{L}^n(B_r(x))$ *then* $\Omega \cap B_r(x) \subset W$*. Then there exists a positive constant* $C = C(n) > 0$ *such that*

$$\mathscr{L}^n(V) \leqslant C\varepsilon \mathscr{L}^n(W).$$

We will only state Lemma 2 for convenience. You can see the paper [27] for further details description of this Covering Lemma.

Continuously, we will show the Lemma 3 which has been stated and proved in the paper of Tran, Nguyen, and Pham [32, Lemma 2.9].

LEMMA 3. *Let* $u \in \mathbb{K}$ *be a weak solution of* (1.4) *with* $\mathbf{F} \in L^{\mathscr{H}}(\Omega)$ *and two constraint functions* $\psi_1, \psi_2 \in W^{1,\mathscr{H}}(\Omega)$ *satisfies* $\psi_1 \leqslant \psi_2$ *almost everywhere in* Ω *and* $\psi_1 \leqslant 0 \leqslant \psi_2$ *almost everywhere on* $\partial\Omega$*. Then, for each* $\xi \in \overline{\Omega}$ *and* $r > 0$*, we can find* $v \in W^{1,\mathscr{H}}(\Omega_{2r}(\xi))$ *such that*

$$\begin{aligned} \fint_{\Omega_{2r}(\xi)} \mathscr{H}(\nabla u - \nabla v)dx \leqslant \delta \fint_{\Omega_{2r}(\xi)} \mathscr{H}(\nabla u)dx \\ + C\delta^{-K} \fint_{\Omega_{2r}(\xi)} \mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla \psi_1) + \mathscr{H}(\nabla \psi_2)dx, \end{aligned} \tag{3.1}$$

for each $\delta \in (0,1)$*, for some* $K > 0$*,* $C = C(\mathbf{input}) > 0$*,* $\Omega_{2r}(\xi) = B_{2r}(\xi) \cap \Omega$*. More than that,*

$$\int_{\Omega} \mathscr{H}(\nabla u)dx \leqslant C \int_{\Omega} \mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla \psi_1) + \mathscr{H}(\nabla \psi_2)dx. \tag{3.2}$$

On the other hand, for every $\gamma > 1$ *there exists a constant* $\varepsilon_0 = \varepsilon_0(\mathbf{input}) > 0$ *such that*

$$\left(\fint_{\Omega_r(\xi)} \left[\mathscr{H}(\nabla v)\right]^{\gamma} dx \right)^{\frac{1}{\gamma}} \leqslant C \fint_{\Omega_{2r}(\xi)} \mathscr{H}(\nabla v)dx, \tag{3.3}$$

where $C = C(\mathbf{input}, \gamma) > 0$.

4. Statement of main results

In this section, we will infer main result via two theorems. We start stating and proving Theorem 1 (good-λ inequality), is proved by Lemma 3 and some tool in section 2. Based for Theorem 1, we use quasi-norm $\|\cdot\|_{L^{s,t}(\Omega)}$ in Lorentz's definition 3 to implies Theorem 2. And our expectations for regularity result, will be given a good picture of regularity in Lorentz spaces.

THEOREM 1. *Let $u \in \mathbb{K}$ be a weak solution to problem* (1.7) *and the boundary $\partial\Omega$ is of class C^{1,σ^+} for some $\alpha^+ \in [\alpha,1]$, $\sigma \in (0,1]$. Then, for every $a \in (0,1)$, there exists $\varepsilon_0 = \varepsilon_0(\mathbf{input}) > 0$ such that the following inequality*

$$\begin{aligned} &\mathscr{L}^n\left(\left\{\mathbf{M}_\alpha[\mathscr{H}(\nabla u)] > \varepsilon^{-a}\lambda\right\}\right) \\ &\leqslant C\varepsilon\mathscr{L}^n\left(\left\{\mathbf{M}_\alpha[\mathscr{H}(\nabla u)] \geqslant \lambda\right\}\right) \\ &\quad + \mathscr{L}^n\left(\left\{\mathbf{M}_\alpha[\mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla\psi_1) + \mathscr{H}(\nabla\psi_2)] \leqslant \varepsilon^b\lambda\right\}\right), \end{aligned} \tag{4.1}$$

holds for all $\varepsilon \in (0,\varepsilon_0)$ and $\lambda > 0$. Here the constants $b = b(\varepsilon, a, \mathbf{input})$ and $C = C(a,b,\mathbf{input})$.

THEOREM 2. *Under the assumptions of Theorem* 1*, let $\mathscr{N}, \mathscr{T} : \Omega \to \mathbb{R}^+$ be two maps defined by*

$$\mathscr{N} := \mathscr{H}(\nabla u) \ \text{ and } \ \mathscr{T} := \mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla\psi_1) + \mathscr{H}(\nabla\psi_2). \tag{4.2}$$

Then, for every $s \in (0,\infty)$ and $0 < t \leqslant \infty$, there exists a constant $\varepsilon_0 = \varepsilon_0(s,t,\mathbf{input}) > 0$ such that the following estimate

$$\|\mathbf{M}_\alpha\mathscr{N}\|_{L^{s,t}(\Omega)} \leqslant C\|\mathbf{M}_\alpha\mathscr{T}\|_{L^{s,t}(\Omega)}. \tag{4.3}$$

Here, the constant $C = C(s,t,\mathbf{input})$.

5. Proof of main theorems

5.1. Proof of Theorem 1

Proof. For each $\lambda > 0$ and $\varepsilon > 0$ small enough, let us set

$$V_\varepsilon = \left\{\mathbf{M}_\alpha[\mathscr{H}(\nabla u)] > \varepsilon^{-a}\lambda\right\} \cap \left\{\mathbf{M}_\alpha[\mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla\psi_1) + \mathscr{H}(\nabla\psi_2)] \leqslant \varepsilon^b\lambda\right\},$$

and

$$W = \left\{\mathbf{M}_\alpha[\mathscr{H}(\nabla u)] > \lambda\right\},$$

where a, b are two constants that will be chosen later. Moreover, we shall write

$$\mathscr{N} := \mathscr{H}(\nabla u) \ \text{ and } \ \mathscr{T} := \mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla\psi_1) + \mathscr{H}(\nabla\psi_2).$$

It is clear to see that

$$V_\varepsilon = \left\{\mathbf{M}_\alpha\mathscr{N} > \varepsilon^{-a}\lambda\right\} \cap \left\{\mathbf{M}_\alpha\mathscr{T} \leqslant \varepsilon^b\lambda\right\} \ \text{ and } \ W = \left\{\mathbf{M}_\alpha\mathscr{N} > \lambda\right\}.$$

Firstly, by Covering Lemma 2, we need to check that V_ε satisfies (i). Indeed, if $V_\varepsilon = \emptyset$, then it is obviously true in case $\mathscr{L}^n(V) \leqslant \varepsilon\mathscr{L}^n(B_r(x))$, for all $\varepsilon > 0$ is small enough and $\lambda > 0$. Without loss of generality, assume that $V_\varepsilon \neq \emptyset$, then there exists $x_1 \in \Omega$ such that $\mathbf{M}_\alpha \mathscr{T}(x_1) \leqslant \varepsilon^b \lambda$. It gives

$$\rho^\alpha \fint_{B_\rho(x_1)} \mathscr{T}(x) \leqslant \varepsilon^b \lambda, \quad \forall \rho > 0. \tag{5.1}$$

Based on the boundedness of $\mathbf{M}_\alpha$ in Lemma 1, we get

$$\mathscr{L}^n(V_\varepsilon) \leqslant \mathscr{L}^n\big(\{\mathbf{M}_\alpha \mathscr{N}(x) > \varepsilon^{-a}\lambda\}\big) \leqslant \left(\frac{C}{\varepsilon^{-a}\lambda}\int_\Omega \mathscr{N}(x)dx\right)^{\frac{n}{n-\alpha}}. \tag{5.2}$$

Using estimate (3.2) in to (5.2), we obtain that

$$\mathscr{L}^n(V_\varepsilon) \leqslant \left(\frac{C}{\varepsilon^{-a}\lambda}\int_\Omega \mathscr{T}(x)dx\right)^{\frac{n}{n-\alpha}}. \tag{5.3}$$

Furthermore, taking $D = \operatorname{diam}(\Omega)$, choose $\mathbf{B} = B_D(x_1)$ such that $\Omega \subset \mathbf{B}$. We deduce

$$\mathscr{L}^n(V_\varepsilon) \leqslant \left(\frac{C}{\varepsilon^{-a}\lambda}\int_\Omega \mathscr{T}(x)dx\right)^{\frac{n}{n-\alpha}} \leqslant \left(\frac{C}{\varepsilon^{-a}\lambda}\int_{\mathbf{B}} \mathscr{T}(x)dx\right)^{\frac{n}{n-\alpha}}. \tag{5.4}$$

There exists $x_1 \in \Omega$ such that substituting $\mathbf{M}_\alpha \mathscr{T}(x_1) \leqslant \varepsilon^b \lambda$ into the evaluation (5.1), that allows us to write

$$\int_{\mathbf{B}} \mathscr{T}(x)dx \leqslant C\big[\mathscr{L}^n(\Omega)\big]^{1-\frac{\alpha}{n}} \mathbf{M}_\alpha \mathscr{T}(x_1). \tag{5.5}$$

Combining (5.4) and (5.5), we have

$$\mathscr{L}^n(V_\varepsilon) \leqslant C\varepsilon^{(a+b)\frac{n}{n-\alpha}} \mathscr{L}^n(\Omega).$$

We choose the parameter b so that $b > 1$. There exists $\varepsilon > 0$ small enough to satisfy

$$\mathscr{L}^n(V_\varepsilon) \leqslant C\varepsilon^{(a+b)\frac{n}{n-\alpha}} \left(\frac{D}{r}\right)^n \mathscr{L}^n(B_r(x_0)) < \varepsilon \mathscr{L}^n(B_r),$$

for all $\varepsilon \in (0, \varepsilon_0)$, and we complete the proof of (i).

Next, we present the steps of proving (ii) by contradiction: for every $x \in \Omega$, if $\mathscr{L}^n(V_\varepsilon \cap B_r(x)) \geqslant \varepsilon \mathscr{L}^n(B_r(x))$ then $B_r(x) \cap \Omega \subset W$. Let us assume that $\mathscr{L}^n(V_\varepsilon \cap B_r(x)) \neq \emptyset$. Then, there exist x_2 and x_3 such that $x_2 \in \Omega \cap B_r(x) \cap W^c$ and $x_3 \in V_\varepsilon \cap B_r(x)$. Therefore,

$$\mathbf{M}_\alpha \mathscr{N}(x_2) \leqslant \lambda \text{ and } \mathbf{M}_\alpha \mathscr{T}(x_3) \leqslant \varepsilon^b \lambda. \tag{5.6}$$

In the first step, we will show that

$$\mathscr{L}^n\big(V_\varepsilon \cap B_r(x)\big) \leqslant \mathscr{L}^n\big(\{\mathbf{M}_\alpha \mathscr{N}(x) > \varepsilon^{-a}\lambda\} \cap B_r(x)\big).$$

Indeed, from Definition 4, it allows us to write

$$\mathbf{M}_\alpha^r \mathscr{N}(x) = \sup_{r>\rho>0} \rho^\alpha \fint_{B_\rho(x)} \mathscr{N}(x)dx,$$

and

$$\mathbf{T}_\alpha^r \mathscr{N}(x) = \sup_{\rho\geqslant r} \rho^\alpha \fint_{B_\rho(x)} \mathscr{N}(x)dx.$$

Then, we can conclude

$$\mathbf{M}_\alpha \mathscr{N}(x) = \max\left\{\mathbf{M}_\alpha^r \mathscr{N}(x); \mathbf{T}_\alpha^r \mathscr{N}(x)\right\}.$$

On the other hand, for all $y \in \Omega$, we have

$$\begin{aligned} \mathscr{L}^n(V_\varepsilon \cap B_r(x)) \leqslant \mathscr{L}^n\left(\left\{y \in B_r(x) : \mathbf{M}_\alpha^r \mathscr{N}(y) > \varepsilon^{-a}\lambda\right\}\right) \\ + \mathscr{L}^n\left(\left\{y \in B_r(x) : \mathbf{T}_\alpha^r \mathscr{N}(y) > \varepsilon^{-a}\lambda\right\}\right). \end{aligned} \tag{5.7}$$

At this moment, for every $y \in B_r(x)$, as $\rho \geqslant r$ we get $z \in B_\rho(y)$, it is easy to check that $z \in B_{3\rho}(x_2)$ and $B_\rho(y) \subset B_{3\rho}(x_2)$. From this, we shall write

$$\begin{aligned} \mathbf{T}_\alpha^r \mathscr{N}(y) &= \sup_{\rho\geqslant r} \rho^\alpha \fint_{B_\rho(x)} \mathscr{N}(z)dz \\ &= \sup_{\rho\geqslant r} \rho^\alpha \frac{\mathscr{L}^n\left(B_{3\rho}(x_2)\right)}{\mathscr{L}^n\left(B_\rho(y)\right)} \frac{1}{\mathscr{L}^n\left(B_{3\rho}(x_2)\right)} \int_{B_\rho(y)} \mathscr{N}(z)dz \\ &\leqslant \sup_{\rho\geqslant r} \rho^\alpha \frac{\mathscr{L}^n\left(B_{3\rho}(x_2)\right)}{\mathscr{L}^n\left(B_\rho(y)\right)} \fint_{B_{3\rho}(y)} \mathscr{N}(z)dz \\ &\leqslant 3^{-\alpha} \frac{(3\rho)^n}{\rho^n} \mathbf{M}_\alpha \mathscr{N}(x_2) = 3^{n-\alpha} \mathbf{M}_\alpha \mathscr{N}(x_2). \end{aligned}$$

So, it concludes that $\mathbf{T}_\alpha^r \mathscr{N}(y) \leqslant 3^{n-\alpha} \mathbf{M}_\alpha \mathscr{N}(x_2)$, for all $y \in B_r(x)$, meanwhile $\mathbf{M}_\alpha \mathscr{N}(x_2) \leqslant \lambda$, so we have

$$\mathbf{T}_\alpha^r \mathscr{N}(y) \leqslant 3^{n-\alpha}\lambda, \quad \forall y \in B_r(x).$$

Then, for all $\varepsilon > 0$ such that $\varepsilon^{-a} > 3^{n-\alpha}$ or $\varepsilon < 3^{-(n-\alpha)/a}$, we can see that

$$\left\{y \in B_r(x) : \mathbf{T}_\alpha^r \mathscr{N}(y) > \varepsilon^{-a}\lambda\right\} = \emptyset.$$

For $y \in B_r(x)$ and $z \in B_r(y)$, it is clear to observe that $z \in B_{2r}(x)$. So $B_r(y) \subset B_{2r}(x)$. We may obtain

$$\begin{aligned} \mathbf{M}_\alpha^r \mathscr{N}(y) &= \sup_{0<\rho<r} \rho^\alpha \fint_{B_\rho(y)} \mathscr{N}(z)dz \\ &= \sup_{0<\rho<r} \rho^\alpha \fint_{B_\rho(y)} \chi_{B_{2r}(x)} \mathscr{N}(z)dz \\ &= \mathbf{M}_\alpha^r\left[\chi_{B_{2r}(x)} \mathscr{N}(z)\right](y). \end{aligned}$$

All in all, we get that

$$\mathscr{L}^n(V_\varepsilon \cap B_r(x)) \leqslant \mathscr{L}^n\left(\left\{y \in B_r(x) : \mathbf{M}_\alpha^r\left[\chi_{B_{2r}(x)}\mathscr{N}(y)\right] > \varepsilon^{-a}\lambda\right\}\right), \tag{5.8}$$

with every $\varepsilon \in (0,1)$.

In the next step, we use comparison inequality and reverse Hölder in Lemma 3 to prove

$$\mathscr{L}^n\left(V_\varepsilon \cap B_r(x)\right) \leqslant \varepsilon \mathscr{L}^n\left(B_r(x)\right). \tag{5.9}$$

We now divide into two cases when x belongs to the interior domain, i.e., $B_{4r}(x) \Subset \Omega$, and otherwise when x is closed to the boundary $\partial\Omega$, i.e., $B_{4r}(x) \cap \partial\Omega \neq \emptyset$. In the first case, $B_{4r}(x) \Subset \Omega$, let us consider v the unique solution of the following reference problem:

$$\begin{cases} -\mathrm{div}\mathscr{A}(x,\nabla v) = 0 \text{ in } B_{4r}(x), \\ \qquad\qquad v = u \text{ on } \partial B_{4r}(x). \end{cases} \tag{5.10}$$

Applying Lemma 3, for every $\gamma > 1$, there exists a positive constant C such that the following inequality holds:

$$\left(\fint_{B_{2r}(x)} \left[\mathscr{H}(\nabla v)\right]^\gamma dx\right)^{\frac{1}{\gamma}} \leqslant C\left(\fint_{B_{4r}(x)} \mathscr{H}(\nabla v)dx\right). \tag{5.11}$$

On the other hand, a comparison estimate between ∇u and ∇v over $B_{4r}(x)$ can find by (3.1) in Lemma 3 for every $\varepsilon_1 \in (0,1)$, there exists $K > 0$ such that:

$$\begin{aligned} &\fint_{B_{4r}(x)} \mathscr{H}(\nabla u - \nabla v)dx \\ \leqslant\ &\varepsilon_1^{1-a} \fint_{B_{4r}(x)} \mathscr{H}(\nabla u)dx \\ &+ C\varepsilon_1^{-(1-a)\max\left\{0,\frac{2-p}{p-1}\right\}} \fint_{B_{4r}(x)} \mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla\psi_1) + \mathscr{H}(\nabla\psi_2)dx \\ \leqslant\ &\varepsilon_1^{1-a} \fint_{B_{4r}(x)} \mathscr{N}(z)dz + C\varepsilon_1^{1-K} \fint_{B_{4r}(x)} \mathscr{T}(z)dz, \end{aligned} \tag{5.12}$$

where $K = 1 + \max\left\{0, (1-a)(2-p)/(p-1)\right\}$. Moreover, as $x_2 \in B_r(x)$ so $B_{4r}(x) \subset B_{5r}(x_2)$, from $z \in B_{4r}(x)$ we have that $d(z-x_2) \leqslant d(z-x) + d(x-x_2) < 5r$. Thus,

$$\begin{aligned} \fint_{B_{4r}(x)} \mathscr{N}(z)dz &= \frac{\mathscr{L}^n\left(B_{5r}(x_2)\right)}{\mathscr{L}^n\left(B_{4r}(x)\right)} \frac{1}{\mathscr{L}^n\left(B_{5r}(x_2)\right)} \int_{B_{4r}(x)} \mathscr{N}(z)dz \\ &\leqslant \left(\frac{5}{4}\right)^n \fint_{B_{5r}(x_2)} \mathscr{N}(z)dz \\ &\leqslant \left(\frac{5}{4}\right)^n (5r)^{-\alpha}\mathbf{M}_\alpha\mathscr{N}(x_2) \leqslant Cr^{-\alpha}\lambda. \end{aligned}$$

Similarly, $x_3 \in B_r(x)$ so $B_{4r}(x) \subset B_{5r}(x_2)$ and we get

$$\begin{aligned}\fint_{B_{4r}(x)} \mathscr{T}(z)dz &= \frac{\mathscr{L}^n\big(B_{5r}(x_3)\big)}{\mathscr{L}^n\big(B_{4r}(x)\big)} \frac{1}{\mathscr{L}^n\big(B_{5r}(x_3)\big)} \int_{B_{4r}(x)} \mathscr{T}(z)dz \\ &\leqslant \left(\frac{5}{4}\right)^n \fint_{B_{5r}(x_3)} \mathscr{T}(z)dz \\ &\leqslant \left(\frac{5}{4}\right)^n (5r)^{-\alpha}\mathbf{M}_\alpha \mathscr{N}(x_3) \leqslant Cr^{-\alpha}\varepsilon^b\lambda.\end{aligned}$$

All in all, we shall get

$$\begin{aligned}\fint_{B_{4r}(x)} \mathscr{H}(\nabla u - \nabla v)dx &\leqslant \varepsilon_1^{1-a} \fint_{B_{4r}(x)} \mathscr{N}(z)dz + C\varepsilon_1^{1-K} \fint_{B_{4r}(x)} \mathscr{T}(z)dz \\ &\leqslant C\Big(\varepsilon_1^{1-a} + \varepsilon_1^{1-K}\varepsilon^b\Big) r^{-\alpha}\lambda.\end{aligned}$$

At this stage, we choose $\varepsilon_1^{1-a} \in (0,1)$ such that $\varepsilon_1^{1-a} = \varepsilon_1^{1-K}\varepsilon^b$ then $\varepsilon_1 = \varepsilon^{b/(K-a)}$ to get

$$\fint_{B_{4r}(x)} \mathscr{H}\big(\nabla u - \nabla v\big)dx \leqslant C\varepsilon^{\frac{b}{K-a}} r^{-\alpha}\lambda.$$

In the same manner as the above argument, it yields

$$\fint_{B_{4r}(x)} \mathscr{H}(\nabla u)dx \leqslant Cr^{-\alpha}\lambda.$$

Thanks to (5.11) one has

$$\left(\fint_{B_{2r}(x)} \big[\mathscr{H}(\nabla v)\big]^\gamma dx\right)^{\frac{1}{\gamma}} \leqslant C\left(\fint_{B_{4r}(x)} \mathscr{H}(\nabla v)dx\right). \tag{5.13}$$

On the other hand, it is easy to obtain

$$|\nabla v|^p = |(\nabla u - \nabla v) - \nabla u|^p \leqslant 2^{p-1}(|\nabla u - \nabla v|^p + |\nabla u|^p),$$

and

$$|\nabla v|^q \leqslant 2^{q-1}(|\nabla u - \nabla v|^q + |\nabla u|^q).$$

Therefore,

$$\begin{aligned}\fint_{B_{4r}(x)} \mathscr{H}(\nabla v)dx &\leqslant C\left(\fint_{B_{4r}(x)} \mathscr{H}(\nabla u - \nabla v)dx + \fint_{B_{4r}(x)} \mathscr{H}(\nabla u)dx\right) \\ &\leqslant C\Big(C\varepsilon^{\frac{b}{K-a}} r^{-\alpha}\lambda + r^{-\alpha}\lambda\Big) \\ &< C\Big(1 + \varepsilon^{\frac{b}{K-a}}\Big) r^{-\alpha}\lambda \\ &< C_1 r^{-\alpha}\lambda.\end{aligned} \tag{5.14}$$

Combining (5.14) with (5.13), it gets

$$\left(\fint_{B_{2r}(x)}\left[\mathscr{H}(\nabla v)\right]^{\gamma}dx\right)^{\frac{1}{\gamma}} \leqslant Cr^{-\alpha}\lambda.$$

At this point, we also write

$$|\nabla u|^p = |(\nabla v-\nabla u)-\nabla v|^p \leqslant 2^{p-1}(|\nabla v-\nabla u|^p+|\nabla v|^p),$$

and

$$|\nabla u|^q = |(\nabla v-\nabla u)-\nabla v|^q \leqslant 2^{q-1}(|\nabla v-\nabla u|^q+|\nabla v|^q).$$

Hence,

$$\mathbf{M}^r_\alpha\left[\chi_{B_{2r}(x)}\mathscr{H}(\nabla u)\right] \leqslant C\mathbf{M}^r_\alpha\left[\chi_{B_{2r}(x)}\mathscr{H}(\nabla u-\nabla v)\right]+C\mathbf{M}^r_\alpha\left[\chi_{B_{2r}(x)}\mathscr{H}(\nabla v)\right],$$

and the estimate (5.8) can be rewritten as

$$\begin{aligned}\mathscr{L}^n(V_\varepsilon\cap B_r(x)) &\leqslant C\mathscr{L}^n\Big(\left\{\mathbf{M}^r_\alpha\left[\chi_{B_{2r}(x)}\mathscr{H}(\nabla u-\nabla v)\right]>\varepsilon^{-a}\lambda\right\}\Big)\\ &\quad +C\mathscr{L}^n\Big(\left\{\mathbf{M}^r_\alpha\left[\chi_{B_{2r}(x)}\mathscr{H}(\nabla v)\right]>\varepsilon^{-a}\lambda\right\}\Big).\end{aligned} \tag{5.15}$$

Applying the boundedness property of $\mathbf{M}_\alpha$ in Definition 4 with $s=1$ and $s=1/\gamma$, we get:

$$\begin{aligned}&\mathscr{L}^n(V_\varepsilon\cap B_r(x))\\ &\leqslant \left(\frac{C}{\varepsilon^{-a}\lambda}\int_{B_{2r}(x)}\mathscr{H}(\nabla u-\nabla v)dx\right)^{\frac{n}{n-\alpha}}+\left(\frac{C}{(\varepsilon^{-a}\lambda)^{\gamma}}\int_{B_{2r}(x)}\left[\mathscr{H}(\nabla v)\right]^{\gamma}dx\right)^{\frac{n}{n-\alpha\gamma}}\\ &\leqslant \left(\frac{Cr^n}{\varepsilon^{-a}\lambda}\fint_{B_{4r}(x)}\mathscr{H}(\nabla u-\nabla v)dx\right)^{\frac{n}{n-\alpha}}+\left(\frac{Cr^n}{(\varepsilon^{-a}\lambda)^{\gamma}}\fint_{B_{2r}(x)}\left[\mathscr{H}(\nabla v)\right]^{\gamma}dx\right)^{\frac{n}{n-\alpha\gamma}}.\end{aligned} \tag{5.16}$$

It is easy to check that $B_{4r}(x)\subset B_{5r}(x_2)\cap B_{5r}(x_3)$ and it follows from (5.6) that

$$\fint_{B_{4r}(x)}\mathscr{H}(\nabla u)dx \leqslant \left(\frac{5}{4}\right)^n\fint_{B_{5r}(x_2)}\mathscr{H}(\nabla u)dx \leqslant Cr^{-\alpha}\mathbf{M}_\alpha\left[\mathscr{H}(\nabla u)\right](x_2) \leqslant Cr^{-\alpha}\lambda. \tag{5.17}$$

Similarly, thanks to (5.12), it holds that

$$\begin{aligned}\fint_{B_{4r}(x)}\mathscr{H}(\nabla u-\nabla v)dx &\leqslant C\varepsilon_1^{1-a}\fint_{B_{4r}(x)}\mathscr{N}(z)dz+C\varepsilon_1^{1-K}\fint_{B_{4r}(x)}\mathscr{T}(z)dz\\ &\leqslant C\varepsilon_1^{1-a}r^{-\alpha}\mathbf{M}_\alpha\mathscr{N}(x_2)+C\varepsilon_1^{1-K}r^{-\alpha}\mathbf{M}_\alpha\mathscr{T}(x_3),\end{aligned} \tag{5.18}$$

which implies from (5.6) that

$$\fint_{B_{4r}(x)} \mathscr{H}(\nabla u - \nabla v)dx \leqslant C\left(\varepsilon_1^{1-a} + \varepsilon_1^{1-K}\varepsilon^b\right)r^{-\alpha}\lambda \leqslant C\varepsilon^{\frac{b}{K-a}}r^{-\alpha}\lambda. \tag{5.19}$$

where $\varepsilon_1 \in (0,1)$ satisfying $\varepsilon_1 = \varepsilon^{b/(K-a)}$.

On the other hand, we use reverse Hölder in Lemma 3

$$\begin{aligned} \fint_{B_{2r}(x)} \left[\mathscr{H}(\nabla v)\right]^\gamma dx &\leqslant C\left(\fint_{B_{4r}(x)} \mathscr{H}(\nabla v)dx\right)^\gamma \\ &\leqslant C\left(\fint_{B_{4r}(x)} \mathscr{H}(\nabla u) + \mathscr{H}(\nabla u - \nabla v)dx\right)^\gamma. \end{aligned} \tag{5.20}$$

Substitution (5.17) and (5.19) into (5.20), it gives

$$\fint_{B_{2r}(x)} \left[\mathscr{H}(\nabla v)\right]^\gamma dx \leqslant C\left(1 + \varepsilon^{\frac{b}{K-a}}\right)^\gamma r^{-\alpha\gamma}\lambda^\gamma \leqslant Cr^{-\alpha\gamma}\lambda^\gamma,$$

Thus, we deduce that

$$\begin{aligned} \mathscr{L}^n\left(V_\varepsilon \cap B_r(x)\right) &\leqslant \left(\frac{Cr^n}{\varepsilon^{-a}\lambda} C\varepsilon^{\frac{b}{K-a}} r^{-\alpha}\lambda\right)^{\frac{n}{n-\alpha}} + \left(\frac{Cr^n}{\left(\varepsilon^{-a}\lambda\right)^\gamma} Cr^{-\alpha\gamma}\lambda^\gamma\right)^{\frac{n}{n-\alpha\gamma}} \\ &\leqslant Cr^n\left[\varepsilon^{\left(a+\frac{b}{K-a}\right)\left(\frac{n}{n-\alpha}\right)} + \varepsilon^{\frac{a\gamma n}{n-\alpha\gamma}}\right]. \end{aligned}$$

And here, we choose γ such that $a\gamma n/(n-\alpha\gamma) \geqslant 1$ and $\varepsilon \in (0,1)$ such that $\mathrm{C}\,\varepsilon^{a\gamma n/(n-\alpha\gamma)} < \varepsilon$ to get that

$$\mathscr{L}^n(V_\varepsilon \cap B_r(x)) \leqslant \varepsilon \mathscr{L}^n(B_r(x)),$$

which makes the contradiction. Therefore, $\mathscr{L}^n(V_\varepsilon \cap B_r(x)) > \varepsilon \mathscr{L}^n(B_r(x))$.

We finally concentrate on the second case when x is close to the boundary of domain $\partial\Omega$, which means $B_{4r}(x) \cap \partial\Omega \neq \emptyset$. In this case, we select $x_4 \in B_{4r}(x) \cap \partial\Omega$ such that $d(x_4, x) < 4r$. We denote $\Omega_{6r}(x_4) = B_{6r}(x_4) \cap \Omega$ and consider v as the unique solution to the following equation:

$$\begin{cases} -\mathrm{div}\mathscr{A}(x,\nabla v) = 0 \text{ in } \Omega_{12r}(x_4), \\ \qquad\qquad v = u \text{ on } \partial\Omega_{12r}(x_4). \end{cases} \tag{5.21}$$

Because $B_{2r}(x) \subset B_{6r}(x_4)$, the estimate in (5.8) can be rewritten as

$$\mathscr{L}^n\left(V_\varepsilon \cap B_r(x)\right) \leqslant \mathscr{L}^n\left(\left\{y \in B_r(x) : \mathbf{M}_\alpha^r\left[\chi_{B_{6r}(x_4)}\mathscr{N}(y)\right] > \varepsilon^{-a}\lambda\right\}\right),$$

which yields

$$\begin{aligned} \mathscr{L}^n(V_\varepsilon \cap B_r(x)) \leqslant\, & C\mathscr{L}^n\left(\left\{(\mathbf{M}_\alpha^r\left[\chi_{B_{6r}(x_4)}\mathscr{H}(\nabla u - \nabla v)\right] > \varepsilon^{-a}\lambda\right\}\right) \\ & + C\mathscr{L}^n\left(\left\{\mathbf{M}_\alpha^r\left[\chi_{B_{6r}(x_4)}\mathscr{H}(\nabla v)\right] > \varepsilon^{-a}\lambda\right\}\right). \end{aligned}$$

Similarly (5.16), we deduce that

$$\mathscr{L}^n(V_\varepsilon \cap B_r(x)) \leqslant \left(\frac{Cr^n}{\varepsilon^{-a}\lambda} \fint_{B_{12r}(x_4)} \mathscr{H}(\nabla u - \nabla v)dx\right)^{\frac{n}{n-\alpha}} + \left(\frac{Cr^n}{(\varepsilon^{-a}\lambda)^\gamma} \fint_{B_{6r}(x_4)} \mathscr{H}(\nabla v)^\gamma dx\right)^{\frac{n}{n-\alpha\gamma}}. \tag{5.22}$$

By (3.3) in Lemma 3, we obtain the Reverse Hölder inequality for the boundary case as follows:

$$\left(\fint_{B_{6r}(x_4)} \left[\mathscr{H}(\nabla v)\right]^\gamma dx\right)^{\frac{1}{\gamma}} \leqslant C\left(\fint_{B_{12r}(x_4)} \mathscr{H}(\nabla v)dx\right).$$

Application of (3.1) in Lemma 3 enables us to get the following comparison estimate

$$\fint_{B_{12r}(x_4)} \mathscr{H}(\nabla u - \nabla v)dx \leqslant \varepsilon_1^{1-a} \fint_{B_{12r}(x_4)} \mathscr{N}(z)dz + C\varepsilon_1^{1-K} \fint_{B_{12r}(x_4)} \mathscr{T}(z)dz.$$

Using these estimates, we proceed similarly as the previous case to show that

$$\fint_{B_{12r}(x_4)} \mathscr{H}(\nabla u)dx \leqslant Cr^{-\alpha}\mathbf{M}_\alpha\left[\mathscr{H}(\nabla u)\right](x_4) \leqslant Cr^{-\alpha}\lambda,$$

$$\begin{aligned}\fint_{B_{12r}(x_4)} \mathscr{H}(\nabla u - \nabla v)dx &\leqslant C\left(\varepsilon_1^{1-a} r^{-\alpha}\mathbf{M}_\alpha \mathscr{N}(x_4) + \varepsilon_1^{1-K} r^{-\alpha}\mathbf{M}_\alpha \mathscr{T}(x_4)\right)\\ &\leqslant C\left(\varepsilon_1^{1-a} + \varepsilon_1^{1-K}\varepsilon^b\right) r^{-\alpha}\lambda \leqslant C\varepsilon^{\frac{b}{K-a}} r^{-\alpha}\lambda,\end{aligned}$$

and

$$\fint_{B_{6r}(x_4)} \left[\mathscr{H}(\nabla v)\right]^\gamma dx \leqslant C\left(\fint_{B_{12r}(x_4)} \mathscr{H}(\nabla u) + \mathscr{H}(\nabla u - \nabla v)dx\right)^\gamma \leqslant Cr^{-\alpha\gamma}\lambda^\gamma.$$

All in all, we are able to conclude the same result as the previous case by taking into account these inequalities to (5.22). The proof of Theorem 1 is therefore complete. □

5.2. Proof of Theorem 2

Proof. Firstly, let us consider the case when $t \in (0,\infty)$ and $s \in (0,\infty)$. Let us take $a \in \left(0, \min\left\{1, \frac{1}{s}\right\}\right)$. Based on Theorem 1, there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}&\mathscr{L}^n(\{\mathbf{M}_\alpha[\mathscr{H}(\nabla u)] > \varepsilon^{-a}\lambda\})\\ &\leqslant C\varepsilon\mathscr{L}^n(\{\mathbf{M}_\alpha[\mathscr{H}(\nabla u)] \geqslant \lambda\})\\ &\quad + \mathscr{L}^n(\{\mathbf{M}_\alpha\left[\mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla\psi_1) + \mathscr{H}(\nabla\psi_2)\right] \leqslant \varepsilon^b\lambda\}).\end{aligned}$$

Let us also denote $\mathscr{N}, \mathscr{T} : \Omega \to \mathbb{R}^+$ two mappings defined as

$$\mathscr{N} := \mathscr{H}(\nabla u) \text{ and } \mathscr{T} := \mathscr{H}(\mathbf{F}) + \mathscr{H}(\nabla \psi_1) + \mathscr{H}(\nabla \psi_2).$$

By changing variables in Lorentz spaces 3, we get that

$$\begin{aligned}
\|\mathbf{M}_\alpha \mathscr{N}\|^t_{L^{s,t}(\Omega)} &= \varepsilon^{-at} s \int_0^\infty \lambda^t \mathscr{L}\Big(\{\mathbf{M}_\alpha \mathscr{N} > \varepsilon^{-a}\lambda\}\Big)^{\frac{t}{s}} \frac{d\lambda}{\lambda} \\
&\leqslant C\varepsilon^{-at+\frac{t}{s}} \int_0^\infty \lambda^t \mathscr{L}\Big(\{\mathbf{M}_\alpha \mathscr{N} > \lambda\}\Big)^{\frac{t}{s}} \frac{d\lambda}{\lambda} \\
&\quad + C\varepsilon^{-at} s \int_0^\infty \lambda^t \mathscr{L}\Big(\{\mathbf{M}_\alpha \mathscr{T} > \varepsilon^{b}\lambda\}\Big)^{\frac{t}{s}} \frac{d\lambda}{\lambda} \\
&\leqslant C\varepsilon^{t\left(\frac{1}{s}-a\right)} \|\mathbf{M}_\alpha \mathscr{N}\|^t_{L^{s,t}} + C\varepsilon^{-t(a+b)} \|\mathbf{M}_\alpha \mathscr{T}\|^t_{L^{s,t}(\Omega)}.
\end{aligned}$$

Since $t\Big(1/s - a\Big) > 0$, we can choose $\varepsilon \in \big(0, \varepsilon_0\big)$ satisfying $C\varepsilon^{t\left(1/s-a\right)} \leqslant 1/2$, which completes the proof. The same condition can be drawn for the case $t = \infty$. □

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